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Topological characterizations of ω_μ-metrizable spaces

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Abstract

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This paper is a detailed elaboration of a talk given by the second author at the Oxford conference in June 1989. It presents necessary and sufficient conditions for a topological space to be ω_μ-metrizable ($\mu > 0$), i.e., linearly uniformizable with uncountable uniform weight. In other words, such spaces are exactly those which can be metrized by a distance function taking its values in a totally ordered Abelian group with cofinality ω_μ. (For $\omega_\mu = \omega_0$, we obtain characterizations of strongly zero-dimensional metric spaces, i.e., nonarchimedeanly metrizable spaces.)

It turns out that (strong) suborderability and the existence of a σ-discrete (respectively ω_μ-discrete) dense subspace are the most interesting properties in this respect, whenever $\omega_\mu > \omega_0$, or $\omega_\mu = \omega_0$ and $\dim X = 0$. Therefore, a main part of the paper is devoted to the study of GO-*spaces having a σ-discrete (ω_μ-discrete) dense subspace* (Section 3). The last section (Section 4) is concerned with the characterization of ω_μ-metrizability in the realm of generalized metric spaces, in particular, by using g-functions.

Since all our spaces are *zero-dimensional*, the paper also contributes results to this important class of spaces, in particular, to the class of nonarchimedean topological spaces.

Keywords: ω_μ-metrizability, linear uniformizability, generalized metric spaces, linearly stratifiable spaces, GO-spaces, σ-discrete subset, strong suborderability, zero-dimensional spaces, nonarchimedean metrics, nonarchimedean topological spaces, GO-spaces with ω_μ-discrete dense subspaces.

AMS (MOS) Subj. Class.: 54E35, 54E15, 54E20, 54F05.

Introduction and survey

Recently, (e.g. by papers of Hodel [6], Hušek [7], Kopperman [10, 4], and others), the theory of nonnumerical distance functions has gained new interest, see also [21].

One of the most interesting kinds of such distance functions are "metrics" d which take their values in linearly ordered Abelian groups $(G; +, \leqslant)$. A topological space (X, τ) is *metrizable over G* if there is a (distance) function $d: X^2 \to G$ satisfying

(1) $d(x, y) \geqslant 0\ \forall x, y \in X;\ d(x, y) = 0 \Leftrightarrow x = y$,

(2) $d(x, y) = d(y, x)\ \forall x, y \in X$,

(3) $d(x, z) \leqslant d(x, y) + d(y, z)\ \forall x, y, z \in X$,

and such that, for any $x \in X$, the balls $B_\varepsilon(x)$, $\varepsilon \in G$, $\varepsilon > 0$, form a local base at x. The following equivalences are well known:

(1) (X, τ) is metrizable over a totally ordered Abelian group $(G, \leqslant)$ with cofinality $\operatorname{cof} G = \omega_\mu$.

(2) (X, τ) is metrizable either over $\mathbb{R}$ ("ω_0-metrizable" = "metrizable") or – if $\mu > 0$ – over $G = \prod\{\mathbb{Z}_i \mid i < \omega_\mu\}$[1] ordered lexicographically (X is "ω_μ*-metrizable* for $\mu > 0$") (see e.g. [12, 14, 21, 22]).

(3) τ can be generated by a *uniformity U* on X which has a *totally ordered base* $\mathscr{B} = \{B_i \mid i < \omega_\mu,\ B_j \subset B_i \Leftrightarrow i \leqslant j\}$, i.e., (X, τ) is "*linearly uniformizable*" and its uniform weight is ω_μ, which – in that case – is also equal to the pseudoweight $\varphi(\Delta X)$ of the diagonal ΔX of X (see e.g. [7]).

Several authors have contributed *topological characterizations* of ω_μ-metrizable spaces for $\mu > 0$ (see e.g. [6] and the bibliography in [7]), most of them generalizing classical metrization theorems to higher cardinals, This situation, however, is not completely satisfying, since, for $\mu > 0$, the class of ω_μ-metrizable spaces is not just a direct and straightforward generalization of metric spaces, it has many autonomous and independent aspects which do not have analogues for metrizable spaces in general (see e.g. [7, 9, 17, 22, 23] and others). It is this aspect which shall be emphasized in the following (which in some sense fills up the work in [7, 15, 20], where Purisch defines "almost n.a. spaces").

For $\omega_\mu > \omega_0$, ω_μ-metrizable spaces are *strongly zero-dimensional* and – even stronger – they are *nonarchimedean (n.a.) topological spaces*, i.e., T_1-spaces with a topological base $\mathscr{B}$ such that, for all disjoint $B_1, B_2 \in \mathscr{B}$, either $B_1 \subset B_2$ or $B_1 \supset B_2$ (for more details, see [2, 7, 12, 14, 15, 19, 20, 23]).[2] Hence they are *hereditarily paracompact* and *suborderable.* If a n.a. topological space is metrizable, it is nonarchimedeanly metrizable ($d(x, z) \leqslant \max[d(x, y), d(y, z)]\ \forall x, y, z \in X$). And – as it has been proved e.g. by de Groot [1] – (X, τ) is n.a. metrizable iff $\dim X = 0$. Hence, for $\mu > 0$, ω_μ-metrizable spaces typically generalize strongly zero-dimensional metric spaces.

Another topological property typical for ω_μ-metrizable spaces if $\mu > 0$, is that all these spaces are *strongly suborderable* (Section 2). Therefore, in the following, we want to characterize ω_μ-metrizable spaces ($\mu > 0$) as a *subclass of*

(1) all *nonarchimedean* topological spaces (Section 1) and

[1] $\mathbb{Z}_i$ denotes the set of all integers, for every i.

[2] Equivalently, these are exactly those T_1-spaces having a *tree base* (with respect to inclusion of basic sets. (For this reason, n.a. spaces also were called "spaces with a ramified base" (D. Kurepa, P. Papič; see [7, 23]).

(2) all *strongly suborderable* spaces (Section 2).

Generally, n.a. topological or strongly suborderable spaces X are not ω_μ-metrizable if they have "too many isolated points which are too close by the set X' of all nonisolated points of X", as the Michael line typically shows [7, 15]. Therefore, similarly as in [7], we have to look for conditions *excluding this situation.* The main tool used hereby will be κ-discrete dense subspaces of X. Therefore, in Section 3, we generally characterize GO-spaces having κ-discrete dense subspaces, for $\kappa \geq \omega_0$. Finally, in Section 4, we shall indicate how ω_μ-metrizable spaces could be *characterized in terms of g-functions.*

Note that specializing our theorems to $\omega_\mu = \omega_0$ yields topological characterizations of strongly zero-dimensional metric spaces, i.e., nonarchimedeanly metrizable spaces, which are interesting (under a different point of view) by themselves. Historical sketches of the whole theory can be found in [7, 17, 21, 23] and in other papers listed there.

1. Nonmetrizable ω_μ-metrizable spaces as a subclass of nonarchimedean spaces

Let X be a T_1-space. For any nonisolated $x \in X$, let $\varphi(x)$ denote the pseudoweight of x, i.e., the smallest infinite cardinal κ such that $\{x\}$ is the intersection of κ many open sets in X. $\varphi(\Delta X)$ denotes the pseudoweight of $\Delta X = \{(x, x) \mid x \in X\}$ in X^2. $D \subset X$ is *σ-discrete* ("ω_0-discrete") if D is the union of countably many closed discrete subsets $D_i \subset X$, $i < \omega_0$. For $\mu > 0$, *ω_μ-discrete* subsets are defined analogously, for $i < \omega_\mu$. Now we can prove

Theorem 1.1. *The following are equivalent*:

(i) *X is ω_μ-metrizable for $\mu > 0$, or X is metrizable and* $\dim X = 0$.

(ii) *X is a nonarchimedean topological space, $\varphi(x) = \omega_\mu$ for any nonisolated $x \in X$, and X has an ω_μ-discrete dense subset $D \subset X$.*

(iii) *X is a nonarchimedean topological space,* ad $X \geq \omega_\mu$, *and X has an ω_μ-discrete dense subset.* (*Here,* ad X *is the smallest* (*infinite*) *cardinal κ such that the intersection of fewer than κ many open sets is open.*)

Proof. (i) $\Rightarrow$ (ii): Consider (X, τ) to be linearly uniformizable. We can assume that τ is generated by a uniform structure $\mathcal{U}$ on X having a well-ordered base $\mathcal{B} = \{B_i \mid i < \omega_\mu\}$ for its entourages; $B_j \subset B_i \Leftrightarrow i \leq j$. Moreover, we can assume that $B_i \circ B_i = B_i$, i.e., B_i is an equivalence relation on X. This can be done because if B_i is any entourage in $\mathcal{B}$, and $\mu > 0$, then there is a sequence of entourages $B_i = B_{i(0)} \supset B_{i(1)} \supset B_{i(2)} \supset \cdots$ such that $B_{i(n+1)} \circ B_{i(n+1)} \subset B_{i(n)}$ and hence $\bigcap \{B_{i(n)} \mid n = 1, 2, \ldots\} =: B^{(i)}$ is an entourage of $\mathcal{U}$ again satisfying $B^{(i)} \circ B^{(i)} = B^{(i)}$. If $\mu = 0$, our assumption follows because $\dim X = 0$. Now for any $x \in X$, let $B_i[x] = \{y \mid (x, y) \in B_i\}$. Then for all $i < \omega_\mu$, $\mathcal{C}_i = \{B_i[x] \mid x \in X\}$ is a clopen partition of X, and $\mathcal{C}_j$ refines $\mathcal{C}_i$ whenever $j \geq i$. Finally, fix $i < \omega_\mu$, and from each $B_i[x] \in \mathcal{C}_i$, take one point $y \in B_i[x]$.

These points - all together - form a closed discrete set $D_i \subset X$. Since $\mathcal{U}$ generates the topology τ of X, the union $D = \bigcup \{D_i \mid i < \omega_\mu\}$ is dense in X, and we are done.

Conversely, let us show (ii) $\Rightarrow$ (i): Let X satisfy the conditions of (ii) for some regular cardinal ω_μ, and let $\mathcal{C}$ be a *nonarchimedean base* for τ; then all $B \in \mathcal{C}$ are clopen automatically. Now let $D = \bigcup \{D_i \mid i < \omega_\mu\}$ be an ω_μ-discrete subset of X. Then - at any $x \in X$ - those $C \in \mathcal{C}$ containing x form a monotone local orthobase [12, 14], i.e., any intersection is open or consists of x only. By technical reasoning we can assume that the union of any chain $C_1 \subset C_2 \subset \cdots$, $C_k \in \mathcal{C}$, belongs to $\mathcal{C}$, too (compare e.g. [15] or [2]).

Now, for any $x \in X$, let $B_1(x)$ be the maximal element of $\mathcal{C}$ containing x and *at most one* point of D, i.e., $|B_1(x) \cap D_1| \leqslant 1$. $\{B_1(x) \mid x \in X\} =: \mathcal{B}_1$ is a cover of X and in fact a clopen partition of X. Since for any nonisolated $x \in X$, $\{C \mid x \in C \in \mathcal{C}\}$ is monotone, and $\varphi(x) = \omega_\mu$, the intersection of fewer than ω_μ many open sets in X is open again. Hence $D^{(i)} = \bigcup \{D_j \mid j \leqslant i < \omega_\mu\}$ is closed and discrete again, and we can proceed inductively. For $x \in X$ and $i < \omega_\mu$, define $B_i(x)$ to be the maximal basis set in $\mathcal{C}$ containing x and at most one point of $\bigcup \{D_j \mid j \leqslant i\}$, i.e., $|B_i(x) \cap (D_1 \cup \cdots \cup D_i)| = 1$. Again, $\mathcal{B}_i = \{B_i(x) \mid x \in X\}$ is a clopen partition of X and $\mathcal{B}_j$ refines $\mathcal{B}_i$ whenever $j \geqslant i$. Therefore, the family of all $\mathcal{B}_i' = \mathcal{B}_i \cup \{\{x\} \mid x$ is an isolated point, $x \in D_i\}$ constitute a totally ordered base of a covering uniformity $\mathcal{U}$ on X. To conclude the proof we have to show that $\mathcal{U}$ generates the originally given topology τ. To see this, let $x \in X$ be a nonisolated point and U an open set containing x. We can assume that $U \in \mathcal{C}$. Now, since X is T_1, $U \backslash \{x\}$ is open and hence contains infinitely many points, especially two distinct points of D, say $x_k \in D_k$ and $x_n \in D_n$, $k < n$ (remember that every D_k is closed, hence $X \backslash D_k$ is open). Further, $B_n(x)$ must be contained in U, since otherwise - by the nonarchimedean property of $\mathcal{C}$ - $B_n(x) \supset U$, hence $x_k \in B_n(x)$ which contradicts the fact that $|B_n(x) \cap (D_1 \cup \cdots \cup D_k \cup \cdots \cup D_n)| < 1$. Summarizing $\{B_i(x) \mid i < \omega_\mu, x \in X\}$ constitutes a base for τ, and we are done. In this proof we essentially used only the fact that, by our conditions, ad $X = \omega_\mu$. Hence, automatically, (iii) $\Rightarrow$ (i) is proved by the same argument.

(ii) $\Rightarrow$ (iii): Also in the above argument it is shown that (ii) implies ad $X \geqslant \omega_\mu$. □

Corollary 1.2 [2]. *A nonarchimedean topological space X is metrizable iff it has a σ-discrete dense subspace.*

The Michael line (which is gotten from the reals by turning the irrational points into isolated points) obviously is a nonarchimedean topological space which does not have a σ-discrete dense subspace (compare [7]).

Remark 1.3. From Theorem 1.1 we can conclude most of the characterizations of ω_μ-metric spaces ($\mu > 0$) given or summed up in [7]. Compare also the examples and counterexamples given there.

Remark 1.4. It is crucial for the above theorem to ask that all $D_i \subset D$ are closed in X. The one-point Lindelöfization $D \cup \{x\}$ of a cardinality $\aleph_1$ discrete space D is nonarchimedean but not metrizable since it is not first countable. (More exactly, $D = (\omega_1 + 1) - \{\alpha \mid \lim \alpha < \omega_1\}$.) On the other hand, D is a nonclosed discrete subset of X. (Obviously, there is an analogue of this example for every $\omega_\mu > \omega_0$.)

2. ω_μ-metrizable spaces as special kinds of strongly suborderable spaces

By a theorem of Herrlich [5], a totally disconnected metric space X is orderable iff $\dim X = 0$[3] (hence every nonarchimedean metrizable space is orderable). Unfortunately, for every $\omega_\mu > \omega_0$, there exists a *nonorderable* ω_μ-metrizable space [23]. However, we can show that, for $\mu > 0$, any ω_μ-metrizable space X is *strongly suborderable*, i.e., there exists a linear order $\leqslant$ on X such that open intervals form a local base at any nonisolated point $x \in X$ (see [7, p. 178], which is a direct generalization of a method described in [16]). Therefore, it seems natural to conversely ask which strongly suborderable spaces are ω_μ-metrizable.

Theorem 2.1. *The following conditions are equivalent:*

(i) *X is ω_μ-metrizable for $\mu > 0$ or X is metrizable and* $\dim X = 0$.

(ii) *X is strongly suborderable,* (*weakly*) *zero-dimensional,* $\operatorname{ad} X = \varphi(\Delta X)(= \omega_\mu)$ *and X has an ω_μ-discrete dense subset.*

Proof. By the comments above and the proof of Theorem 1.1 we only have to show (ii) $\Rightarrow$ (i): Let $D = \bigcup \{D_i \mid i < \omega_\mu\}$ be ω_μ-discrete and dense in X, and if X has a first or last point, let them be in D. By our assumption, for any nonisolated point $x \in X$, we have $\varphi(x) = \varphi(\Delta X) = \omega_\mu$, i.e., the intersection of fewer than ω_μ many open sets is open, and X has a monotone local base of cardinality ω_μ. Moreover, for each $j < \omega_\mu$, $\bigcup \{D_i \mid i \leqslant j\} = D^{(j)}$ is closed and discrete, and, obviously, $D^{(j)} \supset D^{(i)}$ if $j \geqslant i$; $D^{(1)} = D_1$. Finally, let $\Delta X = \bigcap \{U_i \mid i < \omega_\mu\}$, where U_i is open in X^2.

Now, since X is suborderable, it follows from our assumptions that X is paracompact and strongly zero-dimensional, i.e., $\dim X = 0$ (see e.g. [7, (i), (iii), p. 176]). Now, by induction we construct a topologically compatible covering - uniformity similarly to the method in the proof of Theorem 1.1.

For each $j < \omega_\mu$, let $\mathscr{B}_j$ be a partition of X into clopen sets B such that (i) $B \times B \subset U_j$; (ii) each $B \in \mathscr{B}_j$ contains at most one point of $D^{(j)}$; (iii) for all isolated $x \in D^{(j)}$, $\{x\} \in \mathscr{B}_j$; (iv) for $i \leqslant j$, $\mathscr{B}_j$ refines $\mathscr{B}_i$; and (v) each B is *convex* in the chosen strong suborder $\leqslant$; that is, if $x < y < z$ and $\{x, z\} \subset B$, then $y \in B$. All this is a routine application of ultraparacompactness, strong suborderability, and the fact that $\operatorname{ad} X = \omega_\mu$. Indeed, by $\operatorname{ad} X = \omega_\mu$, we can obtain an open cover satisfying (i)–(iv), with the additional property that each isolated $x \in D^{(j)}$ is in exactly one member of the cover,

[3] Independently proved by P. Papič, too.

namely $\{x\}$. Then refine this cover to a clopen partition $\mathcal{P}$, and replace each $P \in \mathcal{P}$ by its convexity components, i.e., equivalence classes with respect to the relation $x \equiv y$ iff x, y, and all intermediate points are in P. The fact that these equivalence classes are both open and closed follows easily from the same properties of P (compare the proof of Theorem 3.6(1) below).

For each $x \in X$ and each $i < \omega_\mu$, let $C_i(x)$ be the unique member of $\mathcal{B}_i$ containing x. Since the equivalence relation associated with $\mathcal{B}_i$ is finer than U_i, and $\Delta X = \bigcap \{U_i \mid i < \omega_\mu\}$, it follows that $\bigcap \{C_i(x) \mid i < \omega_\mu\} = \{x\}$. Since X is strongly suborderable and $\{x\} \in \mathcal{B}_i$ for some i whenever x is isolated, it follows that $\{\mathcal{B}_i \mid i < \omega_\mu\}$ is a base for a uniformity on X. □

Remark 2.2. It is important to note that X was supposed to be *strongly suborderable.* Replacing this assumption by "suborderable" in general we cannot conclude our results as seen e.g. from [7, Example 3, p. 182]. By this example, for every regular ω_μ, there is a dense in itself paracompact suborderable, zero-dimensional space X, $\varphi(x) = \varphi(\Delta X) = \omega_\mu$, such that intersections of fewer than ω_μ many open sets are open, but X is not ω_μ-metrizable (even not nonarchimedean).

Corollary 2.3. *A T_1-space is nonarchimedeanly metrizable iff X is strongly suborderable, zero-dimensional, X has a G_0-diagonal and a σ-discrete dense subset.*

It may be interesting to recall a result of Wouwe [24] who showed that any suborderable space with a σ-discrete dense subset is perfectly normal, and to recall the 15 years old open problem whether every perfectly normal nonarchimedean topological space is metrizable (and hence nonarchimedeanly metrizable); see [2, 12, 14, 15]. (We know, that the answer is "no" if there is a Suslin line [7]. And, by a result of Purisch, every perfectly normal n.a. topological space is orderable [19].)

3. GO-spaces with ω_μ-discrete dense subspaces

In the preceding paragraphs, when characterizing ω_μ-metrizable spaces, the most important properties were the two named in the heading above. Therefore, in the following, we want to investigate these properties in more detail and finally, in this realm, we present a new and very general characterization of ω_μ-metrizable spaces.[4] Moreover, the following result illuminates the difference between *suborderable* and *strongly suborderable* spaces, as well. We start by systematically collecting definitions and facts (either well known or cited in the preceding paragraphs) which will be used in the proof of the main theorem.

[4] Besides, this section throws new light on a paper of Purisch [20], which in some sense will be "completed" by the ideas below.

Fact 3.1. *Every strongly zero-dimensional metrizable space (X, τ) is orderable. For $\mu > 0$, any ω_μ-metrizable space (X, τ) is strongly zero-dimensional and (strongly) suborderable. In other words, there is a linear order $\leqslant$ on (X, τ) such that $(X, \leqslant, \tau)$ is a* GO*-space. Moreover, for any nonisolated point x, the open intervals (a, b), $a < x < b$, form a local base at x. We say that $\leqslant$ "fits" the ω_μ-metrizable topology τ. For every $\omega_\mu > \omega_0$, there is a nonorderable ω_μ-metric space* [23].

Fact 3.2. *There is an ω_μ-analogue of the Nagata–Smirnov metrizability theorem: a regular space X is ω_μ-metrizable iff it is* (1) *ω_μ-additive (i.e., the intersection of fewer than ω_μ many open sets is open), and* (2) *X has an ω_μ-locally finite base* ([11]; *see also* [6, 17]).

Definition 3.3. Let $(X, \leqslant, \tau)$ be a GO-space. A point x is *half-isolated* if X has a local base of nondegenerate half-open intervals of which x is one endpoint. It is *naturally half-isolated* if it has an immediate predecessor or an immediate successor, and *artificially half-isolated* otherwise. (Because of nondegeneracy it cannot have both an immediate predecessor and an immediate successor.) The Sorgenfrey line, for example, consists of artificially half-isolated points.

In the literature, naturally half-isolated points are also called "*jump points*", whereas artificially half-isolated points are often called "*pseudogap points*". Here, moreover we exclude nonisolated endpoints from being defined as pseudogap points.

Definition 3.4. Let $(X, \leqslant)$ be a totally ordered set and let $S \subset X$. Then $A(X, S)$ – as a set – is gotten from X by replacing each $x \in S$ by two elements, x^+ and x^-, i.e., $A(X, S) = (X \backslash S) \cup \{x^- \mid x \in S\} \cup \{x^+ \mid x \in S\}$. Now a linear *order* on $A(X, S)$ is defined as follows: For $x \in X \backslash S$, let $x^\# = \{x\}$, if $x \in S$, let $x^\# = \{x^-, x^+\}$, $x^- < x^+$. Now, if $x < y$, we define $x^\# < y^\#$. (So, for example, if $x < y$, $x \in S$, $y \in S$, we have $x^- < x^+ < y^- < y^+$.) Briefly, we can say $A(X, S)$ is the splitting of the points S in X: $A(X, S) = (X \times \{0\}) \cup (S \times \{1\})$ with the lexicographic order.

Definition 3.5. If $(X, \leqslant, \tau)$ is a GO-space and $S \subset X$, then the *canonical topology on* $A(X, S)$ is defined by describing local bases for the "new" points $x^-, x^+ \in A(X, S)$ whenever $x \in S$:

(1) If x is isolated in τ, let x^- and x^+ be isolated points in $A(X, S)$.

(2) *If* x is half-isolated on the right in X, let x^+ be isolated in $A(X, S)$, and the intervals $(w, x^-]$ form a local base at x^-.

(3) If x is half-isolated on the left in X, similarly let x^- be isolated in $A(X, S)$, and the intervals $[x^+, w)$ form a local base at x^+.

(4) If x is *neither* isolated *nor* half-isolated in X, the intervals $(w, x^-]$ form a local base at x^-, while the intervals $[x^+, w')$ form a local base at x^+ in $A(X, S)$.

Now let us formulate the main *results* of this section.[5]

Theorem 3.6. *Let $(X, \leq, \tau)$ be a strongly zero-dimensional ω_μ-metrizable* GO*-space. Then*

(1) *X has a uniformity $\mathcal{U}$ with a totally ordered base of partitions into τ-clopen intervals, and*

(2) *the collection of half-isolated points $x \in X$ forms an ω_μ-discrete subspace of X.*

Theorem 3.7. *Let Y be a strongly zero-dimensional space. Then the following are equivalent:*

(i) *Y is ω_μ-metrizable.*

(ii) *Y is strongly suborderable by a total order $\leq$, Y has an ω_μ-discrete dense subspace which includes all (naturally) half-isolated points of $(Y, \leq, \tau)$, and Y is ω_μ-additive.*

(iii) *Y is a* GO*-space w.r.t. a total order $\leq$, such that Y has an ω_μ-discrete dense subspace which includes all half-isolated points of $(Y, \leq, \tau)$, and Y is ω_μ-additive.*

Theorem 3.8. *Let Y be a strongly zero-dimensional suborderable space. Then the following are equivalent:*

(i) *Y has an ω_μ-discrete dense subspace, and Y is ω_μ-additive.*

(ii) *Y embeds in some $A(X, S)$ with the canonical topology, for some $(X, \leq)$ where $\leq$ fits some ω_μ-metrizable topology on X.*

(iii) *Y embeds in some $A(X, S)$ with the canonical topology for some ω_μ-metrizable* GO*-space $(X, \leq, \tau)$.*

Theorem 3.9. *Let Y be a strongly zero-dimensional space. Then the following are equivalent:*

(i) *Y is strongly suborderable, Y has an ω_μ-discrete dense subspace, and Y is ω_μ-additive.*

(ii) *$Y \approx A(X, S)$ with the canonical topology for some ω_μ-metrizable* GO*-space $(X, \leq, \tau)$ such that $\leq$ fits τ (in particular, (X, τ) is strongly suborderable).*

The proofs. The proofs of Theorems 3.6–3.9 will now be given in *one* bunch, because the main ideas and the main steps are similar and they can be combined in a certain way. Moreover, the "hierarchy" of the above results will become clearer by arranging the proofs in the way we chose in the following.

Proof of Theorem 3.6(1). Let $\mathcal{U}$ be a uniformity with base $\{\mathcal{B}_\alpha \mid \alpha < \omega_\mu\}$ of partitions of X into clopen sets with $\mathcal{B}_\alpha$ refining $\mathcal{B}_\beta$ whenever $\beta < \alpha$. For each x and each

[5] Note that specializing the following results to the countable case, $\omega_\mu = \omega_0$, we obtain chacterizations of GO-spaces having a *σ-discrete dense subspace.*

α, let $\mathscr{B}_\alpha(x)$ be the unique member of $\mathscr{B}_\alpha$ containing x. Given any $\mathscr{B}_\alpha$ and any $x \in X$, let $\mathscr{C}_\alpha(x) = I_\alpha^-(x) \cup \{x\} \cup I_\alpha^+(x)$, where

$$I_\alpha^-(x) = \{y \mid y \in \mathscr{B}_\alpha(x),\ y < x, \text{ and } z \in \mathscr{B}_\alpha(x) \text{ whenever } y < z < x\},$$

$$I_\alpha^+(x) = \{y \mid y \in \mathscr{B}_\alpha(x),\ y > x, \text{ and } z \in \mathscr{B}_\alpha(x) \text{ whenever } x < z < y\}.$$

Claim. *$\mathscr{C}_\alpha(x)$ is clopen.*

Once the claim is proven, let $\mathscr{C}_\alpha = \{\mathscr{C}_\alpha(x) \mid x \in X\}$.

Now, $\mathscr{C}_\alpha$ is a partition of X into convex sets, meaning that if $z, y \in \mathscr{C}_\alpha(x)$ and $z < w < y$, then $w \in \mathscr{C}_\alpha(x)$, and if $z \in \mathscr{C}_\alpha(x) \cap \mathscr{C}_\alpha(y)$ for (say) $x < y$, then $\mathscr{B}_\alpha(x) = \mathscr{B}_\alpha(y) = \mathscr{B}_\alpha(z)$ (because $\mathscr{B}_\alpha$ is a partition) and hence $\mathscr{C}_\alpha(x) = \mathscr{C}_\alpha(y) = \mathscr{C}_\alpha(z)$ (since $\mathscr{C}_\alpha(z)$ is easily seen to be the union of all subintervals of $\mathscr{B}_\alpha(z)$ containing z, we have $(\mathscr{C}_\alpha(x) \cup \mathscr{C}_\alpha(y)) \subset \mathscr{C}_\alpha(z)$, and by similar arguments, $\mathscr{C}_\alpha(x) = \mathscr{C}_\alpha(y) = \mathscr{C}_\alpha(z)$).

$\mathscr{C}_\alpha$ refines $\mathscr{C}_\beta$ whenever $\mathscr{B}_\alpha$ refines $\mathscr{B}_\beta$ and so $\{C_\alpha(x) \mid \alpha < \omega_\mu\}$ is a totally ordered basis for a (topologically compatible) uniformity on (X, τ). Hence (1) follows as soon as we prove the claim:

Proof of the claim. $\mathscr{C}_\alpha(x)$ is *open*: since $\mathscr{C}_\alpha(x)$ is the union of all subintervals of $\mathscr{B}_\alpha(x)$ containing x, and $\mathscr{B}_\alpha(x)$ is open, any endpoint of $\mathscr{C}_\alpha(x)$ has a nbhd V contained in $\mathscr{B}_\alpha(x)$ which is also an interval, so V is a subset of $\mathscr{C}_\alpha(x)$. Hence $\mathscr{C}_\alpha(x)$ is open. On the other hand, $\mathscr{C}_\alpha(x)$ is *closed*, because any supremum y it has in its closure is in $\mathscr{B}_\alpha(x)$, hence by definition of $I_\alpha^+(x)$, $y \in I_\alpha^+(x) \subset \mathscr{C}_\alpha(x)$, and similar for infima.

Proof of Theorem 3.6(2). Consider the usual inductive process of giving X a total order using $\{\mathscr{C}_\alpha \mid \alpha < \omega_\mu\}$ (described in [7] and indicated in [16, 23] and (older) papers of Kurepa and (possibly) Papič). For any $x \in X$, we have $\{x\} \subset \bigcap \{\mathscr{C}_\alpha(x) \mid \alpha < \omega_\mu\}$, and $x < y$ iff there exists $\alpha < \omega_\mu$ such that $\mathscr{C}_\alpha(x) < \mathscr{C}_\alpha(y)$ in the total order imposed on $\mathscr{C}_\alpha$. Morover, $\{\mathscr{C}_\alpha(x) \mid \alpha < \omega_\mu\}$ is a local base at x, so that if x is half-isolated, it is actually an endpoint of some $\mathscr{C}_\alpha(x)$. But any $\mathscr{C}_\alpha(x)$ has at most two endpoints and – as shown above – $\{\mathscr{C}_\alpha(x) \mid x \in X\}$ is a discrete collection for any fixed $\alpha < \omega_\mu$. Hence the set of *all* half-isolated points is (either empty or) an ω_μ-discrete subset of X.

Proof of Theorem 3.7. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii): Think of Fact 3.1 and combine Theorem 2.1 with Theorem 3.6(2). By Theorem 2.1, (Y, τ) has an ω_μ-discrete subspace $D = \bigcup \{D_i \mid i < \omega_\mu\}$ which, by Theorem 3.6, can be made to contain all (naturally) half-isolated points w.r.t. any order $\leqslant$ that fits τ. Clearly, $(Y, \leqslant, \tau)$ is a GO-space. Further any half-isolated point is *naturally* half-isolated, since Y is *strongly* suborderable.

The proof of (iii) $\Rightarrow$ (i) will be combined with the proofs of parts (i) $\Rightarrow$ (ii) of Theorems 3.8 and 3.9 later on.

Proof of Theorem 3.8. (ii) $\Rightarrow$ (iii): Obvious.

(iii) $\Rightarrow$ (i): By the preceding theorems, we know, that $(X, \leqslant)$ has an ω_μ-discrete subspace $D = \bigcup\{D_i \mid i < \omega_\mu\}$ which contains all half-isolated and isolated points. Then $\bigcup\{d^\# \mid d \in D\} = D^\#$ is dense and ω_μ-discrete, because if F is closed (respectively discrete) in X, then $\bigcup\{f^\# \mid f \in F\}$ is closed (respectively discrete) in $A(X, S)$, by definition of the canonical topology. And these properties carry over to $Y \cap D$ as a subspace of Y. Moreover, the fact that X is ω_μ-additive carries over routinely, first to $A(X, S)$ and then to Y.

(i) $\Rightarrow$ (ii): Will be proved below, together with (i) $\Rightarrow$ (ii) of Theorem 3.9 and (iii) $\Rightarrow$ (i) of Theorem 3.7.

Proof of Theorem 3.9. (ii) $\Rightarrow$ (i): Essentially the same as Theorem 3.8 (ii) $\Rightarrow$ (i).

Finally, (i) $\Rightarrow$ (ii) of Theorem 3.9 will be given by the following reasoning.

Combined proof (of (iii) $\Rightarrow$ (i) of Theorem 3.7 and (i) $\Rightarrow$ (ii) of Theorems 3.8 and 3.9). Let $D = \bigcup\{D_i \mid i < \omega_\mu\}$ be the dense subset hypothesized, with every D_i closed and discrete in Y. We may assume that any (possible existing) endpoint of Y is in D_0. Moreover, we can assume $D_i \subset D_j$ whenever $i < j$, since Y is ω_μ-additive (compare the proof of Theorem 1.1).

Step 1. D_0 induces a partition of Y into τ-clopen intervals each of which meets it in at most one point, as follows:

(a) If $d \in D_0$, let $I^+(d)$ be a τ-open, τ-closed on the right interval defined as follows. If d is *isolated*, or *half-isolated on the right*, we make $I^+(d) = \emptyset$. If d is *neither*, but has an *immediate D_0-successor d'*, let $I^+(d)$ be an initial segment of the interval (d, d'); this can be done using strong zero-dimensionality of Y. If d is *neither* and does *not* have an immediate D_0-successor, let $I^+(d) = \{y \in Y \mid y > d$ and there is no $d' \in D_0$ such that $d < d' \leqslant y\}$. Since D_0 is τ-closed and discrete, $I^+(d)$ is τ-open then, and because, in this case now, $\{d' \in D_0 \mid d' > d\}$ has no least element, $I^+(d)$ is τ-closed on the right.

Similarly, for $d \in D_0$, let $I^-(d)$ be a τ-open τ-closed on the left interval constructed as follows: if d is *isolated* or *half-isolated on the left*, let $I^-(d) = \emptyset$. If d is *neither* but has an *immediate D_0 predecessor d''*, let $I^-(d) = (d'', d) \setminus I^+(d'')$. If d is *neither* and does *not* have an immediate D_0-predecessor, let $I^-(d) = \{y \in Y \mid y < d$ and there is no $d' \in D_0$ such that $y \leqslant d' < d\}$. Similar to the above, $I^-(d)$ is τ-open and τ-closed on the left.

Now, for each $d \in D_0$, let $I(d) = I^-(d) \cup \{d\} \cup I^+(d)$. It is easy to see that $I(d)$ is a τ-clopen interval of T. Also, if $d_1, d_2 \in D_0$ and $d_1 < d_2$, then $I(d_1) \cap I(d_2)$ is empty. This is clear by construction if d_1 is the immediate D_0-predecessor of d_2; otherwise, there exists $d \in D_0$ such that $d_1 < d < d_2$, and then $I^+(d_1)$ does not contain d (by construction of $I^+(d)$) and neither does $I^-(d_2)$, and since the $I(d_i)$ are intervals they must be disjoint.

(b) Now if $y \in Y \setminus D_0$ and y has both an *immediate D_0-predecessor d* and an *immediate D_0-successor d'*, then d' is the immediate D_0-successor of d and, therefore,

y is in either $I^+(d)$ or $I^-(d')$ *unless we have both*: $I^+(d)$ *and* $I^-(d')$ are empty. In that case, y is not in any $I(d'')$, $d'' \in D_0$.

If, secondly, $y \in Y \backslash D_0$ and has an immediate D_0-successor (respectively D_0-predecessor) d but *no* immediate D_0-predecessor (respectively D_0-successor), then $y \in I^-(d)$ (respectively $y \in I^+(d)$) *unless* $I^-(d)$ (respectively $I^+(d)$) *is empty*, in which case y is not in any $I(d')$, $d' \in D_0$.

Thirdly, if $y \in Y \backslash D_0$ and y has *neither* an immediate D_0-successor *nor* an immediate D_0-predecessor, then $y \notin \bigcup \{I(d) \mid d \in D_0\}$. And, for any such y, let $I(y)$ be the union of all intervals containing y and missing D_0, i.e., $I(y) = \{y' \in Y \mid \text{either } y' \geq y$ and there is no $d \in D_0$ s.t. $y \leq d \leq y'$, or $y' \leq y$ and there is no $d \in D_0$ s.t. $y' \leq d \leq y\}$. Then it is easy to see that $\{I(y) \mid y \notin \bigcup \{I(d) \mid d \in D_0\}\}$ is a partition of $Y \backslash \bigcup \{I(d) \mid d \in D_0\}$ into open intervals. So, together with $\{I(d) \mid d \in D_0\}$ we have a *partition of* Y into τ-open, hence τ-clopen, intervals, which finishes Step 1.

Step 2. Now assume that we are given a partition $\mathcal{U}_\alpha$, $\alpha < \omega_\mu$, of Y into clopen intervals. Each of these meets D_α in a closed discrete subspace, so we can use the intersection of D_α with each member of $\mathcal{U}_\alpha$ to subdivide it into τ-clopen subintervals, just as D_0 subdivided all of Y into τ-clopen subintervals (Step 1). In particular, each subinterval contains *at most one* member of D_α (compare the proof of Theorem 1.1).

Let $\mathcal{U}_1$ be the partition obtained from D_0. In general, if $\mathcal{U}_\beta$ has been defined for all $\beta \leq \alpha$, apply D_α to $\mathcal{U}_\alpha$ as prescribed, and let $\mathcal{U}_{\alpha+1}$ be the resulting refinement of $\mathcal{U}_\alpha$. If $\mathcal{U}_\beta$ has been defined for all $\beta < \alpha$ and α is a limit ordinal $< \omega_\mu$, let $\mathcal{U}_\alpha$ be the common refinement of the preceding $\mathcal{U}_\beta$. By the fact that intersection of less than ω_μ many open sets is open (ω_μ-additivity was hypothesized), $\mathcal{U}_\alpha$ is a partition into τ-(cl)open intervals.

Step 3. Let $\mathcal{P}_{\omega_\mu}$ be the common refinement of the $\mathcal{U}_\alpha$, $\alpha < \omega_\mu$. Because D is dense, and each interval of $\mathcal{U}_{\alpha+1}$ contains at most one member of D_α for all α (see Step 2), and D is the union of the D_α, it follows that *each member* of $\mathcal{P}_{\omega_\mu}$ has at most two points.

Every isolated point d of Y is in D_α for some $\alpha < \omega_\mu$. Thus the analogues of $I^+(d)$ and $I^-(d)$ are empty, and so $\{d\} \in \mathcal{U}_{\alpha+1}$. Consequently, if d is isolated in Y, then $\{d\} \in \mathcal{P}_{\omega_\mu}$.

If $\{y, y'\} \in \mathcal{P}_{\omega_\mu}$, and $y < y'$, then neither point is isolated by what we have just said, and y' is the immediate successor of y, so that both y and y' are *naturally* half-isolated.

If, finally, $\{y\} \in \mathcal{P}_{\omega_\mu}$ and y is neither isolated nor half-isolated, then the open intervals around y form a local base there, and y is not an endpoint of the unique member $I_\alpha(y)$ of (each) $\mathcal{U}_\alpha$ containing y, so that *these* open intervals form a *local base at* y.

Step 4 (construction of $A(X, S)$). Let $Z = \{y \in Y \mid \exists y' \in Y$ such that $\{y, y'\} \in \mathcal{P}_{\omega_\mu}$ and $y' > y\}$, and let $X = Z \cup \{y \in Y \mid \{y\} \in \mathcal{P}_{\omega_\mu}\}$. Further, let S be the union of Z with all those points of Y which are artificially half-isolated in Y. Finally, let X be given the order topology augmented by all singletons that are isolated in Y.

Claim. $(X, \leq)$ *is* ω_μ*-metrizable in this topology.*

(The proof of the claim will be given after the next step, so the "red thread" will not be cut.)

Once the claim is proven, we can embed Y into $A(X, S)$ as follows:

Step 5. Let $f: Y \to A(X, S)$ be defined as follows: If $y \in Z$, then $y \in S$ and we let $f(y) = y^-$. If y is the immediate successor of some $z \in Z$, then $\{z, y\} \in \mathcal{P}_{\omega_\mu}$ and $z \in S$, and we let $f(y) = z^+$. If y is artificially half-isolated in Y, say the intervals $(y', y]$ form a local base at y, although y has no immediate successor, then let $f(y) = y^-$. Similarly, if the intervals $[y, y')$ from a local base at y although y has no immediate predecessor, then let $f(y) = y^+$. The only remaining case is where $y \notin S$ and in this case we have $f(y) = y$.

We now show that *f is an embedding*: This is obvious in case of those points which are isolated and also in case of those points which are neither isolated nor half-isolated. The half-isolated points $y \in Y$ break up into *three cases*:

Case 1: *y is artificially half-isolated.* Then $y \in X$, and y is neither isolated nor half-isolated in X. So y^- has a local base of sets of the form $(w, y^-]$ in $A(X, S)$ with $w < y^-$ and w not the immediate predecessor of y^-, while y^+ has a similar local base of sets $[y^+, w)$. No matter which way y is half-isolated in Y, f sends it to the "correct" member of $\{y^-, y^+\}$.

Case 2: *y is half-isolated and* $\{y, y'\} \in \mathcal{P}_{\omega_\mu}$ *for some* $y' \neq y$. (Clearly, in this case, y is naturally half-isolated.) Then either y or y' is in X, and is neither isolated nor half-isolated there (i.e., in X) and we can argue similarly to Case 1.

Case 3: *y is naturally half-isolated and* $\{y\} \in \mathcal{P}_{\omega_\mu}$. Say y has an immediate successor but no immediate predecessor in Y, i.e., its intervals are of the form $(w, y]$ in Y. Now there is some $\mathcal{U}_\alpha$ in which y and its immediate successor y' are in separate members of $\mathcal{U}_\alpha$, and then y is the greatest point of its interval while y' is the least point of its interval. This makes y naturally half-isolated in the order topology in X, and since $y \notin S$ (by definition of S), y keeps that status in $A(X, S)$.

Proof of the claim. Note that $Y \backslash X$ consists of $Z' = \{y \in Y \mid y$ has an immediate predecessor in $Z\}$, these points being naturally half-isolated have no immediate successors in Y; similarly, the points in Z do not have immediate predecessors in Y, and so if $z \in Z$, then z is not an endpoint of the interval of $\mathcal{U}_\alpha$ $(\alpha < \omega_\mu)$ containing it, neither is its immediate successor (which follows by definition of Z). As α ranges over ω_μ, the *trace* of these intervals *on* X close down on $\{z\}$, and form a local base there in the order (LOTS) topology on X.

For *artificially half-isolated points* of Y, the argument is similar; for them too, the $\mathcal{U}_\alpha$ will trace on X a local base in the order topology. This is also true of *naturally half-isolated points* of $X \backslash Z$, except that these will be the upper (or lower) endpoint of some members of some $\mathcal{U}_\alpha$, and then for all $\mathcal{U}_\gamma$, $\gamma > \alpha$, of course.

Finally, *isolated points of* X are isolated in Y and hence their singleton sets get into some $\mathcal{U}_\alpha$, automatically. (Note that the only pseudogap points of X are isolated pseudogap points of Y. Hence X is strongly suborderable.)

Altogether, the traces of the $\mathcal{U}_\alpha$, $\alpha < \omega_\mu$, form a covering uniformity on X of the kind we are looking for which is compatible with the (order) topology of X by construction. In other words, the order $\leqslant$ on X fits an ω_μ-metrizable topology on X.

Conclusion. This now completes the proof of (i) $\Rightarrow$ (ii) in Theorem 3.8. It also completes the proof of (i) $\Rightarrow$ (ii) in Theorem 3.9, since there we have a *strong* suborder on Y, and so there are *no artificially half-isolated points*, so that f is a surjection as well as an embedding, i.e., $Y \approx A(X, S)$. To see how (iii) $\Rightarrow$ (i) of Theorem 3.7 follows, we have to add some more arguments: When we form $A(X, S)$ then S is an ω_μ-discrete subspace in X by the assumption in (iii) of Theorem 3.7. (D includes all half-isolated points of Y.) But, since Y is embedded into $A(X, S)$, and ω_μ-metrizability is hereditary, we will be done as soon as we have shown the following

Lemma 3.10. *If* $(X, \leqslant, \tau)$ *is an* ω_μ*-metrizable* GO*-space and* S *is an* ω_μ*-discrete subspace of* X, *then* $A(X, S)$ *is* ω_μ*-metrizable in the canonical topology.*

Proof. By the ω_μ-analogue of the Nagata–Smirnov theorem (see "Fact 3.3" at the beginning of Section 3 and [6, 11]), (X, τ) has an ω_μ-locally finite base, and since S is ω_μ-discrete, it follows from the definition of the natural topology (which $A(X, S)$ is endowed with) that $A(X, S)$ has such a base, too. Hence, $A(X, S)$ is ω_μ-metrizable, and we are done. □

(Analogously, this latter result can be derived from the ω_μ-analogue of Bing's metrization theorem, which says that a regular space is metrizable iff it has a σ-discrete base.)

Remark 3.11. If in the above proof of (iii) $\Rightarrow$ (i) of Theorem 3.7, we additionally hypothesize that Y is *strongly* suborderable (which, in fact, is condition (ii)), then S is empty, because there are no artificially half-isolated points, $Z = \emptyset$ always and $A(X, S) = X$. Hence (i) follows immediately, and no lemma is needed, indeed.

Remark 3.12. It would be natural to inquire whether the following statement – analogous to Theorem 3.8(iii) – is equivalent to the two statements in Theorem 3.9:

$Y \approx A(X, S)$ with the canonical topology for some ω_μ-metrizable GO-space $(X, \leqslant, \tau)$.

The difficulty is in showing that $A(X, S)$, when defined using $\leqslant$, is strongly suborderable. Of course, we can reorder X to give it a strong suborder, and the resulting $A(X, S)$ would also be strongly suborderable, but the topology on $A(X, S)$ depends upon the order as well as the topology. Therefore, more generally, we can ask:

Problem. If X is a strongly suborderable space, and $\leqslant$ is an order making X a GO-space, and $S \subset X$, is $A(X, S)$ with the canonical topology (w.r.t. $\leqslant$) strongly suborderable?

4. Characterization of ω_μ-metrizable spaces in terms of g-functions

g-functions are an important tool in the theory of generalized metric spaces. Recently, Nagata [11] has published a survey paper on this subject, which also contains new results.

For any topological space (X, τ) a function $g: \mathbb{N} \times X \to \tau$ is a "*g-function*" [11]. We can assume that for any $n \in \mathbb{N}$ and $x \in X$, $g(n, x)$ is a (not necessarily open) neighborhood of x. g-functions can play a major role in characterizing various classes of *generalized metric spaces* and in describing the *difference* between certain classes of such spaces and metrizable spaces [11]. Therefore it would be worthwhile to characterize ω_μ-metrizable spaces in terms of g-functions, too. First, let us take care of the case $\omega_\mu = \omega_0$.

Theorem 4.1. *The following are equivalent*:

(i) *X is a n.a. metrizable space*;

(ii) *X is a n.a. topological space having a g-function satisfying*

$$\text{if } p \in g(n, x_n) \text{ for all } n \in \mathbb{N}, \text{ then } \langle x_n \rangle \to p; \tag{$*$}$$

(iii) *X is a n.a. topological space having a g-function satisfying*

$$\text{if } p \in g(n, x_n) \text{ and } x_n \in g(n, y_n) \text{ for all } n, \text{ then } \langle y_n \rangle \to p. \tag{$**$}$$

Proof. Clearly n.a. metric spaces satisfy (ii) and (iii), since $\mathcal{B}_n = \{B_{1/n}(x)\} = \{\{y \mid d(x, y) < 1/n\} \mid x \in X\}$ is a clopen partition of X and we can take $g(n, x) = B_{1/n}(x)$.

Further, by a result of Heath, condition $(**)$ characterizes σ-spaces, and by [15, Theorem 13] any n.a. σ-space is metrizable and hence n.a. metrizable. Finally, condition $(*)$ characterizes semistratifiable spaces. But any n.a. topological space is monotonically normal and any monotonically normal semistratifiable space is stratifiable and hence [3] a σ-space. Therefore, condition $(*)$ implies $(**)$ in our case, and $(**)$ guarantees metrizability of the given n.a. topological space. □

Remark 4.2. In [25], Vaughan generalized the concept of stratifiable spaces to higher cardinals when defining "linearly stratifiable spaces". By using a result of Nyikos, the authors in [7] have shown that the following are equivalent: (i) X is metrizable and $\dim X = 0$ or X is ω_μ-metrizable for $\mu > 0$; (ii) X is suborderable, zero-dimensional and stratifiable over ω_μ $(> \omega_0)$. From this result and its method of proof it is clear how Theorem 3.6 can be generalized to yield a *characterization of ω_μ-metrizable* spaces for $\mu > 0$. We have to generalize a g-function to be a function

$g:\omega_\mu \times X \to \tau$, and in some cases, we must add ad $X = \omega_\mu$ (which for n.a. topological spaces is equal to $\varphi(x) = \omega_\mu$ for any nonisolated point $x \in X$).

A very typical example for a large number of theorems is the following:

Theorem 4.3. *The following are equivalent*:

(i) *X is ω_μ-metrizable ($\mu > 0$) or X is metrizable and* $\dim X = 0$;

(ii) *X is* (1) *suborderable,* (2) *zero-dimensional, and* (3) *has a g-function $g:\omega_\mu \times X \to \tau$ with property* (*): "*if for all $i < \omega_\mu$, $p \in g(i, x_i)$, then $\langle x_i \mid i < \omega_\mu\rangle \to p$*".

Proof. (i) $\Rightarrow$ (ii): Let (X, d) be ω_μ-metrizable and $d: X^2 \to G$ where $(G, \leqslant, +)$ is a linearly ordered Abelian group and $\langle \varepsilon_i \mid i < \omega_\mu\rangle \to 0$ in G with the order topology ($i < j \Rightarrow \varepsilon_i > \varepsilon_j > 0$). We already know that (1) and (2) are satisfied, and obviously $g(i, x) = B_i(x) = \{y \in X \mid d(x, y) < \varepsilon_i\}$ satisfies property (*). Conversely, let us show (ii) $\Rightarrow$ (i): Similar to the case $\omega_\mu = \omega_0$, the existence of a g-function with property (*) characterizes linear semistratifiability and, because X is monotonically normal, X is linearly stratifiable (in the sense of Vaughan [25]). Now, since by an old result of Nyikos (see e.g. [7]) any linearly stratifiable suborderable space is linearly uniformizable, we are done. □

Note added in proof

Meanwhile, Purisch has shown that the answer to the problem posed just before Section 4 is negative: Let $X = C - \{x \in C \mid x$ is isolated on the left$\}$, where C is the Cantor set. Let S be the set of two-sided limit points of C (neither isolated nor half-isolated). (S is sometimes called the irrational points of C or the inaccessible points of C.) Since X is a totally disconnected set of reals, it is orderable (see I.L. Lynn, Linearly orderable space, Proc. Amer. Math. Soc. (1962); in a more general setting see [5]). But $A(X, S)$ is not strongly suborderable since it has no isolated points and it is not orderable by Example 3 in: S. Purisch, Orderability of GO spaces whose pseudogaps are few or scattered about in: Z. Frolik, ed., Proceedings Sixth Prague Topological Symposium, 1986 (Heldermann, Berlin, 1988).

Note also this example shows that the analogue of Theorem 3.8(iii) in Remark 3.12, is *not* equivalent to the two statements in Theorem 3.9.

References

[1] J. de Groot, Non-archimedean metrics in topology, Proc. Amer. Math. Soc. 9 (1956) 948–953.

[2] S.M. Eure, The role of trees in topology, Thesis, University of South Carolina, Columbia, SC (1985).

[3] R. Heath, Stratifiable spaces are σ-spaces, Notices Amer. Math. Soc. 16 (1969) 761.

[4] M. Henrikson and R. Kopperman, A general theory of structure spaces with applications to spaces of prime ideals, Preprint (1988).

[5] H. Herrlich, Ordnungsfähigkeit total diskontinuierlicher Räume, Math. Ann. 159 (1965) 77–80.
[6] R. Hodel, Extensions of metrization theorems to higher cardinals, Fund. Math. 87 (1975) 219–229.
[7] M. Hušek and H.-C. Reichel, Topological characterizations of linearly uniformizable spaces, Topolgy Appl. 15 (1983) 173–188.
[8] J. Juhász, Untersuchungen über ω_μ-metrisierbare Räume, Ann. Univ. Sci. Budapest. Eôtvôs Sect. Math. 8 (1965) 129–145.
[9] J. Juhász and W. Weiß, On a problem of Sikorski, Fund. Math. 100 (1978) 223–227.
[10] R. Kopperman, All topologies come from generalized metrics, Amer. Math. Monthly 95 (1988) 89–97.
[11] J. Nagata, Metrizability, generalized metric spaces and g-functions, Comment. Math. Univ. Carolin. 29 (1988) 715–722.
[12] P. Nyikos, Some surprising base properties in topology, in: Stavrakas and Allen, eds., Studies in Topology (Academic Press, New York, 1975) 427–450.
[13] P. Nyikos, Some surprising base properties in topology II, in: G.M. Reed, ed., Set-Theoretic Topology (Academic Press, New York, 1977) 277–305.
[14] P. Nyikos, Ordertheoretic base-axioms, in: G.M. Reed, ed., Surveys in General Topology (Academic Press, New York, 1980) 367–398.
[15] P. Nyikos and H.-C. Reichel, On the structure of zerodimensional spaces, Indag. Math. 37 (1975) 120–136.
[16] P. Nyikos and H.-C. Reichel, Topologically orderable groups, Topology Appl. 5 (1975) 195–204.
[17] P. Nyikos and H.-C. Reichel, On uniform spaces with linearly ordered bases II, Fund. Math. 93 (1976) 1–10.
[18] P. Nyikos and H.-C. Reichel, Some results on cardinal functions in metrization theory, Glas. Mat. 15 (35) (1980) 183–202.
[19] S. Purisch, The orderability of non-archimedean spaces, Topology Appl. 16 (1983) 273–277.
[20] S. Purisch, Almost n.-a. spaces, Colloquia Mathematica Societatis János Bolyai 41 (North-Holland, Amsterdam, 1983) 509–516.
[21] H.-C. Reichel, Towards a unified theory of (m-)metric and (m-)semimetric spaces, in: J. van Mill and P.C. Baayen, eds., Topological Structures II, Mathematical Centre Tracts 116 (Math. Centre, Amsterdam, 1979) 209–241.
[22] R. Sikorski, Remarks on some topological spaces of high power, Fund. Math. 37 (1950) 125–136.
[23] S. Todorčević, On a theorem of D. Kurepa, in: H.R. Bennet and D. Lutzer, eds., Topology and Order-Structures 1, Mathematical Centre Tracts 142 (Math. Centre, Amsterdam, 1982) 173–176.
[24] J. van Wouwe, GO-Spaces and a Generalization of Metrizability, Mathematical Centre Tracts 104 (Math. Centre, Amsterdam, 1979).
[25] J. Vaughan, Linearly stratifiable spaces, Pacific J. Math. 43 (1972) 253–266.
[26] Wang, S.-T., Remarks on ω_μ-additive spaces, Fund. Math. 55 (1964) 101–112.