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Constructive algebraic topology ☆

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Abstract

The classical "computation" methods in Algebraic Topology most often work by means of highly infinite objects and in fact *are not* constructive. Typical examples are shown to describe the nature of the problem. The Rubio–Sergeraert solution for Constructive Algebraic Topology is recalled. This is not only a theoretical solution: the concrete computer program *Kenzo* has been written down which precisely follows this method. This program has been used in various cases, opening new research subjects and producing in several cases significant results unreachable by hand. In particular the Kenzo program can compute the first homotopy groups of a simply connected *arbitrary* simplicial set.

Keywords: Algebraic topology; Effective homology; Homotopy groups; Functional programming; Symbolic computation

Introduction

The computation of *homotopy groups* in Algebraic Topology is known as a difficult problem. Every pointed topological space (X, x_0) has a family of homotopy groups $\{\pi_n(X, x_0)\}_{n\geqslant 1}$, $\{\pi_n X\}$ in short, and these groups are abelian for $n \geqslant 2$. The definition was given by Hurewicz in 1935, and for the first non-trivial space from this point of view, namely the 2-sphere S^2, only the groups π_2 and π_3 were known at this time, thanks to Hopf. The group $\pi_4 S^2 = \mathbb{Z}_2$ was determined by Freudenthal in 1937. Thirteen years then passed without any new homotopy group of sphere. The following groups $\pi_n S^2$ were obtained by

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Serre for $5 \leqslant n \leqslant 9$, in 1950. In fact, for $n = 6$, Serre proved the group $\pi_6 S^2$ has twelve elements but did not succeed in choosing between both possible solutions $\mathbb{Z}_{12}$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_6$. Two years later, Barratt and Paechter proved there exists an element of order 4 in $\pi_6 S^2$, so that finally $\pi_6 S^2 = \mathbb{Z}_{12}$. See [21, vol. I, pp. 110 and 113] for details and references.

More generally, Serre obtained a general *finiteness result*.

Theorem 0.1 (Serre, [21, p. 14 and pp. 171–207]). *If X is a simply connected space such that the homology groups $H_n(X;\mathbb{Z})$ are of finite type, then the homotopy groups $\pi_n X$ are also abelian groups of finite type.*

In particular the homology groups of simply connected finite polyhedra, for example the simply connected compact manifolds, are of finite type, so that their homotopy groups are also of finite type. Various methods allow to combinatorially describe the finite polyhedra: these objects may be the *input* of an algorithm. An abelian group of finite type can also be described by some character string: such a group could be the *output* of an algorithm. The following problem therefore makes sense.

Problem 0.2. Does there exist a general algorithm:

- **Input:** A simply connected polyhedron X and an integer $n \geqslant 2$;
- **Output:** The homotopy group $\pi_n X$.

A solution for this computability problem was given by Edgar Brown in 1956 [3]. He used the general organization just defined by Postnikov, now known as the *Postnikov tower*; then the result is not difficult when the homology groups of the space X are *finite*; really finite, not only of finite type: for example this simple method does not work for the 2-sphere S^2 because the homology group $H_2 S^2 = \mathbb{Z}$ is of finite type (one generator), but unfortunately is infinite. The difficult part of the work of Edgar Brown then consisted in overcoming the birth of infinite objects in the Postnikov tower. A complicated and tricky process was used to approximate these infinite objects by finite ones and in this way Edgar Brown succeeded in transforming the finiteness result of Serre into a computability result.

But let us quote Edgar Brown himself in the introduction of his article:

> It must be emphasized that although the procedures developed for solving these problems are finite, they are much too complicated to be considered practical.

Forty years later this appreciation still holds, and will always hold, even with the most powerful computer you can imagine: it is a consequence of the hyper-exponential complexity of the algorithm designed by Edgar Brown.

The problem of finding new *general* algorithms which on the contrary could be *concretely* used in significant cases was not seriously studied up to 1985. This is so true that topologists from time to time meet some difficulty in expressing precisely where the actual nature of a problem is, when in fact it is a matter of computability. Section 1 shows three typical examples of this sort. Let us quote immediately another example found in the introduction of [12]:

> The book by Cartan and Eilenberg contains essentially all the constructions of homological algebra that constitute its computational tools, namely standard resolutions and spectral sequences.

Strictly speaking, this statement is correct, but it is also very misleading. In the "general" domain of Homological Algebra, it is true, but if you intend to apply these "computational" tools in Algebraic Topology, then you realize an enormous *gap* is in front of you, mainly when you have to determine the higher differentials of the spectral sequences you are working with; and if you succeed in finding them, a collection of hard extension problems can be waiting at the abutment. The present paper is essentially devoted to these questions.

One of the examples of Section 1 asserts a computability problem in homotopy theory is "widely" open. In fact three complete solutions are available for several years. Section 2 is devoted to a quick description of these solutions, to their nature and what can be hoped about their concrete use for computer calculations.

So far, only the Rubio–Sergeraert solution has led to a reasonably complete computer program which has been used in significant cases. The main tool is standard algebraic topology combined with *functional programming* and Section 3 uses a didactic example to explain how a functional programming method can be used to obtain efficient algorithms, even for solving problems where there is a function neither in the input nor in the output.

The main ingredient in our solution is the notion of *object with effective homology*. Such an object is a subtle combination via chain equivalences of traditional *effective* objects on one hand, and of other *locally effective* objects on the other hand. Section 4 describes the essential properties of these objects and what an object with effective homology is.

The main tools of basic algebraic topology, mainly the Serre and Eilenberg–Moore spectral sequences, may then be rewritten in such a way they become *algorithms* computing the desired homology groups when the necessary data are given; such a property does not hold for the classical spectral sequences. Section 5 contains the main statements and describes how they can be used for example to compute the homotopy groups of simply connected simplicial sets with effective homology. A simple solution is so obtained for the computability problem of homotopy groups; furthermore its scope is much larger than Edgar Brown's one.

Section 7 describes how these theoretical results led to a concrete program named *Kenzo*.[1] It is a Lisp program of 16000 lines (joint work with Xavier Dousson), now www-available [11] with a rich documentation (340 pp.) written by Yvon Siret.

These results open new research fields; in Computer Science because of the original type of functional programming which is required, but in theoretical Algebraic Topology as well: the objects that are processed by the Kenzo program are much too complicated to be studied by hand, specially around *algebraic fibrations*. These questions are considered in Section 8.

Finally Section 9 gives a few examples of calculations.

[1] Kenzo is the name of the author's *cat* and C.A.T. = Constructive Algebraic Topology; the next version of our program will therefore be called *Simba*, the daughter of Kenzo.

1. Three examples

A preprint by Karoubi [14], distributed in 1993, begins as follows:

> The problem of finding a "computable algebraic model" for the homotopy type of a CW-complex X remains a widely open problem in topology.

The notion of *computable algebraic model* for a homotopy type is not precisely defined in the text, but taking account of the rest of the paper, and also of other related papers by the same author, it is clear the following meaning is the right one:

Definition 1.1. A *computable algebraic model* for the homotopy type of a space X is an additional structure $\mathcal{H}$ over the chain complex C_*X such that the pair $(C_*X, \mathcal{H})$ "contains" the homotopy type of X.

Two spaces X and Y have the same homotopy type if there exist two continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to identity maps; from the point of view of Algebraic Topology, both spaces are "equal", even if they are quite different: for example a point and the infinite unit sphere $S^\infty \subset \ell^2$ have the same homotopy type: this sphere is in fact contractible.

As usual, the additional data $\mathcal{H}$ must be *natural* with respect to X, that is, the mapping $X \mapsto (C_*X, \mathcal{H})$ should be a functor. Several contexts are possible. If the chain complex C_*X is the *singular* chain complex, then it is easy to give the required additional structure (the canonical distinguished generators, namely the singular simplices, and the simplicial operators), but the singular chain complex is a functional space which is so enormous that no program can handle it: the object so obtained is not *computable*. The same in the simplicial context as soon as the simplicial model is infinite, which is frequent. Karoubi wants a chain complex of finite type in any dimension, for example the cellular chain complex $C_*^{\mathrm{cell}}(X)$ if X is presented as a CW-complex of finite type in any dimension; much information about X is lost in this chain complex and Karoubi searches an additional structure over this chain complex which captures the homotopy type of X at least. The structures studied by Karoubi intensively use the notion of *non-commutative differential forms* and are interesting, but to our knowledge, the goal defined by Karoubi is not yet reached by his method.

In fact three solutions now exist for Constructive Algebraic Topology, and two of them exactly have the form that Karoubi looked for. In Justin Smith' solution [23,24], the cellular chain complex $C_*^{\mathrm{cell}}(X)$ is provided with a m-structure which, in appropriate context, is a computable algebraic model for the homotopy type of X. In our solution, the same chain complex is completed with two other chain complexes and a few operators which give the same result. The solution by Rolf Schön [17] is not presented in this way but finally is equivalent to both previous ones.

Let us quote now a paper by Carlsson and Milgram [6, p. 545] in James' Handbook of Algebraic Topology [13]:

In Section 5 we showed that for a connected CW complex with no one cells one may produce a CW complex, with cell complex given as the free monoid on generating cells, each in one dimension less than the corresponding cell of X, which is homotopy equivalent to [the loop space of X] ΩX. To go further one should study similar models for double loop spaces, and more generally for iterated loop spaces.

In principle this is direct. Assume X has no i-cells for $1 \leqslant i \leqslant n$ then we can iterate the Adams–Hilton construction of Section 5 and obtain a cell complex which represents $\Omega^n X$. However the question of determining the boundaries of the cells is very difficult as we already saw with Adam's solution of the problem in the special case that X is a simplicial complex with $sk_1(X)$ collapsed to a point. It is possible to extend Adams' analysis to $\Omega^2 X$, but as we will see there will be severe difficulties with extending it to higher loop spaces except in the case where $X = \Sigma^n Y$.

The paper by Carlsson and Milgram is an excellent presentation of Adams' model for a loop space of a simply connected CW complex and related questions. You see the authors here consider a problem whose solution *in principle is direct*, but new *severe* difficulties are soon announced which can in fact be overcome only if the space X in an iterate suspension $\Sigma^n X$.

In fact the actual problem is a *computability* problem. The following theorem can easily be deduced from Adams' construction. In the statement, the operator s^{-1} is the desuspension of the "augmentation ideal" : the base generator is removed and the degree n of a generator becomes $n-1$; the operator T associates to a chain complex its tensor algebra, another chain complex provided with a multiplicative structure.

Theorem 1.2. *If X is a CW complex with one* 0*-cell, without any i-cell* $(1 \leqslant i \leqslant n)$*, then there exists for the chain complex*:

$$G^n X = (T_s^{-1})^n C_*^{\text{cell}}(X)$$

a new differential δ such that the chain complex $(G^n X, \delta)$ is the cellular chain complex of a CW model of the iterate loop space $\Omega^n X$.

The *existence* of the differential δ can be easily proved thanks to Adams' work about the CW model of the first loop space (cf. also [2]), but the existence proof is not constructive: it is made of a mixture of combinatorial and topological arguments and certainly there are at least "severe difficulties" to translate the topological constructions into the combinatorial constructions that are necessary if you intend to obtain a constructive existence proof for the differential δ. The problem of *iterating the cobar construction* is the *heart* of Algebraic Topology: the main computability problems can be reduced to this one, and it is not amazing this problem is a little severe. The three current solutions [11,17,19,23,24] for Constructive Algebraic Topology are firstly solutions for the problem of iterating the cobar construction.

John McCleary tries in his book [15] to express the same idea in the context of spectral sequences:

> [p. 6] "Theorem". There is a spectral sequence with $E_2^{*,*}$ = "something computable" and converging to H*, something desirable. The important observation to make about the statement of the theorem is that it gives an E_2-term of the spectral sequence but says nothing about the successive differentials d_r. Though $E_2^{*,*}$ may be known, without d_r or some further structure, it may be impossible to proceed.
>
> ... ...
>
> [p. 28] It is worth repeating the caveat about differentials mentioned in Chapter 1: knowledge of $E_r^{*,*}$ and d_r determines $E_{r+1}^{*,*}$ but not $d_r + 1$. If we think of a spectral sequence as a black box, then the input is a differential bigraded module, usually $E_1^{*,*}$, and, with each turn of the handle, the machine computes a successive homology according to a sequence of differentials. If some differential is unknown, then some other (any other) principle is needed to proceed. From Chapter 1, the reader is acquainted with several algebraic tricks that allow further calculation. In the non-trivial cases, it is often a deep geometric idea that is caught up in the knowledge of a differential.

It is in fact again a matter of computability. The higher differentials of a spectral sequence are *mathematically* defined, but, in most cases, their definition *is not* constructive: the differentials are not *computable* with the provided information. For example the result of Adams' work about the first loop space is nothing but an algorithm computing the higher differentials and solving the extension problems at abutment of the corresponding Eilenberg–Moore spectral sequence, thanks to the coalgebra structure over the initial cellular chain complex. But this does not compute the coalgebra structure for the CW model of the loop space so that you cannot continue: this is nothing but the "severe" difficulty above observed by Carlsson and Milgram. See the nice work of Baues [2] to go a little further, but this does not give a solution for the general problem of "iterating the cobar construction".

2. Three complete solutions for the computability problem

In fact three solutions are now available to work in a *constructive* context in Algebraic Topology. This section describes the main ingredients of the solutions that are due to Rolf Schön [17] and Justin Smith [23,24]. The rest of the paper is devoted to our solution and the corresponding Kenzo program.

2.1. Rolf Schön's solution

Schön's solution [17] is a systematic reorganization of Edgar Brown's special work [3] around the computation of homotopy groups. Frequently in Homological Algebra, we work with large chain complexes, the homology groups of which are of finite type; for example the singular chain complex of a compact manifold is not at all of finite type, but the homology groups of this chain complex on the contrary are. The same in a simplicial context; for example a simplicial *group* version of the circle S^1 necessarily has an infinite number of simplices in any positive dimension, but the homology groups are null or with

only one generator. When you work with the traditional tools of homological algebra, you must frequently handle highly infinite chain complexes even if you know the final result is of finite type.

Edgar Brown designed an approximation process which has been skilfully generalized by Rolf Schön. Let X be a simplicial set, described as the limit of a sequence (X_n) of finite approximations. Then the homology group $H_p(X)$ is the inductive limit of the groups $(H_p(X_n))_n$, so that the following definition could be useful.

Definition 2.1. A Schön $\mathbb{Z}$-module G is a triple

$$\big((G_n)_{n\geqslant 0}, (\phi_n)_{n\geqslant 0}, \alpha\big),$$

where the following conditions are satisfied. Every G_n is a $\mathbb{Z}$-module of finite type, and $\phi_n : G_n \to G_{n+1}$ is a morphism of $\mathbb{Z}$-module; the sequence $(G_n, \phi_n) : G_n \to G_{n+1})$ is an inductive system and its limit G is again of finite type. The third component α precisely describes how the limit is reached; $\alpha : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is as follows: if $\alpha(i) = (j,k)$, then $i \leqslant j \leqslant k$ and the canonical morphism $\operatorname{Im} G_j \to G$ is in fact an isomorphism:

$$\alpha : i \mapsto \left\{ \begin{array}{c} G_j \to \operatorname{Im} G_j \subset G_k \\ \cong\downarrow \\ G \end{array} \right\}.$$

The *existence* of such a map α is implied by the finiteness property of the inductive limit G which is assumed, but an *effective* knowledge of this map is required. Because you do not know a priori what approximations X_n of X will be later required for some calculation, the value $\alpha(i)$ must be *computable* for any i. We call the map α the *convergence descriptor.*

The books of Homological Algebra are full of theorems of this sort:

Theorem 2.2. *There is an exact sequence*:

$$\cdots \to A \xrightarrow{f} B \to C \to D \xrightarrow{g} E \to \cdots.$$

The underlying idea is that if you know the $\mathbb{Z}$-modules A, B, D and E, then you should be able to guess the unknown module C. Of course you must in fact also know the maps $f : A \to B$ and $g : D \to E$ to determine the modules $\operatorname{Coker}(f)$ and $\operatorname{Ker}(g)$, giving a simpler exact sequence:

$$0 \to \operatorname{Coker}(f) \to C \to \operatorname{Ker}(g) \to 0,$$

and now you could have an extension problem in front of you, about which the exact sequence says nothing at all! The situation is analogous with the spectral sequences but usually much more complicated. It was exactly the problem encountered by Serre when he was looking for the group $\pi_6 S^2$: the unknown group was in an exact sequence at the end of a spectral sequence between two groups $\mathbb{Z}_2$ and $\mathbb{Z}_6$, and a new idea is necessary to terminate.

On the contrary such a problem is entirely solved in the framework designed by Rolf Schön. The situation is now the following: the modules A, B, D and E are four known *Schön modules*; the map f is in fact a morphism of inductive systems and in particular

for every n a morphism f_n is defined satisfying the usual properties; the same between the other components of the exact sequence. For the *unknown* Schön module C, the underlying inductive system is known but its convergence descriptor *is not.* You know there is an *exact* sequence between the limits A, B, C, D and E, but at the nth stage of the inductive systems, you have only a "differential" sequence:

$$A_n \xrightarrow{f_n} B_n \xrightarrow{f'_n} C_n \xrightarrow{g'_n} D_n \xrightarrow{g_n} E_n$$

where two successive maps have a null composition, but this sequence is not necessarily exact.

$$\begin{array}{l}
(A_n, \phi_n),\ \alpha_A : i \mapsto (j,k) \\
\quad \downarrow (f_n) \\
(B_n, \phi'_n),\ \alpha_B : i \mapsto (j,k) \\
\quad \downarrow (f'_n) \\
(C_n, \chi_n),\ ???????????? \\
\quad \downarrow (g'_n) \\
(D_n, \psi'_n),\ \alpha_D : i \mapsto (j,k) \\
\quad \downarrow (g_n) \\
(E_n, \psi_n),\ \alpha_E : i \mapsto (j,k)
\end{array}$$

Theorem 2.3 (Schön [17]). *With the previous data, an algorithm can compute the convergence descriptor of the intermediate Schön module C.*

Once the missing descriptor α_C is available, then you can compute the limit C. But, and maybe this is more important, the process is *stable:* the object $C = ((C_n), (\chi_n), \alpha_C)$ which is returned by Schön's algorithm is again a Schön module and can be a part of the input for another call of the same algorithm. Rolf Schön explains in his nice paper [17] how this method allows to entirely transform classical Homological Algebra into a *constructive* theory.

To our knowledge, Schön's work has not yet led to concrete machine programs. It is a pity: his general framework is quite original and interesting with respect to what is usually done in computational algebra. The opinion of the present author is that concrete implementation of Schön's results must absolutely be done and should give new insights into several fields: at least in symbolic computation, in computational algebra and also in algebraic topology.

2.2. Justin Smith' solution

This second solution is quite different from the previous one. In a sense it is exactly the solution of the problem stated by Karoubi (cf. Section 1). Let X be a *simplicial* complex. The main problem in Algebraic Topology comes from the non-commutativity of the Alexander–Whitney diagonal. If you intend to send an interval I onto the diagonal

of a square $I \times I$, using only the bisimplicial structure of this square, that is, using *only* its four boundary edges, then you can join one vertex to the opposite one turning around the square in two different ways:

These paths are different but they are homotopic. This homotopy is quite important and leads to this diagram:

$$\begin{array}{ccc} C_*(X^2) & \xrightarrow{\Delta} & C_*(X) \otimes C_*(X) \\ \downarrow{\scriptstyle \pi} & \overset{h}{\searrow} & \downarrow{\scriptstyle \pi} \\ C_*(X^2) & \xrightarrow{\Delta} & C_*(X) \otimes C_*(X) \end{array}$$

The chain complex $C_*(X^2)$ is obtained from the canonical *simplicial* structure of X^2: on the contrary the other chain complex $C_*(X) \otimes C_*(X)$ comes from the canonical *bisimplicial* structure of the same space. If for example X is the interval $I = [0, 1]$, in the first case a square is presented as the union of two triangles joined along a diagonal; in the second case no diagonal in the square, only the boundary edges, the square is simply the product of two intervals. Both presentations are related by the Alexander–Whitney map Δ. Furthermore both components of X^2 can be swapped, and this leads to the vertical canonical (different) maps π. Then the diagram is not commutative: $\Delta \circ \pi \neq \pi \circ \Delta$. Nevertheless the homotopy operator h explains both maps are homotopic. But the same difficulty occurs now for the homotopy h which in turn is not compatible with the symmetry of its source and its target, but again a homotopy can be constructed and so on. This process roughly explained here for both factors works also for an arbitrary number of factors X^n and all the homotopies are related by a very rich structure called a *coalgebra structure with respect to the symmetric operad* $\mathfrak{S}$.

Using an appropriate modified model for the symmetric operad $\mathfrak{S}$ and also a corresponding notion of coalgebra called m-structure, Justin Smith succeeded firstly in iterating the cobar construction [23], and more recently [24] in proving that a chain complex carrying an m-structure contains a homotopy type, so that such a structure can be used as the $\mathcal{H}$ component (cf. Definition 1.1) for the computable algebraic model demanded by Karoubi.

While preparing this paper, the second author received a message of Justin Smith announcing a partial programming work was just starting around the symmetric operad $\mathfrak{S}$. So that we can hope Justin Smith' solution finally leads also to a concrete computer program. The situation here is also interesting because of the original environment where work is to be undertaken: it is probably the first time an operad structure is implemented. Certainly, at least because they solve the same problem (!), Justin Smith' program and ours will be strongly related. Probably the structure of Justin Smith's solution is richer than for our solution; the latter works essentially like a blackbox, because of its highly functional process which in a sense hides what actually happens during the execution. When both solutions will be available, determining what exactly the relations between them are will be still more interesting!

2.3. A quick sketch of the Rubio–Sergeraert solution

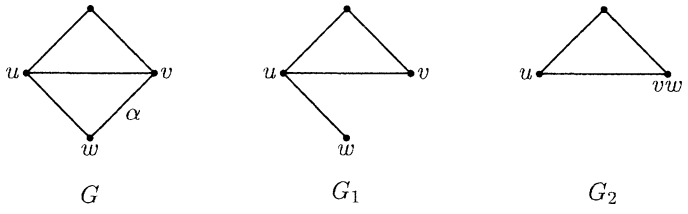
The nature of this "third"[2] solution is not so far from Justin Smith' one. In our framework, any reasonable homotopy type is described as follows: firstly a free $\mathbb{Z}$-chain complex of finite type in any dimension EC_*X is given; then a further structure $\mathcal{H}$ is added to this chain complex in such a way a homotopy type is finally so defined; in fact this homotopy type can be realized as a CW-complex, the cellular complex of which being EC_*X; it is well known this cellular complex does not define a homotopy type, but the added structure $\mathcal{H}$ gives the missing information. What is quite original with respect to the traditional organization in Algebraic Topology is the deeply *functional* nature of the structure $\mathcal{H}$, the main subject of the rest of this paper.

3. A didactic example of functional programming

We briefly recall in this section a typical situation where it is much better to work with functional objects carrying an enormous information, instead of working with data close to those that are looked for.

Let G be a finite graph $G = (V, E)$; the set V is the vertex set and E is the set of the edges. A *good* colouring of G consists in defining a colour for each vertex so that two adjacent vertices have different colours. The *chromatic number* $\chi(G)$ is the minimal number of colours that are necessary. It is not so easy to design a program computing this chromatic number. The traditional backtracking methods work but are quite inefficient.

If you think of a recursive method, you cannot design such a method if you work only with the chromatic number. Let $\alpha \in E$ be an edge between the vertices $v, w \in V$. You would like for example to deduce $\chi(G)$ from $\chi(G')$ where G' is the graph G without the edge α. In fact two interpretations of G' make sense. The first one G_1 has the same vertex set as G, and α is simply removed from E. The second interpretation G_2 consists in collapsing the edge α over one vertex coming for both vertices v and w; in particular if we previously had two different edges uv and uw starting from another vertex u and going respectively to v and w, both edges give only one edge in G_2: both G-edges are now identified in G_2. For example if G is a complete graph of order n, then G_1 is the same with only one edge removed, but G_2 is the complete graph of order $n-1$. And very simple cases show the knowledge of $\chi(G_1)$ and $\chi(G_2)$ is not sufficient to compute $\chi(G)$: the chromatic number does not contain enough information; *we need more*.



G G_1 G_2

[2] The first announcement goes back to 1987 [18]; the first computer program computing an iterate cobar construction started in 1990 [16].

Let us consider the *chromatic polynomial* $P_G(X)$; it is a polynomial with one variable defined as follows: if n is a positive integer, then $P_G(n)$ is the number of good colourings of G that are possible with n colours. Now the situation is good:

1) A recursive relation holds and it is simple: $P_G(X) = P_{G_1}(X) - P_{G_2}(X)$; in fact, let us consider a good colouring of G_1; then, depending on whether both colours of v and w are the same or not, you obtain a good colouring for G_2 or G, and the relation between P_G, P_{G_1} and P_{G_2} follows; starting with graphs without any edge, you obtain in particular that P_G actually is a polynomial!
2) The polynomial P_G contains an *infinite* number of elementary data: how many good colourings exist with 1 color, with 2 colours, and so on; we now have enough information;
3) These data are coded in a *functional* way: of course you cannot store in your machine all the values $P_G(n)$; but it is sufficient to store the degree and the coefficients of P_G: a polynomial is a *finite* object which is nothing but a *program* ready to compute the value $P_G(n)$ for every integer n in the *infinite* set $\mathbb{N}$;
4) The chromatic number $\chi(G)$ is a *by-product* of the polynomial P_G: it is sufficient to compute $P_G(1), P_G(2), \ldots$, until you find the first integer n satisfying $P_G(n) > 0$; then $\chi(G) = n$.

It is then easy to write down a recursive program computing the chromatic number; it is more efficient than a program using backtracking, but however it has an exponential complexity; the problem of finding a polynomial time algorithm computing the chromatic number is open: it is a special case of the general *NP*-complete problem.

Our solution for constructive Algebraic Topology is quite similar. The role of the chromatic number $\chi(G)$ is played by an *effective* chain complex EC_*X, which is the cellular chain complex of some CW-model of the homotopy type we intend to algebraically define. The situation is the same: the information given in this chain complex is in general too poor to process new objects deduced from this one and others; *we need more.* We will define new ingredients, in general containing an *infinite* number of elementary data and which completely define a homotopy type; but these ingredients will be coded in a *functional* way so that a machine program will be able to handle them as easily as polynomials[3] and to compute the corresponding ingredients for a new homotopy type constructed from others which were defined by means of such data.

4. Objects with effective homology

4.1. Effective chain complexes

A chain complex is a sequence of $\mathbb{Z}$-modules and homomorphisms:

$$\cdots \leftarrow C_{n-1} \leftarrow C_n \leftarrow C_{n+1} \leftarrow \cdots$$

[3] At least if your programming language allows you to use *functional programming*.

where the composition of two successive arrows is null.

From now on 4.1. All the chain groups C_n of a chain complex C_* are *free* $\mathbb{Z}$*-modules with distinguished basis.*

In the following definition, the set $\mathcal{U}$ is the "machine universe": any machine object is an element of $\mathcal{U}$; the set List $\subset \mathcal{U}$ is the subset of all *lists*, in other words the finite sequences of elements of $\mathcal{U}$.

Definition 4.2. An *effective chain complex* is defined as a pair of algorithms:

- $\beta : \mathbb{Z} \to$ List;
- $d : \mathbb{Z} \tilde{\times} \mathcal{U} \to$ List;

where:

1. The output $\beta(n)$ is the given basis of the free $\mathbb{Z}$-module C_n; this basis is a list and in particular is finite;
2. A pair (n, g) is in $\mathbb{Z} \tilde{\times} \mathcal{U}$ if g is a generator of C_n, that is, if $g \in \beta(n)$;
3. The output $d(n, g)$ is a list representing the differential $d_n(g) \in C_{n-1}$.

If an effective chain complex C_* and an integer n are given, a program can compute the boundary matrices in dimensions $n+1$ and n, and an elementary algorithm then determines the homology group $H_n(C_*)$. The *global* nature of an effective chain complex C_* is reachable for any dimension n.

4.2. Locally effective chain complexes

Definition 4.3. A *locally effective chain complex* C_* is defined as a pair of algorithms:

- $\beta' : \mathbb{Z} \times \mathcal{U} \to$ Boolean;
- $d : \mathbb{Z} \tilde{\times} \mathcal{U} \to$ List;

where:

1. The output $\beta'(n, \gamma)$ is the Boolean true if and only if the object γ is a generator of the chain group C_n;
2. The sub-product $\mathbb{Z} \tilde{\times} \mathcal{U}$ interpreted as in the previous definition and the differential d as well.

It is explained in the handbooks of set theory there are two different methods to define a set S. You can give the element list of S; in a computational framework, such a list is necessarily finite. You can also define the set S by means of a characteristic property of its elements. For example you can require an element of S must be an integer and must be odd. Then such a set may be infinite. Do not object the set of actual elements that can

actually be processed on your machine is finite; consider for example this Lisp definition of the set N_{odd}:

```
> (setf odd-integers
      #'(lambda (object)
           (and (integerp object)
           (oddp object))))
```

This string of 82 characters is finite and *defines* the infinite set of odd integers.

In the same way the generators of our locally effective chain complexes are defined by means of a characteristic property, so that now our chain groups are not necessarily of finite type. This looks like an advantage with respect to the notion of effective chain complex, but there is an important drawback: in general no global information is reachable for such a machine chain complex; in particular the homology groups in general are not computable. This is an avatar of the main incompleteness theorem (Gödel, Church, Turing, Post). The key point of our solution for Constructive Algebraic Topology consists in combining effective *and* locally effective chain complexes, connecting them by *reductions.*

4.3. Reductions

Definition 4.4. A *reduction* $\rho : D_* \Rightarrow C_*$ between two chain complexes is a triple $\rho = (f, g, h)$ where:

1. The components f and g are chain complex morphisms

$$f : D_* \to C_* \quad \text{and} \quad g : C_* \to D_*;$$

2. The component h is a homotopy operator $h : D_* \to D_*$ (degree 1);
3. The following relations are satisfied:
   (a) $f \circ g = \mathrm{id}_{C_*}$; $g \circ f + d_{D_*} \circ h + h \circ d_{D_*} = \mathrm{id}_{D_*}$;
   (b) $f \circ h = 0$; $h \circ g = 0$; $h \circ h = 0$.

In these formulas, d_{D_*} denotes the differential of the chain complex D_*. These formulas have a simple interpretation: the chain complex C_*, the small one, is isomorphic to a subcomplex of D_*, the big one, and a decomposition $D_* = C_* \oplus E_*$ is given where the summand E_* is acyclic and provided with an explicit homological contraction. This implies both chain complexes C_* and D_* have the same homology.

Frequently in our context, the big chain complex D_* is locally effective, so that its homology groups are not computable; on the contrary, the small chain complex C_* is effective, so that its homology groups are computable. In such a situation, the reduction can be understood as a provided description of the global homological properties of D_*. In particular if you are interested by the explicit value of $H_n(D_*)$, you can obtain the result by $H_n(C_*)$; furthermore an explicit representative for any homology class can be deduced in D_n; if z is a cycle of D_n, the homology class of z can be determined, and if null, a

chain $c \in D_{n+1}$ can be found such that $dc = z$. In a word you know *everything* about the homological properties of D_*.

Definition 4.5. An *equivalence* $\varepsilon : C_* \Longleftrightarrow E_*$ is a pair $\varepsilon = (\rho_\ell, \rho_r)$ of reductions ρ_ℓ: $D_* \Rightarrow C_*$ and $\rho_r : D_* \Rightarrow E_*$.

Again, frequently the chain complexes C_* and D_* are only locally effective and the third one E_* is effective; so that the equivalence ε describes the homological properties of C_* thanks to E_*.

4.4. Objects with effective homology

Definition 4.6. An *object with effective homology* is a pair (X, ε) where X is some locally effective object and ε is an equivalence between the chain complex "canonically" associated to X and some effective chain complex.

The associated chain complex depends on the context. For example if X is a simplicial set, then $C_*(X)$ could be the normalized chain complex defining its simplicial homology. The simplicial set X should be also locally effective; in other words some algorithm is given as the characteristic property of the n-simplices of X; if σ is such a simplex, another algorithm can compute the faces $\partial_i(\sigma)$. The equivalence:

$$\varepsilon : C_*(X) \overset{\rho_\ell}{\Leftarrow} D_*X \overset{\rho_r}{\Rightarrow} E_*X$$

entirely describes the homological properties of X, because the chain complex E_*X is effective. In general there is no way to *deduce* this equivalence from the locally effective object X. Most often we start with effective objects where such an equivalence is trivial, and also with special objects for which the particular situation gives such an equivalence; the Eilenberg–MacLane spaces $K(\pi, 1)$ are of this sort if the group π is abelian of finite type. Then the *effective homology version* of the "classical" construction methods of Algebraic Topology allow you to obtain new objects with effective homology. For example the Eilenberg–MacLane space $K(\pi, 2)$ is the classifying space of $K(\pi, 1)$, so that the effective homology version of the classifying space construction, available in the program Kenzo, will give you a copy of $K(\pi, 2)$ with effective homology. You can trivially iterate the process and obtain versions with effective homology of the Eilenberg–MacLane spaces $K(\pi, n)$'s. Proceeding in the same way with the loop space construction, a very simple solution for *iterating the cobar construction* is obtained.

5. The spectral sequences revisited

Many constructions in algebraic topology can be organized as solutions of fibration problems. In particular the classifying space BG of a topological group G is the solution for a fibration $BG \times_\tau G$ where the fiber space is the given group G, the base space is the classifying space BG and the product $BG \times G$ is twisted in such a way the total space $BG \times_\tau G$ is contractible. The same idea where the base space X is given and the fibre

space is unknown leads to the loop space ΩX and the contractible total space $X \times_\tau \Omega X$. The handbooks of Algebraic Topology more or less explain the Eilenberg–Moore spectral sequence can be used to "compute" the homology groups of the new objects BG and ΩX if the homology groups of G or X are known. In fact this spectral sequence is in general unable to give you the new homology groups, unless you are in a very special situation.

The Serre spectral sequence works in the third situation, when you are looking for the homology groups of a total space $B \times_\tau F$ if the homology groups of B and F are known; but in general you meet the same difficulties with the higher differentials and the extension problems at abutment.

The Serre and Eilenberg–Moore spectral sequences have *effective homology* versions which work when the data are simplicial sets with effective homology. We detail a little the organization and the proof for the Serre spectral sequence.

Theorem 5.1. *There exists an algorithm*:

- **Input:** *Two simplicial sets B and F with effective homology and a twisting operator τ defining a fibration $F \to B \times_\tau F \to B$;*
- **Output:** *A version* with effective homology *of the total space $T = B \times_\tau F$.*

The same with the Eilenberg–Moore spectral sequences when you are looking for the effective homology of the base space B (resp. the fiber space F), if versions with effective homology of the total space T and the fiber space F (resp. the base space B) are given. These effective homology versions of the Serre and Eilenberg–Moore spectral sequences are available in the program Kenzo.

The main ingredient for the proof of the effective homology version of the Serre spectral sequence is the *Basic Perturbation Lemma* [5].

Theorem 5.2 (Basic Perturbation Lemma). *Let $\rho : D_* \Rightarrow C_*$ be a chain complex reduction and $\delta_{D_*} : D_* \to D_*$ a* perturbation *of the differential d_{D_*} satisfying the nilpotency condition. Then a general algorithm can compute a new reduction $\rho' : D'_* \Rightarrow C'_*$ where the underlying graded modules of D_* and D'_* (resp. C_* and C'_*) are the same, but the differentials are perturbed*:

$$d_{D'_*} = d_{D_*} + \delta_{D_*},$$
$$d_{C'_*} = d_{C_*} + \delta_{C_*}.$$

The perturbation δ_{D_*} for the differential of the big chain complex is *given*; on the contrary the perturbation δ_{C_*} for the small one is *computed* by the algorithm. In a sense, the perturbation of the big chain complex is also *reduced*. This is possible thanks to the nilpotency condition: let $h : D_* \to D_*$ be the homotopy component of the reduction ρ; then the nilpotency condition is satisfied if the composition $\nu = h \circ \delta_{D_*}$ is pointwise nilpotent, that is, $\nu^n(x) = 0$ for an $n \in \mathbb{N}$ depending on x.

A typical application of the basic perturbation lemma is the following. Let $T = B \times_\tau F$ be a fibration with the base space B and the fiber space F. Let us assume two reductions $\rho_B : C_*(B) \Rightarrow EB_*$ and $\rho_F : C_*(F) \Rightarrow EF_*$ are given, describing the homology of both

spaces by means of the *effective* chain complexes EB_* and EF_*; then it is easy, thanks to Eilenberg–Zilber, to compute a *non-twisted* product reduction:

$$\rho_B \times \rho_F : C_*(B \times F) \Rightarrow EB_* \otimes EF_*.$$

The underlying graded modules of $C_*(T) = C_*(B \times_\tau F)$ and $C_*(B \times F)$ are the same but the differentials are not; the difference is a perturbation of the big chain complex. If the base space B is 1-reduced (no edge, the geometry begins in dimension 2), then the nilpotency condition is satisfied and applying the Basic Perturbation Lemma gives a reduction:

$$\rho_T : C_*(T) = C_*(B \times_\tau F) \Rightarrow EB_* \otimes_t EF_*$$

which describes the homology of the total space of the fibration by means of a twisted tensor product of the chain complexes EB_* and EF_*.

This was already done by Shih [22] and the present work about effective homology is nothing but the following remark: if functional programming is used, then Shih's presentation of the Serre spectral sequence becomes an *algorithm* computing a version *with effective homology* of the total space of a fibration if analogous versions of the fibre and base spaces are given, at least if the base space is simply connected. It is a little more complicated but not very difficult to process in the same way the Eilenberg–Moore spectral sequences to compute a version with effective homology of the base space or the fibre space if such versions of both other components of the fibration are given.

6. Computing homotopy groups

Theorem 6.1. *Let X be a* 1*-reduced* (*one vertex, no edge*) *simplicial set with effective homology. Then the homotopy groups of X are computable.*

This is a strong generalization of Edgar Brown's theorem about the computability of homotopy groups of finite 1-reduced simplicial sets [3]. Furthermore our proof is not difficult and leads to concrete programs actually computing the first homotopy groups of a "reasonable" simplicial set; an example is given in Section 9.

Let $\pi = \pi_n X$ the first non-zero homotopy group. Hurewicz' theorem implies this group is also the first non-trivial homology group $H_n(X, \mathbb{Z}) = \pi$, a group which is computable, because X has effective homology. Then a fundamental cohomology class $\zeta \in H^n(X, \pi)$ is defined, which in turn defines a canonical fibration:

$$K(\pi, n-1) \hookrightarrow X_{n+1} \to X.$$

The group π is of finite type so that starting from $K(\pi, 1)$ and using $(n-2)$ times the version with effective homology of the Eilenberg–Moore spectral sequence gives a copy with effective homology of $K(\pi, n-1)$. Then applying our version of the Serre spectral sequence produces the total space X_{n+1} of our fibration with its effective homology. This total space is the same space as X except that the nth homotopy group is null: $\pi_n X_{n+1} = 0$. Applying again Hurewicz' theorem to X_{n+1} gives $\pi_{n+1} X = \pi_{n+1} X_{n+1} = H_{n+1}(X_{n+1}, \mathbb{Z})$. Iterating the process gives the result.

This sequential process to compute the homotopy groups is known as the *Whitehead tower.* The dual process *(Postnikov tower)* may be used as well, computing also the *Postnikov invariants.*

7. The Kenzo program

The *Kenzo* program implements the main components of the organization that is roughly described in these notes. It is a 16000 lines Lisp program, www-reachable at the address [11], with a rich documentation (340 pp.). It can be used with any Common Lisp system satisfying the ANSI norm.[4] A small typical demonstration is www-visible [11].

It seems difficult to realize the same work with another programming language. At least for four reasons:

- The heart of our programming work is mainly devoted to complex functional programming; this feature forbids to use the so called imperative languages such as C++ or Java with which functional programming is theoretically possible,[5] but practically it is not.
- The structures of Algebraic Topology that are processed by the Kenzo program are rich and complex: chain complexes, differential graded algebras, differential coalgebras, differential Hopf algebras, simplicial sets, Kan simplicial sets, simplicial groups, various morphisms between these objects, reductions, equivalences between chain complexes. In the current context, the modern methods of Object Oriented Programming (OOP) *must be used*. In particular the multi-inheritance feature available in Common Lisp is invaluable: for example a simplicial group is simultaneously a simplicial set and a differential graded algebra, and these classes are both subclasses of the class of chain complexes. In functional programming languages such as ML or Maple-V,[6] the OOP tools that are provided are too weak (or lacking) to work comfortably. On the contrary, from this point of view, Axiom would be satisfactory, but ....
- The time complexity of the algorithms implemented in the Kenzo program is high; more simply, computing time is critical. Common Lisp is a stratified language where the lowest level can be understood as the assembly language of a virtual machine (functions `car`, `cdr`, `cons`, ...) and the Lisp compiler produces very efficient code for the low level functions. So that using this assembly-like language when programming the kernel of a program is an excellent optimization tool. Furthermore the powerful Lisp macrogenerator allows the user to define his own intermediate language. Other good languages such as Axiom, ML, Maple have a too thick interface between the machine and the user to be satisfactory from this point of view.

[4] Mainly Allegro Common Lisp (cf. www.franz.com), LispWorks (www.harlequin.com) and Mac Common Lisp (www.digitool.com).

[5] All languages are "equivalent".

[6] Functional programming is available in Maple-V release 5.

- Lisp is one of the oldest languages still available and his enormous and well organized package of predefined functions, for example to process lists, trees, binary numbers, gives the user powerful tools again not available in the other current high level languages, in particular when dynamically created functions are implied.

No particular difficulty has been met during the programming work. In particular, the rigorous *mathematical* definition of the virtual Common Lisp machine [8,25] gives the programmer a safe and convenient framework.

8. New research fields

Various *new* research fields are open by this work, in computer science and in "pure" mathematics as well. Let us quickly describe two typical examples.

8.1. A new subject in computer science

A Kenzo computation of some homology group, for example a homology group of an iterated loop space $H_p\Omega^n X$ is split in two steps:

1. Constructing a version *with effective homology* of the loop space $\Omega^n X$; during this step, an enormous set of functional objects, something like several hundreds or thousands, are dynamically constructed. They are organized as an oriented graph where the nodes are the functional objects and each node f is connected to several other nodes $f_1, \ldots, f_k$, if a call of f requires the call of $f_1, \ldots, f_k$, to be viewed as auxiliary functions (subroutines), which in turn have other auxiliary functions, and so on. But at this time these functions have not yet worked: the first step is in a sense *macrogeneration* of *object*[7] code;
2. When the computation of $H_p\Omega^n X$ is started, the effective chain complex corresponding to $\Omega^n X$ is examined, two (finite) boundary matrices are constructed, and the homology group is computed. The construction of this boundary matrix is the problem with "severe" difficulties mentioned by Carlsson and Milgram, see Section 1; the "program" written in the step 1 now works and most functions are used.

This situation gives rise to a difficult and interesting problem of memory optimization. When the function f is called and some result $f(x_1, \ldots, x_k)$ has been computed, what about the idea of storing the result? After all, and this is frequent, the same calculation will be again required later. If the calculation is trivial, for example if the map f is constant, or if it is fast, storing the result is expensive in time and space. If on the contrary the computation is long, it is better to store the result to avoid the repetition. But the decisions that are to be taken are not independent from each other: if the calculation of $f(x)$ is long but amounts in fact to calculating $f_1(x')$, storing the result $f_1(x')$ implies the calculation

[7] In fact, this is an illusion: thanks to the *closure* mechanism, only an enormous set of pointers is installed.

of $f(x)$ becomes very fast! Furthermore, after a long work, *experience* can show that in fact some stored result has never been reused, so that it could be thrown away? Yes, but in general the program is unable to prove the result will *certainly* not be re-used. It seems clear only empirical methods can be applied, but nevertheless modelizing and studying simplified models from this point of view should be interesting and useful.

In the Kenzo program, a small set of empirical methods are applied to decide when a result is stored or not, but it is obvious we are far from the "best" choices.

8.2. A new research field in pure mathematics

The complicated calculations which may be undertaken with the help of the Kenzo program give new insights into some fields. The following example is typical. If X is a 1-reduced (one vertex, no edge) simplicial set, the main result which was obtained by Adams [1][8] towards the calculation of the homology groups $H_*\Omega X$ was a morphism of differential graded algebras:

$$\alpha : \mathrm{Cobar}^{C_*X}(\mathbb{Z},\mathbb{Z}) \to C_*\Omega X$$

which is a chain equivalence. In interesting cases, the source of α is of finite type. The computation of $H_*\Omega X$ amounts to considering the chain complex $\mathrm{Cobar}^{C_*X}(\mathbb{Z},\mathbb{Z})$ and its finiteness properties make the homology groups computable. The Kenzo program computes such a map α and also an *explicit* inverse chain equivalence:

$$\beta : C_*\Omega X \to \mathrm{Cobar}^{C_*X}(\mathbb{Z},\mathbb{Z}).$$

Once upon a time, a student implicitly used that β is also a morphism of differential graded algebra. To persuade him he was wrong, the second author used the Kenzo program to give him simple examples showing such a statement is not sensible, but he was rather surprised: the map β automatically constructed by the Kenzo program is, at least for the numerous examples that have been tried, a morphism of algebra! In fact so many cases have been computed that this is now an *experimental* "definitive" *fact.* This is an amazing strong version of Adams' result: there exists a two-sided ideal I in the algebra $C_*\Omega X$ such that Adams' Cobar construction $\mathrm{Cobar}^{C_*X}(\mathbb{Z},\mathbb{Z})$ is nothing but the quotient $C^*\Omega X/I$.

This became the main research subject of this student. Several interesting results in this direction have been obtained, but at this time, the complete result has not yet been proved. In particular it was completely obtained if a new differential is installed on $\mathrm{Cobar}^{C_*X}(\mathbb{Z},\mathbb{Z})$, but it is not clear what the status of this new differential is. See [9,10].

Other amazing experimental results of this sort have been obtained, in particular around the canonical *algebraic* fibration:

$$C_*\Omega X \to X \otimes_t C_*\Omega X \to X.$$

This is the *algebraic* version of the co-universal fibration:

$$\Omega X \hookrightarrow PX \to X,$$

[8] See also [6] for an excellent recent extensive study of the subject.

where the fibre space (resp. total space) is the loop space (resp. the path space) of the pointed space X. The path space is contractible: it is a "unit" space and in a sense, ΩX is an inverse space of X. In the same way, the twisted tensor product $X \otimes_t C_*\Omega X$ is acyclic and an explicit contraction h of this chain complex plays a capital role in effective homology. The *existence* of this contraction is known for a long time [4], but the explicit Kenzo computation of h shows very surprising properties, which imply we are far from mastering the underlying algebraic structure. Let us recall the loop space construction is the heart of Algebraic Topology and that many problems can be reduced to problems about loop spaces; they were *invented* by Jean-Pierre Serre fifty years ago for this reason.

9. Examples of calculations

9.1. $H_5\Omega^3\,\mathrm{Moore}(\mathbb{Z}_2, 4)$

Carlsson and Milgram explain in the paper quoted in Section 1 the computation of $H_*\Omega^n X$ may be undertaken if X is a suspension $X = S^n Y$; then the homology groups $H_*\Omega^n X$ are entirely determined by the homology groups H_*Y thanks to a process where the Dyer–Lashof homology operations play the main role, see [6,7]. For example the Moore space $\mathrm{Moore}(\mathbb{Z}_2, 4)$ is nothing but the third suspension $S^3P^2\mathbb{R}$, so that the homology groups $H_*\Omega^3\,\mathrm{Moore}(\mathbb{Z}_2, 4)$ are entirely determined by the well known groups $H_*P^2\mathbb{R} = (\mathbb{Z}, \mathbb{Z}_2, 0, 0, \ldots)$. The best specialists have been questioned and so far they have not yet been able to compute for example $H_5\Omega^3\,\mathrm{Moore}(\mathbb{Z}_2, 4)$.[9] With the Kenzo program the Moore space $\mathrm{Moore}(\mathbb{Z}_2, 4) = S^3P^2\mathbb{R}$ is constructed as follows:

```
USER(3): (setf moore-2-4 (moore 2 4))
[Kl Simplicial-Set]
```

The (sub-) statement (`moore 2 4`) constructs the Moore space and the statement (`setf ...`) assigns the result to the symbol `moore-2-4`. Lisp explains the result is the Kenzo object #1 (`[Kl ...]`) and this object is a simplicial set. Then the third loop space is constructed and the result is assigned to the symbol `o3-moore-2-4`:

```
USER(4): (setf o3-moore-2-4 (loop-space moore-2-4 3))
[K30 Simplicial-Group]
```

This time, the result is a simplicial *group*. And the group $H_5\Omega^3 X = \mathbb{Z}_2^5$ is obtained in one minute:

```
USER(5): (homology o3-moore-2-4 5)
Computing boundary-matrix in dimension 5.
Rank of the source-module : 23.
```

[9] In a case, two different (!) results were successively proposed but both were wrong ...

```
;; Clock -> 1999-08-10, 14h 19m 56s.
[... ... Lines deleted ... ...]
Computing boundary-matrix in dimension 6.
Rank of the source-module : 53.
[... ... Lines deleted ... ...]

Homology in dimension 5 :
Component Z/2Z
Component Z/2Z
Component Z/2Z
Component Z/2Z
Component Z/2Z
---done---
;; Clock -> 1999-08-10, 14h 20m 50s.
```

The Kenzo program has constructed a chain equivalence between the highly infinite chain complex $C_*\Omega^3 X$ and an effective one EC_* which for example has 53 generators in dimension 5. The boundary matrices can be computed and the corresponding homology group is obtained.

9.2. *A CW-model for* $\Omega^3(P^\infty\mathbb{R}/P^3\mathbb{R})$

Let us now consider an example where the Kenzo program overcomes the "severe difficulties" quoted by Carlsson and Milgram, see again Section 1. In a sense, the first case where their proposed methods fail is the following: what about a CW-model for $\Omega^3 X$ where X is the quotient $X = P^\infty\mathbb{R}/P^3\mathbb{R}$? Let us construct such a model with the Kenzo program; the space X is constructed as follows:

```
USER(6): (setf p4 (r-proj-space 4))
[K405 Simplicial-Set]
```

The statement (`r-proj-space 4`) constructs the infinite real projective space "beginning" only in dimension 4, that is the required quotient $X = P^\infty\mathbb{R}/P^3\mathbb{R}$. The third loop space is constructed as before:

```
USER(7): (setf o3p4 (loop-space p4 3))
[K434 Simplicial-Group]
```

The Kenzo object `o3p4` is a simplicial group with effective homology and the *effective* associated chain-complex can be extracted:

```
USER(8): (setf eff-chain-complex-of-o3p4 (echcm o3p4))
[K794 Chain-Complex]
```

You see $794-434-1=359$ other Kenzo objects (chain complexes with various added structures and chain complex morphisms) have also been constructed to obtain the result. The boundary matrix in dimension 5 of this effective chain complex is computed by the Kenzo program in 30 seconds:

```
USER(9): (chcm-mat eff-chain-complex-of-o3p4 5)
Computing boundary-matrix in dimension 5.
Rank of the source-module : 33.
;; Clock -> 1999-08-10, 14h 22m 30s.
[... ... Lines deleted ... ...]
;; Clock -> 1999-08-10, 14h 22m 57s.

========== MATRIX 13 lines + 33 columns =====
L1=[C1=-2]
L2=[C1=-1]
L3=[C1=-4] [C2=1] [C3=-1] [C4=-2]
L4=[C2=1] [C3=-l] [C6=2]
L5=[C1=6][C4=1][C6=1]
L6=[C1=4] [C4=4] [C6=4] [C7=3]
L7=[C1=4] [C12=-2] [C14=2]
L8=[C1=6][C4=1][C6=1]
L9=[C1=4] [C4=4] [C6=4] [C7=3]
L10=[C8=4] [C10=l] [C11=-1] [C14=-4] [C15=-2] [C20=-2]
L11=[C1=4] [C8=4] [C10=1] [C11=-1] [C16=-4] [C18=-1]
    [C19=1] [C23=-2]
L12=[C12=4] [C13=2] [C16=-4] [C18=-1] [C19=1] [C27=-2]
L13=[C1=-1] [C20=4] [C21=2] [C23=-4] [C24=-2] [C27=4]
    [C28=2]
========== END-MATRIX
```

You must read the result as follows: the non-null $a_{i,j}$ terms of the matrix are $a_{1,1}=-2$, $a_{2,1}=-1,\ldots,a_{13,28}=2$. This is a computer-aided proof that there exists a CW-model for Ω^3X with in particular 13 4-cells and 33 5-cells. This is an easy consequence of Adams' Cobar construction, but the severe difficulties about the differentials are here solved. In particular the boundary of the first 56 cell e_1^5 is $de_1^5=-2e_1^4-e_2^4-4e_3^4+6e_5^4+4e_6^4+4e_7^4+6e_8^4+4e_9^4+4e_{11}^4-e_{13}^4$. This defines only the homology type of the attaching map for e_1^5, but the rest of the Kenzo object contains also its *homotopy* type.

9.3. $\pi_5(\Omega S^3\cup_2 e^3)$

The Kenzo program may compute the first homotopy groups of an *arbitrary* simply connected simplicial set with effective homology. Our last example of Kenzo computation shows the calculation of $\pi_5(\Omega S^3\cup_2 e^3)$: a 3-cell e^3 is attached to the loop space ΩS^3 by a map $\delta e^3=S^2\to\Omega S^3$ of degree 2. The space $X=\Omega S^3\cup_2 e^3$, called `dos3` below, can

be constructed by a process which is not necessary to detail here and which finishes as follows:

```
USER(13): (setf dos3 (disk-pasting os3 3 'new faces))
[K826 Simplicial-Set]
```

In principle the group H_2X should be Z_2:

```
USER(14): (homology dos3 2)
Computing boundary-matrix in dimension 2.
[... ... Lines deleted ... ...]
Homology in dimension 2 :
Component Z/2Z
---done---
```

and the notion of a canonical cohomology class in dimension 2 is defined; the Kenzo program can construct it:

```
USER(15): (setf ch2 (chml-clss dos3 2))
[K947 Cohomology-Class (degree 2)]
```

The canonical fibration $K(\mathbb{Z}_2, 1) \hookrightarrow X_3 \to X$ induced by this cohomology class is then constructed, and the total space of the fibration is extracted:

```
USER(16): (setf f2 (z2-whitehead dos3 ch2))
[K962 Fibration]
USER(17): (setf X3 (fibration-total f2))
[K968 Simplicial-Set]
```

This is the beginning of the classical Whitehead tower, see Section 6. In particular the group $H_3X_3 = \pi_3X_3 = \pi_3X$ can be computed; in fact the Kenzo program has applied the version with effective homology of the Serre spectral sequence:

```
USER(18): (homology X3 3)
Computing boundary-matrix in dimension 3
[... ... Lines deleted ... ...]
Homology in dimension 3 :
Component Z/2Z
---done---
```

so that $\pi_3X = \mathbb{Z}_2$. Continuing in the same way for the following stages of the Whitehead tower, the groups $\pi_4X = \mathbb{Z} + \mathbb{Z}_4$, $\pi_5X = \mathbb{Z}_2^4$ are obtained in less than one hour.

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