The Mislin genus, phantom maps and classifying spaces

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Abstract

For a finite type, nilpotent space $X$, we prove that the cardinality of the set $\text{Ph}(X, Y)$, where $\text{Ph}(-, -)$ denotes homotopy classes of phantom maps, depends only on the Mislin genus of $X$, at least if $Y$ has countable higher homotopy groups. In the special case where $X = BG$, the classifying space of a 1-connected Lie group $G$, and $Y$ is the iterated loop space of a 1-connected, finite CW-complex, we prove the stronger result that the isomorphism class of the group $\text{Ph}(X, Y)$ depends only on the Mislin genus of $X$. The latter strengthening depends on two results of independent interest: (i) Under a fairly mild connectivity condition on $X$, the torsionization of $X$, that is the homotopy fiber of a rationalization map $X \to X_{(p)}$, is a Mislin genus invariant; (ii) the torsionization of $BG$, localized away from a prime $p$, is homotopy equivalent to the plus construction applied to a space of the form $BA$, where $A$ is a suitable locally finite, perfect (discrete) group.

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1. Introduction/statement of the main results

Classifying spaces of Lie groups $G$, denoted $BG$, have been an important part of the homotopy theorist's landscape ever since their discovery (or invention!). At first, these spaces appeared mostly as the target space $Y$ in the homotopy set $[X, Y]$, because of

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Throughout, all spaces are assumed to be pointed CW-spaces.
their central role in bundle theory. However, more recently, they also have been featured as the domain space $X$ and there is now an extensive journal literature about $[BG, Y]$ for various target spaces $Y$. For example, the paper [6] of Friedlander and Mislin deals with $[BG, Y]$ when $\pi_0 G$ is finite and $Y$ is of the form $\Omega^k Z$, $Z$ being a 1-connected finite CW-complex. In [6], it is shown that each element in $[BG, Y]$ is phantom — that is, the restriction to any skeleton $(BG)_n$ is inessential; see [19] for a survey of phantom map theory — and a complete computation of the group $[BG, Y]$ is made in the case $k \geq 1$.

Another context in which the spaces $BG$ arise is that of the Mislin genus. Recall that the Mislin genus of a finite type, nilpotent space $X$, denoted $G(X)$, consists of all (homotopy types of) finite type, nilpotent spaces $X'$ such that the $p$-localizations $X'_{(p)}$ and $X'_{(p)}$ are homotopy equivalent for all primes $p$ (including, of course, $p = 0$). It turns out that $G(BG)$ is almost always an uncountable set — see [16, Theorem 2.3] and the survey article [9].

The first result in this paper is a theorem connecting the Mislin genus and phantom maps, namely:

**Theorem 1.** Let $X$ and $X'$ be finite type, nilpotent spaces having the same genus and let $Y$ be a countable type target. Then $Ph(X, Y)$ and $Ph(X', Y)$ are equivalent as sets, where $Ph(-, -)$ denotes homotopy classes of phantom maps.

Roughly, Theorem 1 asserts that $Ph(X, -)$ is, in a weak sense, a genus invariant. Notice that we do not claim that there is a map $X \to X'$ or a map $X' \to X$ inducing a bijection of $Ph(X, Y)$ with $Ph(X', Y)$; in fact, as we shall see, there need not be such maps, even stably. Nor do we claim, in the case $Y$ is grouplike, that the groups $Ph(X, Y)$ and $Ph(X', Y)$ are isomorphic; again, as we shall note, this need not be the case. However, to emphasize that Theorem 1 does give positive information, we point out that, in general, the full homotopy sets $[X, Y]$ and $[X', Y]$ need not be equivalent as sets. For instance, if $X = BS^3 = Y$ and $X'$ is an element in $G(BS^3)$ distinct from $BS^3$, then $[X, Y]$ has cardinality $> 1$ while $[X', Y] = 0$; cf. [9, Theorem 11] (due to Ishiguro, Møller and Notbohm) and the earlier result [16, Example 3.11].

Theorem 1 may be regarded as a variant of a result in [12]:

**Theorem 2** [12, Corollary 2.1(i)]. Let $X$ and $X'$ be finite type domains, $Y$ a finite type target and suppose there are rational homology equivalences $X \to X'$, $X' \to X$ in both directions. Then $Ph(X, Y)$ and $Ph(X', Y)$ are equivalent as sets. (It actually suffices to suppose there are rational homology equivalences $\Sigma X \to \Sigma X'$, $\Sigma X' \to \Sigma X$ of the suspended spaces; see [13, Corollary 2].)

However, Theorems 1 and 2 differ in significant ways. In the first place, in Theorem 2, the spaces $X$ and $X'$ need not be nilpotent; all that is required is that the integral

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2 Following the terminology of [12], a countable type target, respectively finite type target, is a space whose higher homotopy groups are countable, respectively finitely generated. If $Y$ is assumed to be a finite type target, there is a version of Theorem 1 wherein $X$ and $X'$ are merely required to have the same completion genus [9]; see the appendix.
homology groups of these spaces be finitely generated in each degree. Secondly, Theorem 2 fails if \( Y \) is merely a countable type target rather than a finite type target; for example, failure occurs when \( X = K(\mathbb{Z}/2, 1) \), \( X' = \ast \) and \( Y = \bigvee_n \Sigma \mathbb{R}P^n \) ([13, Example A], due to Gray and McGibbon). Finally, we shall see that the equality of the genus sets, \( G(X) = G(X') \), does not imply the existence of rational homology equivalences \( X \to X' \) or \( X' \to X \) (or even of \( \Sigma^k X \to \Sigma^k X' \), \( \Sigma^k X' \to \Sigma^k X \), \( k \geq 1 \)). Thus, Theorem 1 does not follow from Theorem 2 even in the case that \( Y \) is a finite type target.

Another variant of Theorem 1 easily is extracted from [11]. To state this result, we first recall that \( \text{SNT}(X) \) is the set of all (homotopy types of) spaces \( X' \) such that the Postnikov approximations \( X^{(n)} \) and \( X'^{(n)} \) are homotopy equivalent for all \( n \). Then:

**Theorem 3** (cf. [11], discussion before Example B). Let \( X \) and \( X' \) be finite type domains with \( \text{SNT}(X) = \text{SNT}(X') \) and let \( Y \) be a countable type target. Then \( \text{Ph}(X, Y) \) and \( \text{Ph}(X', Y) \) are equivalent as sets.

The proofs of Theorems 2 and 3 are both based on \( \lim^1 \) descriptions of \( \text{Ph}(-, -) \); see, e.g., [19, Section 3]. Our proof of Theorem 1 in Section 2 is also of this type.

Another approach to \( \text{Ph}(-, -) \), via localization and completion techniques (see, e.g., [19, Section 4]), leads to the description

\[
\text{Ph}(X, Y) = r^*[X(0), Y],
\]

where \( r : X \to X(0) \) is a rationalization map; here, \( X \) and \( Y \) are assumed to be finite type, nilpotent spaces. To properly exploit (1.1), it is convenient to introduce \( X_\tau \), the homotopy fiber of the map \( r \); \( X_\tau \) is a locally finite space associated to \( X \), and is referred to as the "torsionization" of \( X \) in [20]. It turns out [19, (4.2), (4.3)], that, if \( \pi_1 X \) is finite, there is an exact sequence of sets

\[
[X_\tau, Y] \to [X, Y] \xrightarrow{r_\ast} [X(0), Y] \to [\Sigma X_\tau, Y],
\]

resulting from the fact that the mapping cone of the fiber inclusion \( i : X_\tau \to X \) may be identified with \( X(0) \). The second main result in this paper states:

**Theorem 4.** Let \( G(X) = G(X') \) and suppose, moreover, that \( X \) (hence also \( X' \)) is 1-connected and that \( \pi_2 X \) (hence also \( \pi_2 X' \)) is finite. Then \( X_\tau \simeq X'_\tau \).

In other words, under the given connectivity conditions, \( X_\tau \) is a genus invariant. Theorem 4 is particularly well-suited for a study of \( \text{Ph}(X', Y) \) when \( X' \in G(BG) \), \( G \) a 1-connected Lie group, and \( Y = \Omega^k Z \), \( Z \) a 1-connected finite CW-complex. To see why, we first record a result, of independent interest, which follows readily from the ideas in [6]:

**Theorem 5.** Let \( \Phi : B\Lambda \to BG \) be a locally finite approximation away from the prime \( p \) in the sense of [6], where \( G \) is a 1-connected Lie group. Then there is a unique induced
map \( \tilde{\Phi} : B\Lambda \to BG_{[1/p]} \) and \( \tilde{\Phi} \) is the plus construction with respect to the locally finite, perfect group \( \Lambda \).

Theorem 5 leads to a proof of Theorem 3.1 of [6], slightly different from the proof in [6]. Combined with Theorem 4, Theorem 5 additionally provides a much sharpened version of Theorem 1 in the case \( X = BG \), \( G \) a 1-connected Lie group and \( Y = \Omega^k Z \), \( Z \) a 1-connected, finite CW-complex:

**Theorem 6.** Let \( X' \in \mathcal{G}(BG) \), \( G \) a 1-connected Lie group, and let \( Y = \Omega^k Z \), \( Z \) a 1-connected, finite CW-complex. If either \( Z \) is a loop space, or \( k \geq 1 \), we have group isomorphisms

\[
\text{Ph}(BG, Y) \cong [BG, Y] \cong [BG_{(0)}, Y] \cong [X', Y] = \text{Ph}(X', Y)
\]

and \( [BG_{(0)}, Y] \) is either 0 or \( \mathbb{R} \) (a \( \mathbb{Q} \)-vector space of uncountable dimension).

Moreover, let \( j : BG \to K \) and \( j' : X' \to K \) be rational homotopy equivalences with \( K \) a (finite) product of \( K(\mathbb{Z}, n) \)'s. Then

\[
j^* : \text{Ph}(K, Y) \to \text{Ph}(BG, Y), \quad j'^* : \text{Ph}(K, Y) \to \text{Ph}(X', Y)
\]

are both group isomorphisms.

The rest of the paper is organized as follows. In Section 2, we prove Theorem 1 and discuss a number of related results. The latter include an Eckmann–Hilton dual of Theorem 1 and the possible genus invariance of \( WI(X) \), where \( WI(X) \) is the group introduced by the author (see [19, Section 12]; also [11]). Theorems 4 and 5 are proved in Sections 3 and 4 respectively. In Section 5, we prove Theorem 6 and present a curious example relating to the question of whether the stringent condition on the target \( Y \) in Theorem 6 can be relaxed. Finally, in a brief appendix, we present a version of Theorem 1 appropriate to the situation where the domain spaces have the same completion genus.

I am greatly indebted to Chuck McGibbon for reading an earlier version of this paper and pointing out an improvement of my proof of Proposition 2.1; the proof below is presented with his permission.

### 2. Theorem 1 and related results

To prove Theorem 1, we begin by writing

\[
\text{Ph}(X, Y) \cong \lim_{\leftarrow} \left[ X, \Omega Y^{(n)} \right],
\]

\[
\text{Ph}(X', Y) \cong \lim_{\leftarrow} \left[ X', \Omega Y^{(n)} \right];
\]

see [19, Corollary 3.1(ii)]. Abbreviate \( G_n = [X, \Omega Y^{(n)}] \), \( G'_n = [X', \Omega Y^{(n)}] \) and observe that \( G_n \) and \( G'_n \) are countable, nilpotent groups. According to [10, Theorem 2], \( \lim_{\leftarrow} G_n \) either is 0 or has cardinality \( 2^{\aleph_0} \); moreover, \( \lim_{\leftarrow} G_n = 0 \) if and only if the inverse tower \( \{G_n\} \) is Mittag-Leffler. Similar remarks apply, of course, to \( G'_n \). Theorem 1 then reduces to:
Proposition 2.1. \( \{ G_n \} \) is Mittag-Leffler \( \iff \{ G'_n \} \) is Mittag-Leffler.

**Proof.** Recall that \( \{ G_n \} \) is Mittag-Leffler if for each \( m \), there exists \( k \) such that the image of \( G_{m+\ell} \) in \( G_m \) is the same as the image of \( G_{m+k} \) in \( G_m \) for all \( \ell \geq k \). Thus, to prove the implication \( \Rightarrow \), it suffices to show that for \( m, k \) as above, the image of \( G'_{m+\ell} \) in \( G'_m \) is the same as the image of \( G'_{m+k} \) in \( G'_m \) for all \( \ell \geq k \). Fix \( m, k \) and \( \ell \geq k \) and write
\[
I = I_{m,k} = \text{image of } \sigma_{m+k,m} : G_{m+k} \to G_m,
\]
\[
I' = I'_{m,k} = \text{image of } \sigma'_{m+k,m} : G'_{m+k} \to G'_m;
\]
here, \( \sigma_{i,j} \) and \( \sigma'_{i,j} \) denote the structure maps in the towers \( \{ G_n \} \) and \( \{ G'_n \} \). Next write
\[
\bar{\sigma}_{m+\ell,m} : G_{m+\ell} \to I, \quad \bar{\sigma}'_{m+\ell,m} : G'_{m+\ell} \to I'
\]
for the corestrictions of
\[
\sigma_{m+\ell,m} : G_{m+\ell} \to G_m, \quad \sigma'_{m+\ell,m} : G'_{m+\ell} \to G'_m.
\]
We are assuming that \( \bar{\sigma}_{m+\ell,m} \) is an epimorphism and wish to infer that \( \bar{\sigma}'_{m+\ell,m} \) is likewise an epimorphism.

We have an isomorphism of \( p \)-localized towers
\[
\{ \phi_n(p) \} : \{ G_n(p) \} \cong \to \{ G'_n(p) \};
\]
indeed,
\[
G_n(p) = [X, \Omega Y^{(n)}]_{(p)} \cong [X, \Omega Y^{(n)}(p)] \cong [X_{(p)}, \Omega Y^{(n)}_{(p)}] \cong [X'_p, \Omega Y^{(n)}_{(p)}] \cong [X'_p, \Omega Y^{(n)}_{(p)}] \cong [X'_p, \Omega Y^{(n)}_{(p)}] = G'_n(p),
\]
all isomorphisms being compatible with the structure maps in the two \( p \)-localized towers.
Thus we obtain commutative diagrams
\[
\begin{array}{ccc}
G_{m+\ell}(p) & \cong & G'_{m+\ell}(p) \\
\sigma_{m+\ell,m}(p) \downarrow & & \sigma'_{m+\ell,m}(p) \\
I(p) & \cong & I'(p)
\end{array}
\tag{2.2}
\]
where \( \bar{\phi}(p) \) is naturally induced by \( \phi_m(p) \). Now
\[
\bar{\sigma}_{m+\ell,m} \text{ epi } \Rightarrow \bar{\sigma}_{m+\ell,m}(p) \text{ epi for all } p
\]
\[
\Rightarrow \bar{\sigma}'_{m+\ell,m}(p) \text{ epi for all } p, \text{ by (2.2)}
\]
\[
\Rightarrow \bar{\sigma}'_{m+\ell,m} \text{ epi, by [7, Theorem 1.3.12]},
\]
as claimed.

Since the opposite implication \( \Leftarrow \) in Proposition 2.1 is handled in a symmetric fashion, the proof of Proposition 2.1 is completed.

A number of remarks about Theorem 1 are in order but first we wish to record a strengthening of Theorem 1 in the case that \( Y \) is an H-space:
Theorem 2.2. Let $X$ and $X'$ be finite type domains such that $\mathcal{G}(\Sigma X) = \mathcal{G}(\Sigma X')$ and let $Y$ be a countable type target which is an $H$-space. Then $\Phi(X, Y)$ and $\Phi(X', Y)$ are equivalent as sets.

The proof follows the lines of proof of Theorem 1 but starts from the adjoint version of (2.1) — that is,

$$\Phi(X, Y) \approx \lim_{\to} [\Sigma X, Y^{(n)}],$$

$$\Phi(X', Y) \approx \lim_{\to} [\Sigma X', Y'^{(n)}]$$

— rather than (2.1) itself.

Next we revert to the proof of Proposition 2.1 and observe that, in general, there is no direct link between the two towers $\{G_n\}$ and $\{G'_n\}$. This is because the relation $X_{(p)} \simeq X'_{(p)}$ does not imply the existence of $p$-equivalences $X \to X'$ or $X' \to X$, even if $X$ and $X'$ are finite CW-complexes. An example of a "noninvertible" $p$-equivalence $X \to X'$ of 1-connected, finite CW-complexes was first given in [14]. The Mimura–Toda example was reexamined in [17] from a Lie algebraic point of view. It follows from the technique in [17] (though it was not pointed out in [17]) that one may find an example of the Mimura–Toda type with the spaces having arbitrarily high connectivity. Then, if $f: A \to B$ and $g: C \to D$ are noninvertible $p$-equivalences of 1-connected, finite CW-complexes with the connectivities of $C$ and $D$ exceeding the dimensions of $A$ and $B$, it is easy to see that $A \vee D$ and $B \vee C$ satisfy $(A \vee D)_{(p)} \simeq (B \vee C)_{(p)}$ even though there are no rational homotopy equivalences $A \vee D \to B \vee C$, $B \vee C \to A \vee D$ in either direction. It can be shown that for 1-connected, finite CW-complexes $X$ and $X'$, $X_{(p)} \simeq X'_{(p)}$ implies the existence of a third 1-connected, finite CW-complex $Z$, together with a pair of $p$-equivalences $X \to Z \leftarrow X'$; this sharpens part of [17, Remark 2].

A very instructive example (in the context of infinite-dimensional CW-complexes) of spaces $X$ and $X'$ in the same genus but with no rational homotopy equivalences $X \to X'$, $X' \to X$ in either direction will now be presented:

Example 2.3 (cf. [11, Example B] with respect to parts (i) and (iii)). Let

$$X = K(\mathbb{Z}, 2) \times \Omega S^3, \quad X' = Z \times Z',$$

where $Z$ and $Z'$ are obtained as follows: Let $\{I, J\}$ be a partition of the set of primes into nonempty, disjoint subsets; $Z$ is a Zabrodsky mix of $K(\mathbb{Z}, 2)$ localized at $I$ with

3 It is possible to choose the spaces so that such a homotopy equivalence exists for all primes but one.

4 Using a slight variant of this observation, it may also be shown that for any integer $N > 0$, if we truncate the towers $\{G_n\}$ and $\{G'_n\}$ at level $N$ (that is, replace $G_n$ and $G'_n$ by 0 for $n > N$) and assume $X$ and $X'$ are 1-connected, then there exist a sequence of towers $\{H_n(p)\}$, indexed by the primes $p$, and a sequence of tower maps $\{G_n\} \xleftarrow{\alpha_n(p)} \{H_n(p)\} \xrightarrow{\beta_n(p)} \{G'_n\}$, with each $\alpha_n(p)$ and each $\beta_n(p)$ being a $p$-isomorphism. Our original proof of Proposition 2.1 was based on the existence of such auxiliary towers and tower maps.
\( \Omega S^3 \) localized at \( J \), and \( Z' \) is a Zabrodsky mix of \( K(\mathbb{Z}, 2) \) localized at \( J \) with \( \Omega S^3 \) localized at \( I \). Then:

(i) \( \mathcal{G}(X) = \mathcal{G}(X') \);

(ii) for any \( k \), there are no rational homotopy equivalences \( \Sigma^k X \to \Sigma^k X' \), \( \Sigma^k X' \to \Sigma^k X \) in either direction;

(iii) for any \( k \), the groups \( \text{Ph}(\Sigma^k X, S^{k+3}) \) and \( \text{Ph}(\Sigma^k X', S^{k+3}) \) are not isomorphic.

**Proof.** (i) is clear and (iii) is embedded in the computations in [11], at least in the case \( k = 0 \). To prove (ii), in the case \( k = 0 \), we first claim:

(a) \( [K(\mathbb{Z}, 2), Z] = 0 = [K(\mathbb{Z}, 2), Z'] \);

(b) \( [Z, \Omega S^3] \cong [\Sigma Z, S^3] \cong [\Sigma \Omega S^3, S^3], \ [Z', \Omega S^3] \cong [\Sigma Z', S^3] \cong [\Sigma \Omega S^3, S^3] \).

The verification of (a) and (b) may be carried out as in [18]; details are omitted. Now (a) and (b) imply that any map \( f : X \to X' \), respectively \( g : X' \to X \) induces a map \( f_* : \pi_2 X \to \pi_2 X' \) with \( \text{rank}(\text{ker} f_*) \geq 1 \), respectively \( g_* : \pi_2 X' \to \pi_2 X \) with \( \text{rank}(\text{coker} g_*) \geq 1 \). Thus, neither \( f \) nor \( g \) can be a rational homotopy equivalence. In similar fashion, one may show that any map \( f : \Sigma^k X \to \Sigma^k X' \), respectively \( g : \Sigma^k X' \to \Sigma^k X \) induces a map \( f_* : \pi_{2+k} \Sigma^k X \to \pi_{2+k} \Sigma^k X' \) with \( \text{rank}(\text{ker} f_*) \geq 1 \), respectively \( g_* : \pi_{2+k} \Sigma^k X' \to \pi_{2+k} \Sigma^k X \) with \( \text{rank}(\text{coker} g_*) \geq 1 \).

It may be that the nonexistence of rational homotopy equivalences \( X \to X' \), \( X' \to X \) in both directions is tied up with the phenomenon exhibited in (iii) above, i.e. the existence of a grouplike \( Y \) such that

\[ \text{Ph}(X, Y) \not\cong \text{Ph}(X', Y); \]

see [12, Question 2]. In this connection, it is interesting that for \( X' \) a nontrivial element of \( \mathcal{G}(BS^3) \), there do not exist rational homotopy equivalences \( BS^3 \to X' \), \( X' \to BS^3 \) in either direction ([9, Theorem 11]; also [16, Example 3.1]), yet

\[ \text{Ph}(BS^3, Y) \cong \text{Ph}(X', Y), \]

at least if \( Y \) is a grouplike space of the form \( \Omega^k Z \), \( Z \) a 1-connected, finite CW-complex (Theorem 6).

Returning to the particular spaces \( X \) and \( X' \) of Example 2.3, we recall that the authors’ actual purpose in [11, Example B], was to show that the groups \( \text{WI}(X \times S^3) \) and \( \text{WI}(X' \times S^3) \) are not isomorphic, where \( \text{WI}(U) \) denotes the subgroup of the group \( \text{Aut}(U) \) of (homotopy classes of) self-homotopy equivalences consisting of maps weakly homotopic to the identity [19, Section 12]. They used the group isomorphism \( \text{WI}(U) \cong \text{Ph}(U, U) \), valid for grouplike \( U \) with \( \pi_1 U \) finite [19, Theorem 12.2(i)] to reach their conclusion. They also noted that, in general, if \( X \) and \( X' \) are finite type, 1-connected spaces with \( \text{SNT}(X) = \text{SNT}(X') \), then \( \text{WI}(X) \) and \( \text{WI}(X') \) are equivalent as sets; compare with Theorem 3. It therefore seems natural to ask for a comparison of \( \text{WI}(X) \) with \( \text{WI}(X') \) for \( X \) and \( X' \) in the same genus.
Question 2.4. Let $X$ and $X'$ be finite type, nilpotent spaces having the same genus. Are $\text{WI}(X)$ and $\text{WI}(X')$ equivalent as sets?

One may approach Question 2.4 by using descriptions of $\text{WI}(X)$ and $\text{WI}(X')$, namely

$$\text{WI}(X) \approx \varprojlim \pi_1(\text{aut}_1(X^{(n)})) \approx \varprojlim \pi_2(B\text{aut}_1(X^{(n)})),$$

$$\text{WI}(X') \approx \varprojlim \pi_1(\text{aut}_1(X'^{(n)})) \approx \varprojlim \pi_2(B\text{aut}_1(X'^{(n)})).$$

Here, $\text{aut}_1(X^{(n)})$ is the topological monoid consisting of self-homotopy equivalences of $X^{(n)}$ inducing the identity on homotopy groups and $B\text{aut}_1(X^{(n)})$ is its classifying space; similarly for $X'^{(n)}$. Spaces of the form $B\text{aut}_1(P)$, $P$ a nilpotent, finite Postnikov space, were studied in [4], where they were shown to be nilpotent. It is plausible to conjecture that

$$B\text{aut}_1(P)(n) = B\text{aut}_1(P^{(n)}),$$

since, according to [8] and [15],

$$\pi_1(B\text{aut}_1(P))(p) \approx \pi_1(B\text{aut}_1(P^{(n)})).$$

(In other words, it is plausible to conjecture that $G(X) = G(X')$ implies $G(B\text{aut}_1(X^{(n)})) = G(B\text{aut}_1(X'^{(n)}))$.) If this were so, the two towers

$$\{\pi_2(B\text{aut}_1(X^{(n)}))\} \text{ and } \{\pi_2(B\text{aut}_1(X'^{(n)}))\}$$

would be towers of finitely generated, nilpotent groups with the $n$th groups in the two towers having the same genus, and we could proceed as in the proof of Theorem 1.

In any event, we can answer Question 2.4 affirmatively in case $X$ and $X'$ are grouplike spaces with $\pi_1 X$ and $\pi_1 X'$ finite but first we need an Eckmann–Hilton dual of Theorem 1.

**Theorem 1'.** Let $X$ be a finite type domain and let $Y$ and $Y'$ be finite type, nilpotent spaces having the same genus. Then $\text{Ph}(X, Y)$ and $\text{Ph}(X, Y')$ are equivalent as sets.

A proof similar to that of Theorem 1 may be carried out starting from

$$\text{Ph}(X, Y) \approx \varprojlim \Sigma X_n, Y],$$

$$\text{Ph}(X, Y') \approx \varprojlim \Sigma X_n, Y'];$$

see [19, Corollary 3.1(i)].

We then have:

**Theorem 2.5.** Let $X$ and $X'$ be finite type, grouplike spaces with $\pi_1 X$ and $\pi_1 X'$ finite and with $G(X) = G(X')$. Then $\text{WI}(X)$ and $\text{WI}(X')$ are equivalent as sets.

**Proof.**

$$\text{WI}(X) \approx \text{Ph}(X, X), \text{ by [19, Theorem 12.2(i)]}$$

$$\approx \text{Ph}(X', X), \text{ by Theorem 1}$$
3. Proof of Theorem 4

If $X$ is 1-connected (or even if $\pi_1 X$ is finite), the fibration sequence

$$X_{\tau} \xrightarrow{i} X \xrightarrow{r} X_{(0)}$$

consists of nilpotent spaces, so that the $p$-localized sequence

$$X_{\tau(p)} \xrightarrow{i_{(p)}} X_{(p)} \xrightarrow{r_{(p)}} X_{(0)}$$

is again a fibration sequence. In particular, we may identify $X_{\tau(p)}$ with $X_{(p)\tau}$ (torsion-ization commutes with $p$-localization). As a consequence we have:

**Lemma 3.1.** Let $X$ and $X'$ be 1-connected, finite type spaces with $G(X) = G(X')$. Then $X_{\tau(p)} \simeq X'_{\tau(p)}$ for all primes $p$.

We also have a fibration sequence

$$X_{\tau}[\frac{1}{p}] \to X_{\tau} \xrightarrow{e_p} X_{\tau(p)},$$

with $e_p$ a $p$-localization map. A straightforward obstruction argument gives:

**Lemma 3.2.** The map $e_p : X_{\tau} \to X_{\tau(p)}$ admits a unique (homotopy) section

$$s(p) : X_{\tau(p)} \to X_{\tau}.$$

Since $\Sigma X_{\tau}$ is homotopy equivalent to the mapping cone of $r : X \to X_{(0)}$, the reduced homology groups $\tilde{H}_n X_{\tau}$ are torsion groups; similarly, the reduced homology groups $\tilde{H}_n X_{\tau(p)}$ are $p$-torsion groups. Thus the $p$-localization map

$$e_{p*} : \tilde{H}_n X_{\tau} \to \tilde{H}_n X_{\tau(p)}$$

maps the $p$-torsion subgroup of $\tilde{H}_n X_{\tau}$ isomorphically onto $\tilde{H}_n X_{\tau(p)}$ and, by Lemma 3.2,

$$s(p)_* : \tilde{H}_n X_{\tau(p)} \to \tilde{H}_n X_{\tau}$$

is the canonical splitting of $e_{p*}$, that is $s(p)_*$ maps $\tilde{H}_n X_{\tau(p)}$ isomorphically onto the $p$-torsion subgroup of $\tilde{H}_n X_{\tau}$. We readily conclude:

**Lemma 3.3.** The map

$$s : \bigvee_{p} X_{\tau(p)} \to X_{\tau},$$

induced by the various $s(p)$, is a homology equivalence.
To prove Theorem 4, we observe that if $\pi_2 X$ is finite, then $X_\tau$, as well as each of its $p$-localizations $X_{(p)}$, is 1-connected. Then the map $s$ in Lemma 3.3 is a homotopy equivalence.\(^5\) Thus,

$$X_\tau \simeq \bigvee_p X_{(p)}, \text{ by Lemma 3.3}$$

$$\simeq \bigvee_p X'_{(p)}, \text{ by Lemma 3.1}$$

$$\simeq X'_\tau, \text{ by Lemma 3.3},$$

as claimed.

4. Proof of Theorem 5

In this section, $G$ will be a 1-connected Lie group though in [6], the authors work, more generally, with Lie groups having finitely many connected components. A locally finite approximation away from the prime $p$ for $G$ is a map

$$\Phi : B\Lambda \to BG,$$

where $\Lambda$ is a locally finite (discrete) group, possessing various properties, among which are:

(a) If $\Lambda$ is a finite abelian group without $p$-torsion, then the induced map $\Phi^* : H^*(BG; A) \to H^*(B\Lambda; A)$ is an isomorphism (the coefficient group $A$ is constant since $BG$ is 1-connected, indeed 3-connected);

(b) $\widetilde{H}^*(B\Lambda; \mathbb{Z}/p) = 0$.

See [6, Definition 1.11]. According to [6, Theorem 1.31] such maps exist.

To prove Theorem 5, we consider the diagram

\[
\begin{array}{ccc}
G(0) & \xrightarrow{i} & BG_\tau[\frac{1}{p}] \\
\downarrow & & \downarrow \sim \\
B\Lambda & \xrightarrow{\Phi'} & BG[\frac{1}{p}] \\
\downarrow \Phi & & \downarrow \tau \\
BG(0) & & 
\end{array}
\]

\(^5\) On the other hand, if $\pi_2 X$ is infinite, then each $\pi_1 X_{(p)}$ is nontrivial, and so the domain of $s$ is nonnilpotent while the target of $s$ is nilpotent.
Here $\Phi'$ is the composition $e[1/p] \circ \Phi$, $e[1/p]$ being localization away from $p$, $r$ is rationalization and the column is the bottom end of the usual fibration sequence. The existence and uniqueness of the lift $\tilde{\Phi}$ follow from

$$[BA, BG(0)] = 0 = [BA, G(0)],$$

these two vanishing results being easy consequences of the local finiteness of the domain space and the rationality of the target spaces. We prove that

$$\tilde{\Phi}_*: H_*(BA; A) \to H_*(BG_{\tau}[\frac{1}{p}]; A)$$

is an isomorphism for arbitrary $A$, in stages:

(i) $A = \mathbb{Z}/q^n, n \geq 1, q$ a prime $\neq p$:

$$\Phi'^*: H^*(BG_{\tau}[\frac{1}{p}]; \mathbb{Z}/q) \to H^*(BA; \mathbb{Z}/q)$$

is an isomorphism by (a),

$$i^*: H^*(BG_{\tau}[\frac{1}{p}]; \mathbb{Z}/q) \to H^*(BG_{\tau}[\frac{1}{p}]; \mathbb{Z}/q)$$

is an isomorphism since the mapping cone of $i$ is the rational space $BG(0)$, hence

$$\tilde{\Phi}^*: H^*(BG_{\tau}[\frac{1}{p}]; \mathbb{Z}/q) \to H^*(BA; \mathbb{Z}/q)$$

is an isomorphism. It is then clear that

$$\tilde{\Phi}_*: H_*(BA; \mathbb{Z}/q) \to H_*(BG_{\tau}[\frac{1}{p}]; \mathbb{Z}/q)$$

is an isomorphism. An induction on $n$ then shows that

$$\tilde{\Phi}_*: H_*(BA; \mathbb{Z}/q^n) \to H_*(BG_{\tau}[\frac{1}{p}]; \mathbb{Z}/q^n)$$

is an isomorphism, $n \geq 1$.

(ii) $A = \mathbb{Z}/p^n, n \geq 1$:

$$\tilde{H}^*(BA; \mathbb{Z}/p) = 0$$

by (b), hence

$$\tilde{H}_*(BA; \mathbb{Z}/p) = 0$$

as well. An induction on $n$ shows that

$$\tilde{H}_*(BA; \mathbb{Z}/p^n) = 0, \quad n \geq 1.$$ 

Since clearly

$$\tilde{H}_*(BG_{\tau}[\frac{1}{p}]; \mathbb{Z}/p^n) = 0, \quad n \geq 1,$$ 

it follows that

$$\tilde{\Phi}_*: H_*(BA; \mathbb{Z}/p^n) \to H_*(BG_{\tau}[\frac{1}{p}]; \mathbb{Z}/p^n)$$

is an isomorphism, $n \geq 1$.

(iii) $A = \mathbb{Q}/\mathbb{Z}$: Since $\mathbb{Q}/\mathbb{Z}$ is a colimit of finite groups, and since (i) and (ii) together imply that

$$\tilde{\Phi}_*: H_*(BA; A) \to H_*(BG_{\tau}[\frac{1}{p}]; A)$$
is an isomorphism for any finite $A$, we conclude that

$$\tilde{\Phi}_*: H_*(BA; \mathbb{Q}/\mathbb{Z}) \to H_*(BG_{[1/p]; \mathbb{Q}/\mathbb{Z}})$$

is an isomorphism.

(iv) $A = \mathbb{Z}$: Using the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, (iii) and the fact that

$$\tilde{H}_*(BA; \mathbb{Q}) = 0 = \tilde{H}_*(BG_{[1/p]; \mathbb{Q}}),$$

we readily conclude that

$$\tilde{\Phi}_*: H_*(BA; \mathbb{Z}) \to H_*(BG_{[1/p]; \mathbb{Z}})$$

is an isomorphism.

In particular, $H_1(BA) \cong H_1(BG_{[1/p]}) = 0$, as $BG_{[1/p]}$ is 1-connected. That is, $A$ is perfect. It follows (see [5, Theorem 3.6] for a more general result) that $\tilde{\Phi}$ is the plus construction with respect to $A$, proving Theorem 5.

Using Theorem 4, we draw the following corollary to Theorem 5:

**Corollary 4.1.** Let $X'$ have the genus of $BG$, $G$ a 1-connected Lie group. Then, for any prime $p$, $X'[1/p]$ has the homotopy type of a space of the form $(BA)^+$, with $A$ a locally finite, perfect group.

Corollary 4.1 raises the question of which 1-connected spaces have the form $(BA)^+$ for $A$ a locally finite, perfect group. (For finite, perfect groups $A$, the structure of $(BA)^+$ has been studied in [3]. However, finite groups do not satisfy one of the conditions imposed on $A$ by Friedlander and Mislin, namely that $\text{Hom}(A, F) = 0$ for all finite groups $F$.) We pose the following specific question.

**Question 4.2.** Let $H$ be a 1-connected, finite loop space, with classifying space $BH$, and let $p$ be a prime. Does $BH[1/p]$ have the form $(BA)^+$ for a suitable locally finite, perfect group?

5. Theorem 6 and beyond

To prove the first part of Theorem 6, we consider the exact sequence (1.2) for the domains $BG$ and $X'$; thus

$$[BG_{\tau}, Y] \leftarrow [BG, Y] \xleftarrow{\tau^*} [BG_{(0)}, Y] \leftarrow [\Sigma BG_{\tau}, Y],$$

$$[X', Y] \leftarrow [X', Y] \xleftarrow{\tau^*} [X'_{(0)}, Y] \leftarrow [\Sigma X', Y].$$

\[ \text{It has belatedly come to my attention that David Handel already treated Theorem 1 (but not Theorems 2 or 3) of [3] in his paper "Epimorphism plus monomorphism implies equivalence in the homotopy category", J. Pure Appl. Algebra 6 (1975) 357–360, and so, of course, deserves priority for that result. I regret the unfortunate oversight.} \]
Theorem 5 implies that, for any prime \( p \), the induced map
\[
\tilde{\phi}^*: [BG_\tau(\frac{1}{p}), Y] \to [BA, Y]
\]
is an isomorphism. But
\[
[BA, Y] = 0
\]
by H. Miller's theorem on the Sullivan conjecture, so that
\[
[BG_\tau(\frac{1}{p}), Y] = 0. \tag{5.2}
\]
Now the proof of Lemma 3.3 shows that
\[
BG_\tau[\frac{1}{p}] \simeq \bigvee_{a \neq p} BG_{\tau(q)}.
\]
Thus, by (5.2), it follows that
\[
[BG_{\tau(q)}, Y] = 0
\]
for all \( q \neq p \). Since \( p \) is arbitrary, we infer that
\[
[BG_{\tau(p)}, Y] = 0
\]
for all primes \( p \), hence, again by Lemma 3.3,
\[
[BG_{\tau}, Y] \cong \prod_{p} [BG_{\tau(p)}, Y] = 0. \tag{5.3}
\]
Appealing to Theorem 4, we also obtain
\[
[X', Y] = 0. \tag{5.4}
\]
Of course, both (5.3) and (5.4) hold with \( \Omega Y \) in place of \( Y \). Reverting to (5.1), we see that
\[
r^*: [BG(0), Y] \to [BG, Y], \quad r^*: [X'_0, Y] \to [X', Y]
\]
are isomorphisms. Since, obviously, \( BG(0) \simeq X'_0 \), we obtain
\[
\text{Ph}(BG, Y) = [BG, Y] \cong [BG(0), Y] \cong [X', Y] = \text{Ph}(X', Y)
\]
and a standard computation [19, Theorem 4.1], yields
\[
[BG(0), Y] \cong \prod_{n \geq 1} \text{Ext}(H_{n-1}(BG(0), \pi_n Y) \cong 0 \text{ or } \mathbb{R}.
\]
To obtain the second part of Theorem 6, we first consider a more general situation; see [19, Section 6]: Let \( X \) be a finite type, 1-connected, rationally elliptic space and let \( j: X \to L \) be an integral approximation, i.e. a rational homotopy equivalence from \( X \) to a finite Postnikov space \( L \) with torsion-free homotopy groups. (If \( X \) is an \( H_0 \)-space, e.g. \( BG \) or anything in its genus, then \( L \) may be taken to be a finite product of \( K(\mathbb{Z}, n) \)'s.)

**Theorem 5.1.** Let \( Y = \Omega^k Z \), \( Z \) a 1-connected, finite CW-complex, and let \( j: X \to L \) be as above. Then
\[
j^*: \text{Ph}(L, Y) \to \text{Ph}(X, Y)
\]
is onto and
\[ \ker j^* \approx \frac{[\Sigma X, Y]}{(\Sigma i)^*[\Sigma X, Y]}, \]

\( i : X_i \to X \) the fiber inclusion.

**Proof.** From (1.2), we have a commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
L_{(0)}, Y & \xrightarrow{\sim} & L(Y) & \xleftarrow{\sim} & [L(Y), Y] & \xleftarrow{\sim} & \Sigma L_{(0)}, Y & \xleftarrow{\sim} & \Sigma L, Y \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
[1, Y] & \xleftarrow{\sim} & [1, Y] & \xleftarrow{\sim} & [1, Y(0), Y] & \xleftarrow{\sim} & [\Sigma 1, Y] & \xleftarrow{\sim} & [\Sigma 1, Y],
\end{array}
\]

and

\[ [L(Y), Y] = 0 = [\Sigma L_{(0)}, Y] \]

by Zabrodsky’s extension of the aforementioned theorem of H. Miller; see [19, Theorem 4.2]. That \( j^* \) is onto is plain from (1.1) and (5.5). Now any \( \kappa \) in \( \text{Ph}(L, Y) = [L, Y] \) which lies in \( \ker j^* \) lifts uniquely to \( \lambda \) in \( [L(Y), Y] \). Since the image of \( \lambda \) in \( [1, Y(0), Y] \), say \( \tilde{\lambda} \), maps to 0 in \( [1, Y] \), \( \tilde{\lambda} \) lifts to some \( \mu \) in \( [\Sigma X, Y] \). The coset of \( \mu \) modulo \( (\Sigma i)^*[\Sigma X, Y] \) is uniquely determined by \( \kappa \) and conversely, any such coset determines an element in \( \ker j^* \).

**Remarks.** (1) Notice that \( \ker j^* \) is (up to bijection) independent of the integral approximation \( j \); (2) if \( Z \) is grouplike, or \( k \geq 1 \), the bijection in Theorem 5.1 is a group isomorphism.

To complete the proof of Theorem 6, we merely invoke Theorem 5.1, together with the sentence following (5.3) and (5.4), to infer that

\[ j^* : \text{Ph}(K, Y) \to \text{Ph}(BG, Y) \quad \text{and} \quad j'^* : \text{Ph}(K, Y) \to \text{Ph}(X', Y) \]

are epimorphisms with trivial kernels.

Notice that, much as in Theorem 1, the identification of \( \text{Ph}(BG, Y) \) with \( \text{Ph}(X', Y) \) in Theorem 6 does not result from a direct comparison of the two domains. Rather, an intermediate space is required, which serves as the target for rational homotopy equivalences coming out of the two domains. For the first part of Theorem 6, the intermediate space is \( BG(0) \), and for the second part of Theorem 6, the intermediate space is \( K \).

Before continuing our discussion relative to Theorem 6, we pause to point out that Theorem 5.1 may be used to carry out some interesting computations of \( \ker j^* \). Following are two examples, which may be handled using the techniques of [19, Section 9]:

**Example 5.2.** (i) Let \( M \) be the Meier space, obtained by taking a partition \( \{ I, J \} \) as in Example 2.3, and forming a Zabrodsky mix of \( K(Z, 3) \) localized at \( I \) with \( S^3 \) localized at \( J \), and let \( j : M \to K(Z, 3) \) be an integral approximation. Then

\[ \ker \{ j^* : \text{Ph}(K(Z, 3), \Omega S^5) \to \text{Ph}(M, \Omega S^5) \} \]
is isomorphic to $\prod_{p \in J} \hat{\mathbb{Z}}_p$.

(ii) Let $j : \Omega^2 S^5 \to K(\mathbb{Z}, 3)$ be an integral approximation. Then
\[
\ker \{j^* : \text{Ph}(K(\mathbb{Z}, 3), BS^3) \to \text{Ph}(\Omega^2 S^5, BS^3)\}
\]
is isomorphic to the image of $\prod_{p} \hat{\mathbb{Z}}_p$ in $\prod_{p} \hat{\mathbb{Z}}_p/\mathbb{Z}$.

Next, we ask whether Theorem 6 remains true if $Y$ is no longer assumed to be the iterated loop space of a 1-connected, finite CW-complex. We are not certain of the answer for the first part of Theorem 6 but we are able to settle negatively the question of extending the second part of Theorem 6 to arbitrary loop spaces $Y$.

**Example 5.3.** Let $BG, X', j : BG \to K$ and $j' : X' \to K$ be as in Theorem 6, $G$ noncontractible, and let $Y = U$, the “big” unitary group. Then
\[
j^* : \text{Ph}(K, U) \to \text{Ph}(BG, U), \quad j'^* : \text{Ph}(K, U) \to \text{Ph}(X', U)
\]
are not bijective. In fact, $\text{Ph}(K, U)$ is an uncountable group while $\text{Ph}(BG, U) = 0 = \text{Ph}(X', U)$.

Thus, if $U_n$ is the finite unitary group, $n \geq 3$, Theorem 6 gives us sequences of isomorphisms
\[
j^* : \text{Ph}(K, U_n) \to \text{Ph}(BG, U_n), \quad j'^* : \text{Ph}(K, U_n) \to \text{Ph}(X', U_n)
\]
(all the groups being $\mathbb{R}$), which, after passing to the limit as $n \to \infty$, fail to be isomorphisms.

**Proof of Example 5.3.** The uncountability of $\text{Ph}(K, U)$ follows from the computation
\[
\text{Ph}(K(\mathbb{Z}, 2m), U) \cong \mathbb{R}, \quad m \geq 2,
\]
carried out in [2, Theorem II]. On the other hand, the computation
\[
\text{Ph}(BG, U) = 0
\]
is contained in [1]. That
\[
\text{Ph}(X', U) = 0
\]
as well follows immediately from Theorem 1.

**Appendix**

In addition to the Mislin genus, there are two other notions of genus that arise in homotopy theory. If $X$ is, as usual, a finite type, nilpotent space, then we have: $\hat{G}_0(X)$, the set consisting of all (homotopy types of) finite type, nilpotent spaces $X'$ such that (i) the $p$-profinite completions $\hat{\mathbb{F}}_p$ and $\hat{\mathbb{F}}'$ are homotopy equivalent for all primes $p$ and (ii) the rationalizations $X(0)$ and $X'(0)$ are homotopy equivalent; $\hat{G}(X)$, the set consisting
of all (homotopy types of) finite type, nilpotent spaces \( X' \) such that \( \hat{X}_p \) and \( \hat{X}'_p \) are homotopy equivalent for all \( p \) (but where \( X^{(0)} \) and \( X'^{(0)} \) are not necessarily homotopy equivalent). The latter set, \( \hat{G}(X) \), is referred to as the completion genus of \( X \).

It is well known that there are inclusions

\[
\mathcal{G}(X) \subset \hat{\mathcal{G}}_0(X) \subset \hat{\mathcal{G}}(X),
\]

and examples of Belfi and Wilkerson ("Some examples in the theory of \( p \)-completions", Indiana J. Math. 25 (1976) 565–576) show that both inclusions are proper. For further information and references concerning the three notions of genus, see [9].

Since the first inclusion in (A.1) is proper, we see that, in the case that \( Y \) is a finite type target, the following Theorem A.1 properly generalizes Theorem 1. Since the second inclusion in (A.1) is proper, we see that the difference between Theorem A.1 and Theorem 2 is yet greater than the difference between Theorems 1 and 2.

**Theorem A.1.** Let \( X \) and \( X' \) be finite type, nilpotent spaces having the same completion genus and let \( Y \) be a finite type target. Then \( \text{Ph}(X, Y) \) and \( \text{Ph}(X', Y) \) are equivalent as sets.

The proof of Theorem A.1 precisely follows the lines of the proof of Theorem 1. The new ingredient needed is a suitable analog of [7, Theorem 1.3.12], which we now provide:

**Lemma A.2.** A homomorphism \( \phi: G \to K \) of finitely generated nilpotent groups is an epimorphism \( \Rightarrow \hat{\phi}_p: \hat{G}_p \to \hat{K}_p \) is an epimorphism for all \( p \).

**Proof.** The implication \( \Rightarrow \) being evident, we consider the implication \( \Leftarrow \).

We argue indirectly and assume that \( \phi(G) \) is a proper subgroup of \( K \). In that case, we may find a proper normal subgroup \( N \) of \( K \) containing \( \phi(G) \); for instance, \( N \) may be taken to be the normal closure of \( \phi(G) \) in \( K \). Thus we have a short exact sequence

\[
1 \longrightarrow N \longrightarrow K \longrightarrow Q \longrightarrow 1,
\]

and

\[
Q \neq 1.
\]

From (A.2), we derive a short exact sequence

\[
1 \longrightarrow \hat{N} \longrightarrow \hat{K} \longrightarrow \hat{Q} \longrightarrow 1
\]

of the profinitely completed groups. Furthermore, as \( Q \) is finitely generated, nilpotent, we have a natural embedding

\[
\mathcal{Q} \subset \hat{\mathcal{Q}}
\]

It is also true that \( \phi \) is a monomorphism \( \Leftrightarrow \hat{\phi}_p \) is a monomorphism for all \( p \) but we do not need this fact for our present purposes.
and so, by (A.3), we conclude that
\[ \tilde{Q} \neq 1. \]

In sum,
\[ N \neq K \Rightarrow \tilde{N} \neq \tilde{K}. \]

But since it is assumed that
\[ \tilde{\phi}_p(\tilde{G}_p) = \tilde{K}_p \]
for all \( p \), and since
\[ \tilde{G} \cong \prod \tilde{G}_p, \quad \tilde{K} \cong \prod \tilde{K}_p, \]
we see that
\[ \tilde{\phi}(\tilde{G}) = \tilde{K}, \]
that is
\[ \tilde{N} = \tilde{K}. \]

This contradiction establishes Lemma A.2.

References