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The matching polynomial of a regular graph

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Abstract

The matching polynomial of a graph has coefficients that give the number of matchings in the graph. For a regular graph, we show it is possible to recover the order, degree, girth and number of minimal cycles from the matching polynomial. If a graph is characterized by its matching polynomial, then it is called matching unique. Here we establish the matching uniqueness of many specific regular graphs; each of these graphs is either a cage, or a graph whose components are isomorphic to Moore graphs. Our main tool in establishing the matching uniqueness of these graphs is the ability to count certain subgraphs of a regular graph.

1. Introduction

The matching polynomial is an example of a general graph polynomial, as introduced in [3]. Here we prove that many interesting regular graphs are characterized by their matching polynomial. These graphs are either cages (the smallest regular graphs for a given girth and degree) or have components that are cages. Since the matching polynomial is a subpolynomial of many other important graph polynomials (such as the circuit polynomial), these graphs are also characterized by these other polynomials.

We begin by describing a general graph polynomial. The first element in the construction of a graph polynomial is a family of graphs, F, such as all trees, or all circuits. Typically such a family is infinite, and often includes a single vertex and a single edge as members. To each member of this family a weight is assigned. Often this weight is an indeterminate, which is subscripted by either the number of vertices or the number of edges in the graph. Having chosen a family F, and a weighting scheme, we compute the F-polynomial of a graph G by first finding the spanning

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subgraphs of G where each component is an element of F. Such a spanning subgraph is called an F-cover. For each cover, take the product of the weights of the components, and then sum these terms over all F-covers of the graph. The resulting polynomial is the F-polynomial of G. Throughout this paper, we only consider F-polynomials constructed by assigning the indeterminate w_i to a component with i vertices. For more on the general properties of F-polynomials see [3].

Presently, we are interested in the matching polynomial of a graph. We take F to be the family consisting of just a vertex and an edge. In this case, a cover will consist of disjoint edges and isolated vertices — a matching in the graph. The resulting matching polynomial has terms of the form $cw_1^{n-2m}w_2^m$, where n is the number of vertices, and c is the number of matchings in G that have m edges. Thus, finding the matching polynomial of a graph is equivalent to finding the number of m-matchings in the graph, for all m.

Two graphs are co-matching if their matching polynomials are equal. A graph that is characterized by its matching polynomial is said to be matching unique. It is our intention to show that many cages are matching unique.

Many of the families used to construct interesting F-polynomials include a vertex and an edge, in which case all the matchings of the graph are created as F-covers. The terms of the form $cw_1^{n-2m}w_2^m$ in the F-polynomial then coincide with the matching polynomial itself; in other words, the matching polynomial is a subpolynomial of the F-polynomial. The proof of the following theorem should be apparent.

Theorem 1.1. *Suppose that F is a set of graphs that contains a single vertex and a single edge as members. If G is a graph that is matching unique, then G is also F-unique.*

Since we will show that certain graphs are matching unique, this theorem allows us to conclude that these graphs are characterized by F-polynomials for many popular choices of F. The existence of a non-trivial polynomial that characterizes all graphs is still an open question.

2. Counting subgraphs of regular graphs

To find the matching polynomial of a regular graph, we count the number of matchings that contain m edges, for all m. To do this we will use results from [1] that give expressions for the number of many of the subgraphs of a regular subgraph. We will state the required results here, more detail and proofs can be found in [1].

We adopt the following notation: For a fixed graph G, $\Gamma_{m,k,i}$ is an isomorphism class of spanning subgraphs of G, with m edges from G, and k vertices of degree 1. $G_{m,k,i}$ will denote a representative chosen from $\Gamma_{m,k,i}$. Finally, $g_{m,k,i}$ is the cardinality of the set $\Gamma_{m,k,i}$.

Theorem 2.1. *Suppose that G is a regular graph of degree r on n vertices. To each graph* G_{m,k,i^*}, *with* $k \geqslant 1$, *there corresponds a graph* $G_{m-1,f,l}$ *and an equation of the form*

$$a(n,r)g_{m-1,f,l} = \sum_{j=k-2}^{k-1} \sum_i a_{j,i}\, g_{m,j,i} + a^* g_{m,k,i^*},$$

where $f = k, k-1$ *or* $k-2$, $a(n,r)$ *is an expression that depends on G only through n and r, the* $a_{j,i}$ *are integer constants that are independent of G, and* a^* *is a non-zero integer constant that is independent of G.*

For example, the proof of this theorem allows us to construct the equation

$$2(r-1)g_{3,2,1} = 8g_{4,0,1} + 2g_{4,1,1} + 2g_{4,2,2},$$

where the representative subgraphs are: $G_{3,2,1}$ is a path on 4 vertices, $G_{4,0,1}$ is a circuit on 4 vertices, $G_{4,1,1}$ is a triangle with an additional edge joining one vertex of the triangle to a vertex of degree one, and $G_{4,2,2}$ is a path on 5 vertices (isolated vertices have been ignored in these descriptions). Note that this relation is true for any regular graph of degree r.

By forming a complete set of similar equations, and solving the resulting system of linear equations, we obtain the following result.

Theorem 2.2. *Suppose that G is a regular graph of degree r on n vertices. Then for each combination of m, k and* i^* *there exist constants* $a_{j,i} = a_{j,i}(n,r)$, *that depend on G only through n and r, and constants* a_i *that are independent of G, so that*

$$g_{m,k,i^*} = \sum_{j=0}^{m-1} \sum_i a_{j,i}(n,r)\, g_{j,0,i} + \sum_i a_i g_{m,0,i}.$$

For example, applying the proof of this result, we have the following equation for $g_{4,8,1}$. Note that $G_{3,0,1}$ and $G_{4,0,1}$ are circuits on 3 and 4 vertices, respectively, (when the isolated vertices are ignored) and that $g_{0,0,1} = 1$.

$$g_{4,8,1} = \frac{nr}{384}(240 - 960r + 76nr + 1344r^2 - 240nr^2 + 12n^2r^2 - 672r^3$$
$$+ 208nr^3 - 24n^2r^3 + n^3r^3)g_{0,0,1} + \left(-6 + 6r - \frac{nr}{2}\right)g_{3,0,1} + g_{4,0,1}.$$

Since $G_{4,8,1}$ is a subgraph with 4 edges and 8 vertices of degree one, it is a matching, and thus $g_{4,8,1}$ is the coefficient of $w_2^4 w_1^{n-8}$ in the matching polynomial. Again, notice that this relation holds for any regular graph.

We will need to know explicitly one portion of the solutions described in Theorem 2.2. In the expression for the number of m-matchings, one term involves the number of circuits on m edges. We will determine the coefficient for this term, which will be independent of the graph. Towards this end, we state the following lemma. It is an easy exercise when one counts matchings by counting ‘unmatched’ vertices in the matchings, or it can be derived from results in [4].

Lemma 2.1. *Let C_m be a circuit with m vertices.*

(1) *If m is even, then there are 2 matchings in C_m with $m/2$ edges.*
(2) *If m is even, then there are $m^2/4$ matchings in C_m with $(m-2)/2$ edges.*
(3) *If m is odd, then there are m matchings in C_m with $(m-1)/2$ edges.*

Lemma 2.2. *Suppose that G is a regular graph of degree r on n vertices. Let $G_{m,0,1}$ be an isomorphism class representative that consists of a circuit on m vertices together with $n-m$ isolated vertices. Then, in the expression described in Theorem 2.2 for $g_{m,2m,1}$ the coefficient of $g_{m,0,1}$ is $(-1)^m$.*

Proof. The constants described in Theorem 2.2 depend on G at most through the values of n and r, so we will consider two graphs that are regular of the same degree and that have the same number of vertices. Each of these graphs will be composed of two components that are circuits, specifically $G'=C_m \cup C_{m+2}$ and $G''=C_{m+1} \cup C_{m+1}$. These two graphs will have slightly different numbers of spanning subgraphs, but the equations from Theorem 2.2 describing these numbers will have identical coefficients. Remember that the coefficient of $g_{m,0,1}$ is independent of the graph G, so it suffices to determine it by looking at specific instances of G.

Both G' and G'' are regular of degree 2 on $2m+2$ vertices. We will now find the values of $g_{l,0,i}$ for these two graphs. As is the case for any graph, $g'_{0,0,1}=g''_{0,0,1}=1$. Any subgraph that has no vertices of degree one, and one or more edges, will not be a forest and thus must contain a circuit. There are no circuits with $m-1$ or fewer edges in either G' or G'' so $g'_{l,0,i}=g''_{l,0,i}=0$ for all $0<l<m$. Examining subgraphs that have no vertices of degree one and that have m edges, we find that $g'_{m,0,1}=1$, $g''_{m,0,1}=0$ and $g'_{m,0,i}=g''_{m,0,i}=0$ for all $i>1$. Since G' and G'' have the same number of vertices and the same degree, the coefficients from Theorem 2.2 will be the same for each graph. Thus, the expressions for $g'_{m,2m,1}$ and $g''_{m,2m,1}$ will only differ in the term corresponding to the contribution from the number of circuits of length m. So a_1, the coefficient of $g_{m,0,1}$, does not depend on the graphs G' and G'' in any way, and we find that,

$$g'_{m,2m,1}-g''_{m,2m,1}=a_1 g'_{m,0,1}-a_1 g''_{m,0,1}=a_1 .$$

The subgraph $G_{m,2m,1}$ is an m-matching in G, so we want to compute the number of m-matchings in G' and G''. There are two cases, first we consider when m is even. An m-matching in G' can be formed by combining $(m/2)$-matchings from C_m and C_{m+2}, or by combining an $((m-2)/2)$-matching from C_m with an $((m+2)/2)$-matching from C_{m+2}. Applying Lemma 2.1 this gives

$$g'_{m,2m,1}=2((m+2)^2/4)+2(m^2/4)=m^2+2m+2 .$$

An m-matching in G'' can only be created by combining $(m/2)$-matchings from each C_{m+1}. This gives

$$g''_{m,2m,1}=(m+1)^2=m^2+2m+1 .$$

Thus,

$$a_1 = g'_{m,2m,1} - g''_{m,2m,1} = 1\ .$$

A similar analysis gives $a_1 = -1$ for the case when m is odd. Thus we have $a_1 = (-1)^m$ for all regular graphs G. $\square$

3. Girth and the matching polynomial

We will show that it is possible to determine the girth (the size of the smallest circuit), and the number of such minimal circuits from the matching polynomial of a regular graph.

Theorem 3.1. *Suppose G is a regular graph of non-zero degree and $g_{m,2m,1}$, the number of m-matchings, is known for each m. Then we can determine the order, degree, girth and number of minimal circuits for G.*

Proof. Suppose G has n vertices and degree r. Then two equations in the form described in Theorem 2.2 are

$$g_{1,2,1} = \frac{nr}{2},$$

$$g_{2,4,1} = \frac{nr(2-4r+nr)}{8}.$$

If $g_{1,2,1} = 0$, then $n = 0$, and the graph G is completely determined. Otherwise, we can solve these equations to find,

$$r = \frac{1}{2g_{1,2,1}}(g_{1,2,1} + g_{1,2,1}^2 - 2g_{2,4,1}),$$

$$n = \frac{2g_{1,2,1}}{r},$$

so it is possible to determine the order and degree of the graph.

Suppose now that G has girth l. A non-trivial subgraph with no vertices of degree one cannot be a tree, and therefore has a circuit. However, if a subgraph has fewer edges than the girth, then it cannot possess a circuit. Thus, $g_{m,0,i} = 0$ when $0 < m < l$. Also, $g_{l,0,1}$ will equal the number of minimal cycles in the graph, while there will not be any other subgraph on l edges with no vertices of degree one, so $g_{l,0,i} = 0$ for $i > 1$.

Let $a_{0,1}^{(m)}$ be the coefficient of $g_{0,0,1}$ in the expression for $g_{m,2m,1}$ described in Theorem 2.2. This coefficient depends on G only through n and r, both of which we have determined, so we can determine this coefficient also. We will examine the difference between $a_{0,1}^{(m)}$ and $g_{m,2m,1}$ — first when $m<l$. (Recall that $g_{0,0,1}=1$ for any graph.)

$$g_{m,2m,1}-a_{0,1}^{(m)}=\sum_{j=0}^{m-1}\sum_{i}a_{j,i}(n,r)g_{j,0,i}+\sum_{i}a_i g_{m,0,i}-a_{0,1}g_{0,0,1}=0\,.$$

In the case that $m=l$, applying Theorem 2.2 yields

$$g_{m,2m,1}-a_{0,1}^{(m)}=a_1 g_{l,0,1}=(-1)^l g_{l,0,1}\neq 0\,.$$

Thus, the smallest value of m such that $g_{m,2m,1}-a_{0,1}^{(m)}\neq 0$ is the girth of G and $|g_{l,2l,1}-a_{0,1}^{(l)}|=g_{l,0,1}$ equals the number of minimal circuits. Notice that in the case that $r=1$, G has no circuits, and this difference will be zero for all m. □

Corollary 3.1. *If G is a regular graph, then its order, degree, girth and number of minimal cycles can be determined from the matching polynomial of the graph.*

Proof. The previous theorem can be used if the degree of G is non-zero, since the matching polynomial will have coefficients equal to the $g_{m,2m,1}$. If G has zero degree, then its matching polynomial is simply w_1^n, and G can be determined uniquely as an empty graph on n vertices. □

4. Comatching regular graphs

In this section, we will prove our main result. First we include a theorem due to Farrell and Guo [5]. A proof is included for completeness' sake.

Theorem 4.1 (Farrell and Guo [5]). *Suppose that H is a regular graph of degree r on n vertices, and that G is comatching with H. Then G is also regular of degree r on n vertices.*

Proof. The matching polynomial of any graph on n vertices has a term of the form w_1^n, corresponding to the one matching with no edges. Since G and H have identical matching polynomials, the equality of this term implies they have the same number of vertices.

For any graph, the next term of the matching polynomial is $qw_1^{n-2}w_2$, where q is the number of edges in the graph. This corresponds to the matchings that have just one edge. The equality of G and H's matching polynomials then implies that they have the same number of edges.

Any graph on q edges has exactly $\binom{q}{2}$ edge-induced subgraphs on 2 edges. These subgraphs come in two flavors: matchings with two edges, and paths on three vertices. If we let d_i denote the degree of vertex i, then the number of paths on 3 vertices, which have i as the central vertex of degree 2, is $\binom{d_i}{2}$. In total, there will be $\sum_{i=1}^{n}\binom{d_i}{2}$ subgraphs on 2 edges that are not matchings. Thus, the coefficient of $w_1^{n-4}w_2^2$ will be

$$\binom{q}{2}-\sum_{i=1}^{n}\binom{d_i}{2}.$$

Furthermore, we know that the number of edges in the graph is given by

$$\frac{1}{2}\sum_{i=1}^{n}d_i.$$

Since G and H are comatching, these last two expressions must be equal for G and H. We will let d_i represent the degrees in G, and since H is regular, we can designate this common degree by r. Equating the last two expressions for G and H, we have

$$\frac{1}{2}\sum_{i=1}^{n}d_i=\frac{1}{2}\sum_{i=1}^{n}r, \tag{1}$$

$$\binom{q}{2}-\sum_{i=1}^{n}\binom{d_i}{2}=\binom{q}{2}-\sum_{i=1}^{n}\binom{r}{2}. \tag{2}$$

Starting with Eq. (2), and applying Eq. (1) in the last step, we have

$$\begin{aligned}
0&=\sum_{i=1}^{n}\binom{d_i}{2}-\binom{r}{2}\\
&=\frac{1}{2}\sum_{i=1}^{n}d_i^2-d_i-r^2+r.\\
0&=\sum_{i=1}^{n}(d_i^2-2rd_i+r^2)-d_i+r+2rd_i-2r^2\\
&=\sum_{i=1}^{n}(d_i-r)^2+(2r-1)(d_i-r)\\
&=\sum_{i=1}^{n}(d_i-r)^2+(2r-1)\sum_{i=1}^{n}(d_i-r)\\
&=\sum_{i=1}^{n}(d_i-r)^2.
\end{aligned}$$

This implies that $d_i=r$ for $1\leqslant i\leqslant n$. In other words, G must be regular of degree r. □

With this result in hand, we can now present our main result.

Theorem 4.2. *Suppose that G is a graph that is comatching with a regular graph H. Then G is a regular graph, and G and H have the same number of vertices, the same degree, the same girth, and the same number of cycles of minimal length.*

Proof. That G is regular of the same degree on the same number of vertices follows from Theorem 4.1. Now that we know that G is a regular graph, we can apply Corollary 3.1. G and H have identical matching polynomials, and therefore we can determine that they have equal girth and an equal number of minimal circuits. □

5. Cages

Having specified that a graph be regular of degree r, and that it has girth g, what is the minimum number of vertices the graph must contain? This problem has received much attention, yet many questions remain. In the cases where the minimum order has been found and the graphs meeting this minimum have been determined, the resulting graphs often have strong regularity properties, such as being distance-regular, or at the very least are visually appealing. Furthermore, a graph achieving the minimum order is often unique. In these cases, we can easily show that the graph is also matching unique.

We start with some basic results on regular graphs of given girth, see [2, Chap. 23] or [8] for additional information. The proof of the first result should be apparent by counting the vertices of a balanced $(r-1)$-ary tree (with a root of degree r) having the depth necessary to achieve the desired girth. This gives a straightforward lower bound on the size of a minimal regular graph of given girth.

Lemma 5.1 (2, Proposition 23.1). *Suppose that G is a regular graph of girth g. Then G must have at least $n(r, g)$ vertices, where*

(1) *If g is odd*:

$$n(r,g)=1+r+r(r-1)+r(r-1)^2+\cdots+r(r-1)^{(g-3)/2}$$
$$=(r(r-1)^{(g-1)/2}-2)/(r-2) \quad \text{when } r>2\,.$$

(2) *If g is even*:

$$n(r,g)=1+r+r(r-1)+r(r-1)^2+\cdots+r(r-1)^{(g-4)/2}+(r-1)^{(g-2)/2}$$
$$=(2(r-1)^{g/2}-2)/(r-2) \quad \text{when } r>2\,.$$

Definition 5.1. An (r,g)-*cage* is the smallest regular graph of degree r with girth g. If the number of vertices in an (r,g)-cage equals the lower bound given in Lemma 5.1, then it is a *Moore graph.*

The next theorem restricts the possibilities for the existence of Moore graphs. It is the combined work of several authors, and a complete description can be found in [2].

Theorem 5.1. *Suppose that G is a Moore graph of girth $g\geqslant 3$ and degree $r\geqslant 3$. Then either*

(1) $g=5$, *and* $r=3, 7,$ *or* 57, *or*

(2) $g=3, 4, 6, 8,$ *or* 12.

For many of the cases allowed by this theorem, examples of Moore graphs have been found, and many of these are unique. However, the existence of a Moore graph with $g=5$ and $r=57$ is still unknown. The list of known unique Moore graphs includes Petersen's graph and the Heawood graph, in addition to circuits, complete graphs and complete bipartite graphs. (Wong's survey article [8] contains a complete description of other known cages.) With the following theorem, and the extensive results on Moore graphs and cages, we are able to determine a great number of matching unique regular graphs.

Theorem 5.2. *Suppose that G is a unique* (r, g)*-cage. Then G is matching unique.*

Proof. Suppose that H is comatching with G. Then by Theorem 4.2, H is regular and has the same degree, girth and order as G. Thus, H is an (r, g)-cage. Because G is unique, H must be isomorphic to G, and hence G is characterized by its matching polynomial. □

6. More matching unique graphs

In this section we determine several more matching unique graphs, each of which has components isomorphic to Moore graphs. We need the following lemma in the proof of the next theorem.

Lemma 6.1 (Farrell [3]). *Suppose that G is a graph with components* G_1 *and* G_2*. If* $p(w)$ *is the F-polynomial of G, and* $p_1(w)$ *and* $p_2(w)$ *are the F-polynomials of the components, then*

$$p(w)=p_1(w)\,p_2(w)\,.$$

Theorem 6.1. *Suppose that G is a regular graph of degree 2. Then G is matching unique.*

Proof. The components of a regular graph of degree 2 are all circuits. Suppose that G has several components and let s denote the order of the smallest component of G. Then the girth of G is s. Let H be a graph that is comatching with G. Since G and H are comatching, H must also be regular of degree 2 with girth s. It follows that one of the components of H is a circuit with s vertices.

Remove these identical components from both G and H. By Lemma 6.1 the matching polynomials of G and H can be factored into the matching polynomial of a circuit on s vertices and the matching polynomials of the remaining graphs. By repeating this procedure, and using the fact that G and H have equal matching polynomials, we can conclude that G and H are isomorphic. □

We will let mG denote a graph that has m components, each isomorphic to G. The proofs of the next two theorems are similar in spirit.

Theorem 6.2. *Suppose $G = mK_{r,r}$, where $K_{r,r}$ is the complete bipartite graph of degree r. Then G is matching unique.*

Proof. The case when $r = 1$ is an easy exercise, since the graph is itself a matching, and the case when $r = 2$ is covered by the previous theorem. Thus, we will assume that $r \geqslant 3$.

Let c_v be the number of circuits in G of minimal length that contain the vertex v. (The minimal length is 4 in this case.) By looking at the r edges emanating from each of the $r-1$ vertices a distance 2 from v, we find that $c_v = (r-1)\binom{r}{2}$ for all v in G.

Let H be a graph that is comatching with G. Then by Theorem 4.2, H has $2mr$ vertices, is regular of degree r, has girth 4 and contains the same number of circuits of length 4 as G does. Since G and H contain the same number of vertices, and the same number of circuits of length 4, the average value of c_v will be the same in both graphs. Because c_v is constant in G, this common average value is $(r-1)\binom{r}{2}$. Now, there must be at least one vertex in H, say w, such that c_w equals or exceeds this average value, so $c_w \geqslant (r-1)\binom{r}{2}$. We will now construct the component of H that contains this vertex w.

The graph H is regular of degree r, so w has r neighbors, $x_1, x_2, \ldots, x_r$. There cannot be any edges in H of the form $\{x_i, x_j\}$ since H has girth 4. Let f_{ij} denote the number of circuits of length 4 that include the edges $\{w, x_i\}$ and $\{w, x_j\}$ for $1 \leqslant i < j \leqslant r$. Then, since there are $(r-1)\binom{r}{2}$ circuits of length 4 through w, we have

$$\sum_{1 \leqslant i < j \leqslant r} f_{ij} = (r-1)\binom{r}{2}.$$

Thus the average value of f_{ij} is $r-1$.

However, we claim that $f_{ij} \leqslant r-1$ for all i and j. Let C be a circuit of length 4 that includes the edges $\{w, x_i\}$ and $\{w, x_j\}$. Then C uniquely determines a fourth vertex y, such that $\{y, x_i\}$ and $\{y, x_j\}$ are edges of H. So for every circuit counted in f_{ij}, we get a new vertex that is a neighbor of x_i. Since x_i is already known to be adjacent to w we can find at most $r-1$ new neighbors for x_i, and thus $f_{ij} \leqslant r-1$.

Because the average value of f_{ij} is equal to an upper bound for f_{ij}, we find that $f_{ij} = r-1$ for all i and j. Since $f_{12} = r-1$, there are $r-1$ vertices, $y_1, y_2, \ldots, y_{r-1}$ so that for each $1 \leqslant k \leqslant r-1$, (x_1, w, x_2, y_k) forms a circuit of length 4. Also since $f_{1j} = r-1$ for each $3 \leqslant j \leqslant r$, there must be $r-1$ vertices $z_{j,1}, z_{j,2}, \ldots, z_{j,r-1}$ so that for each $1 \leqslant k \leqslant r-1$, $(x_1, w, x_j, z_{j,k})$ is a circuit of length 4. We now have edges $\{w, x_1\}$, $\{x_1, y_k\}$ for $1 \leqslant k \leqslant r-1$, and $\{x_1, z_{j,k}\}$ for $1 \leqslant k \leqslant r-1$, while x_1 has degree r. Thus $\{y_1, y_2, \ldots, y_{r-1}\} = \{z_{j,1}, z_{j,2}, \ldots, z_{j,r-1}\}$. Assuming that the elements of these two sets are ordered identically, we see that $\{x_j, z_{j,k}\} = \{x_j, y_k\}$ is an edge of H for all $1 \leqslant j \leqslant r$ and $1 \leqslant k \leqslant r-1$.

We have shown that $\{w, x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_{r-1}\}$ is the vertex set of a component of H that is isomorphic to $K_{r,r}$. The remainder of H will then have $2(m-1)r$ vertices, will be regular of degree r, will possess the same number of circuits of length 4 as $(m-1)K_{r,r}$, and thus will have girth 4. By repeating the process above we will be able to find that the components of H are isomorphic to $K_{r,r}$, and so H is isomorphic to $mK_{r,r}$. Thus, $mK_{r,r}$ is matching unique. $\square$

We now turn our attention to a more general result involving Moore graphs of odd girth.

Theorem 6.3. *Suppose that $G=mL$, where L is the unique Moore graph for a given valency and odd girth. Then G is matching unique.*

Proof. Suppose that L has n vertices, girth $g=2s+1$ and is regular of degree r. For a vertex v, let c_v be the number of circuits of length g that pass through v. Let v be any vertex of L. Then we can find a subgraph of L that is a tree rooted at v, with all its internal vertices of degree r, and $r(r-1)^{s-1}$ vertices of degree one at distance s from v — these degree one vertices will be referred to as the leaves of the tree. Such a tree exists because L has no circuits of length less than g, and furthermore, since L is a Moore graph of odd girth, the tree contains the entire vertex set of L. It can be extended to form L by adding additional edges among the leaves, so that no circuits of length less than g are formed. For every edge joining two leaves, a circuit of length g is formed. The subgraph of L that is induced by the leaves of the tree is a regular graph of degree $r-1$ on $r(r-1)^{s-1}$ vertices, and each edge of this subgraph corresponds to a circuit of length g in L containing v. So $c_v=(1/2)r(r-1)^s$.

Let H be a graph that is comatching with G. Then by Theorem 4.2, H has mn vertices, is regular of degree r, has girth g, and has the same number of circuits of length g as mL. Because G and H have the same number of vertices and circuits of length g, the average value of c_v in H must equal the average value in G. Since the value of c_v is constant in G, the average value in H must equal this constant value, $(1/2)r(r-1)^s$.

There must be a vertex, w, in H such that c_w equals or exceeds the average value, that is $c_w \geqslant (1/2)r(r-1)^s$. We can now find a tree rooted at w, with internal vertices of degree r, a total of n vertices and $r(r-1)^{s-1}$ leaves, each leaf being a distance s away from w. Moreover, this tree must be a subgraph of H, since H has girth g. In order that $c_w \geqslant (1/2)r(r-1)^s$ there must be at least $(1/2)r(r-1)^s$ edges in H that join two of the leaves of the tree. The subgraph of H induced by the leaves of the tree has $r(r-1)^{s-1}$ vertices, and the degree of a vertex cannot exceed $r-1$ in the subgraph. Thus, it has at most $(1/2)r(r-1)^s$ edges. Therefore, the subgraph induced by the leaves has exactly $(1/2)r(r-1)^s$ edges, and is regular of degree $r-1$. Furthermore, we conclude that the vertices of the tree induce a subgraph that is a full component of H, since each vertex has degree r in this induced subgraph.

Thus, we have located a component of H that is regular of degree r, has n vertices, and girth g. With these conditions we can conclude that this component is isomorphic to the unique Moore graph, L. Now consider the remaining portion of H. It has $(m-1)n$ vertices, is regular of degree r, has girth g and possesses the same number of circuits of length g as $(m-1)L$. By repeating the above process, we find each component is isomorphic to L, and thus H is isomorphic to $G=mL$. Therefore, G is matching unique. □

While this last theorem seems quite general, unfortunately, there are very few Moore graphs of odd girth, as indicated by the restrictions in Theorem 5.1. The known unique Moore graphs of odd girth are the complete graphs (K_n), Petersen's graph, and the Hoffman–Singleton graph. The only other possibility for a unique Moore graph of odd girth would have valency 57 and girth 5, resulting in a graph with 3250 vertices. We note that the matching uniqueness of mK_n was first established in [6].

We close by mentioning the following theorem.

Theorem 6.4 (Farrell and Guo [7]). *Suppose that G is matching unique. Then the complement of G is also matching unique.*

Thus, for each graph above that we have determined to be matching unique, we obtain its complement as another matching unique graph.

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