Nielsen numbers for based maps, and for noncompact spaces

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Abstract

In this paper we present two (not independent) applications of the surplus Nielsen number of the complement due to the second author. For the first application we give a based Nielsen number $N_*(f)$ associated with a based map $f : (X, x_0) \rightarrow (X, x_0)$ of a compact connected polyhedron $X$. The number $N_*(f)$ is a based homotopy invariant which is a lower bound for the number of fixed points in the based homotopy class of $f$. Moreover $N_*(f) \geq N(f; X, x_0)$ the relative Nielsen number of the pair $(X, x_0)$. The inequality may be strict, in particular $N_*(f)$ detects the two unremovable fixed points (not detected by $N(f; X, x_0)$) on Jiang's well known example on the figure eight. A minimum theorem is given.

In the second part of the paper we introduce a Nielsen type number $N_\infty(f)$ for noncompact spaces by taking the surplus number of the complement of the point at infinity in the one point compactification of the original space. The number $N_\infty(f)$ is a homotopy invariant with respect to those homotopies which extend to the one point compactification. In particular it is an isotopy invariant for self-homeomorphisms. We also indicate how to extend $N_\infty(f)$ to the relative setting for noncompact spaces. © 1997 Elsevier Science B.V.

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1. Introduction

In this paper we give two (not independent) applications of the surplus Nielsen number of the complement due to the second author [10]. The surplus number is defined in the context of a self-map $f : (X, A) \rightarrow (X, A)$ of pairs, and counts certain fixed point classes
in the complement $X - A$. Both of our applications concern the case that $A = \{x_\ast\}$, a singleton. For the first application we introduce a based Nielsen number $N_\ast(f)$, for a self-map $f: (X, x_\ast) \to (X, x_\ast)$ of a compact connected polyhedron $X$. The number $N_\ast(f)$ can be greater than the relative Nielsen number of $f$ (when the subspace in question is the singleton $\{x_\ast\}$) and of course $N_\ast(f) \leq M(f; X, x_\ast)$, the minimum number of fixed points for any map in the same relative homotopy class of $f$. Our number comes into its own for a class of spaces that are known to be non-Wecken (spaces for which $N(f) < M(f, \emptyset)$), namely those for which the base point is a local cut point. Under mild conditions these spaces turn out to be based Wecken, that is, there is a map $g$ which is based homotopic to $f$ with the property that $g$ has exactly $N_\ast(f)$ fixed points. In order to give our results on based surplus classes, there are a number of adjustments and extensions that need to be made to the theory given in [lo] for the special case that $A = \{x_\ast\}$. For example the minimum theorem in [lo] requires that the space $X$ under consideration not have a local cut point. We make the important adjustment of freeing our minimum theorem of this requirement. This refinement and some others are made possible precisely because we are not dealing with an arbitrary subspace $A$, but rather the special case $A = \{x_\ast\}$.

The second part of the paper also uses surplus theory, and at times the first part of the paper. In fact we use surplus theory in a different but related way, and define a Nielsen type number $N^\infty(f)$ on a self-map $f$ of a noncompact space $Y$. Here we take the Alexandrov one point compactification $Y^\infty$ of a space $Y$, and we then define our Nielsen type number $N^\infty(f)$ to be the surplus Nielsen number of the complement of the point at infinity. Clearly not all self-maps of $Y$ extend to self-maps of $Y^\infty$, so part of our work is to determine which maps and homotopies do extend. In fact in our context, all homeomorphisms (and isotopies of such maps) do extend (see Corollary 4.3 and Proposition 4.9), so our main application is to homeomorphisms. The number $N^\infty(f)$ is a homotopy invariant with respect to those homotopies which extend to the one point compactification. In particular it is an isotopy invariant for self-homeomorphisms. We show that $N^\infty(f)$ satisfies a number of other properties including invariance under conjugation. We also indicate how to extend $N^\infty(f)$ to the relative setting for noncompact spaces.

Our approach to noncompact spaces differs from that of Scholz [7] who worked with homotopies with compact support (i.e., with maps whose fixed point set is compact). We, on the other hand, work with maps and homotopies that extend to the one point compactification. This allows us to deal at times with maps whose fixed point set might not be compact (e.g., Example 4.14).

As is usual in a restricted Nielsen theory neither of the numbers in this paper satisfies all of the properties of ordinary Nielsen fixed point theory, but we investigate the discrepancies, and show which properties do hold.

The philosophy of the first and second parts of the paper are different. In the first part we are using the surplus number to detect possibly more fixed points than the ordinary or appropriate relative theory. In the second part it is not so much that we obtain more fixed points, but that we are able to use Nielsen theory to detect them at all.
The paper is divided as follows: in Section 2, following this introduction, we recall the necessary definitions and make the some adjustments and refinements that are needed for our applications to the special case that the subspace is a singleton. In Section 3 we define and give properties of our based Nielsen number, and prove a minimum theorem for it. In Section 4 we give our Nielsen type number for noncompact spaces, give some properties and examples, and extend the noncompact number to the relative setting.

2. Relative Nielsen theory and refinements of surplus theory

In this section we recall some of the fundamentals of surplus Nielsen theory [10], relative Nielsen theory [6], and make some refinements which we need because we are dealing with the special case that \( A = \{x_*\} \).

Let \( f : (X, A) \to (X, A) \) be a self-map of a pair of compact connected polyhedra \( X \) and \( A \) in which \( A \) has finitely many path components. We will denote the restriction of \( f \) to \( A \) by \( f|A \). We are interested primarily in the case that \( A \) is a singleton which we denote in the first part by \( x_* \), and in the second by \( \infty \) (in the second part the base point is not simply a preselected point, but is taken to be the point at infinity in the one point compactification). We will however recall many of the definitions for arbitrary \( A \). For any \( f : Y \to Y \) we denote the set of fixed points \( \{ y \in Y \mid f(y) = y \} \) by \( \Phi(f) \).

Let \( U \) be a subset of \( X \) which has finitely many path components. Recall that two fixed points \( x, y \in \Phi(f) \) are said to belong to the same fixed point class of \( f \) on \( U \) if \( x, y \in U \), and there is a path \( \alpha \) in \( U \) from \( x \) to \( y \) such that \( \alpha \simeq f(\alpha) \) rel \( \{0, 1\} \). If \( U = X \) then the fixed point classes of \( f \) on \( U \) are just the usual Nielsen classes of \( f \). We shall refer to them as ordinary fixed point classes. As usual the cardinality \( \#(E(f)) \) of the set \( E(f) \), of ordinary essential fixed point classes, is denoted by \( N(f) \). As pointed out in [10], fixed point classes of a relative map \( f \) on \( X - A \) have all the same basic properties of ordinary fixed point classes of \( f \), and each fixed point class of \( f \) on \( X - A \) is contained in an ordinary fixed point class of \( f \) (but a fixed point class of \( f \) may contain more than one fixed point class of \( f \) on \( X - A \), see Example 3.2).

Recall next that a fixed point class of \( F \) of \( f \) on \( X - A \) is said to be a nonsurplus fixed point class of \( f \) on \( X - A \) if there is a point \( x \in F \) and a path

\[
\alpha : (I, 0, I - \{1\}, 1) \to (X, x, X - A, A)
\]

such that \( \alpha \simeq f(\alpha) : (I, \{0\}, \{1\}) \to (X, x, A) \). A fixed point class of \( f \) on \( X - A \) which is not a nonsurplus fixed point class of \( f \) on \( X - A \) is said to be surplus. When \( X \) is compact there are a finite number of such classes. In general, several surplus fixed point classes of \( f \) (or in fact nonsurplus fixed point classes of \( f \)) on \( X - A \) may combine to give a Nielsen class of \( f \) on \( X \) (see Example 3.2).

As we mentioned earlier the fact that we are considering the special case that \( A = \{x_*\} \) allows us to make certain refinements of the theory. For arbitrary \( A \), the end point \( \alpha(1) \) of the path \( \alpha \) in the definition of nonsurplus fixed point class, need not be a fixed point.
However for $A = \{x_\ast\}$ it clearly is. In the next proposition $F_\ast$ will denote the fixed point class of $f$ on $X$ that contains $x_\ast$.

**Proposition 2.1.** Let $f : (X, x_\ast) \to (X, x_\ast)$ be a map, then the class $F_\ast$ contains every nonsurplus fixed point class of $f$ on $X - x_\ast$. Moreover every other fixed point class of $f$ on $X$ is a union of surplus fixed point classes of $f$ on $X - x_\ast$.

As mentioned above when $X$ is compact each surplus fixed point class of $f$ on $X - A$ is compact and so has an index in the usual way (see [10, 3.3 and 3.4]). If the index of a surplus fixed point class of $f$ on $X - A$ is nonzero, we call the class essential and denote by $SN(f : X - A)$ the number of such essential classes.

For a map $f : (X, A) \to (X, A)$ we denote by $M(f ; X, A)$ the minimum number $\#(\Phi(g))$ for $g$ pairwise homotopic to $f$, and by $M(f ; X - A)$ the minimum such points on $X - A$.

**Theorem 2.2** [10, 3.6 and 3.7]. The number $SN(f ; X - A)$ is a nonnegative integer. If $f \simeq g : (X, A) \to (X, A)$ then

$$SN(f ; X - A) = SN(g ; X - A), \quad \text{and}$$

$$SN(f ; X - A) \leq M(f ; X - A).$$

We recall the elements of relative Nielsen theory. If $f : (X, A) \to (X, A)$ is a map of pairs, we denote by $f|A$ the restriction of $f$ to $A$. We use subscripts to distinguish between fixed point classes $F$ of $f$ on $X$, and classes $F_A$ of $f|A$ on $A$ (note that this is an abuse when $A = \{x_\ast\}$, because then we denote the class of $x_\ast$ with respect to $f$ in $X$ by $F_\ast$). If $F_A \cap F \neq \emptyset$, then $F_A \subseteq F$ [6, 2.2]. An essential fixed point class $F_\ast$ of $f : X \to X$, is called an essential common fixed point class of $f$ and $f|A$ if $F$ contains an essential fixed point class of $f|A$. Denote by $N(f,f|A)$ the number of essential common fixed point class of $f$ and $f|A$. The relative Nielsen number $N(f; X, A)$ of $f : (X, A) \to (X, A)$ is defined to be

$$N(f ; X, A) = N(f) + N(f|A) - N(f,f|A).$$

Standard results of relative Nielsen theory include [6, 3.1 and 3.2] that

$$M(f ; X, A) \geq N(f ; X, A),$$

$$N(f ; X, A) \geq N(f|A) \quad \text{and} \quad N(f ; X, A) \geq N(f).$$

Note that when $A = \{x_\ast\}$, $N(f|A) = 1$. If $F_\ast$ is an essential class of $f$ on $X$, then $N(f,f|A) = 1$, if $F_\ast$ is not essential on $X$, then this number is zero. The following proposition then is clear.

**Proposition 2.3.** Let $f : (X, x_\ast) \to (X, x_\ast)$ be a based map where $X$ is a compact connected polyhedron, if $F_\ast$ is essential then $N(f; X, x_\ast) = N(f)$ else $N(f; X, x_\ast) = N(f) + 1$. 
3. The based Nielsen number

We are now ready to define our based Nielsen number and prove some of its properties.

3.1. Definition, properties and estimates

Definition 3.1. Let \( f : (X, x_\ast) \to (X, x_\ast) \) be a based map (i.e., \( f(x_\ast) = x_\ast \)) of a compact connected polyhedron \( X \) then the based Nielsen number is the sum

\[
N_\ast(f) = SN(f; X - x_\ast) + 1.
\]

Example 3.2 (cf. [5] and [10, 5.2]). Let \( X = S^1 \vee S^1 \) and \( x_\ast = S^1 \cap S^1 \) be the wedge point. We use the notation of [5] illustrated in the above diagram. We define \( f : (X, x_\ast) \to (X, x_\ast) \) by \( f(b_1) = b_1^{-1} \), \( f(b_2) = b_1^{-1} \), \( f(b_3) = b_2^{-1}b_1^{-1}b_3 \), and \( f(b_4) = b_4b_3b_4 \). Then, as in [5], \( \Phi(f) = \{x_\ast, y_1, y_2\} \) and \( f \) has two Nielsen classes \( F_1 = \{x_\ast\} \) and \( F_2 = \{y_1, y_2\} \) of \( f \) on \( X \). Simplifying the proof of [10, 5.2], we observe that \( y_1 \) and \( y_2 \) cannot be in the same fixed point class of \( f \) on \( X - A \) since there is no path from \( y_1 \) to \( y_2 \) in \( X - A \). So \( F_2 \) divides into two surplus classes each with a single element. Now \( y_1 \) and \( y_2 \) have index 1 and -1 respectively, so \( SN(f : X - x_\ast) = 2, N_\ast(f) = 3 \) but \( N(f) = 0 \). So then \( SN(f : X - x_\ast) \), and hence \( N_\ast(f) \), detects the two unremovable fixed point classes in Jiang’s well known map on the figure eight.

From Theorem 2.2 and Definition 3.1, we have immediately

Theorem 3.3. If \( f \simeq g : (X, x_\ast) \to (X, x_\ast) \) then \( N_\ast(f) = N_\ast(g) \).

So \( N_\ast(f) \) is a based homotopy invariant. The point of all Nielsen theories is of course the homotopy invariant lower bound property which we now give.

Theorem 3.4. If \( f : (X, x_\ast) \to (X, x_\ast) \) is a based map of a compact connected polyhedron \( X \), then \( N_\ast(f) \leq M(f; X, x_\ast) \).
Proof. By Theorem 3.3, we need only show that \( \Phi(f) \geq N_*(f) \). For a map \( f \) the lower bound property of \( SN(f; X - x_*) \) implies that \( \#(\Phi(f) \cap (X - x_*)) \geq SN(f; X - x_*) \). Since \( x_* \) is a fixed point of \( f \), then

\[
\#(\Phi(f)) = 1 + \#(\Phi(f) \cap (X - x_*)) \geq 1 + SN(f; X - x_*) = N_*(f).
\]

The next theorem, which is the main one for computation and estimates, makes it clear that \( N_*(f) \) comes into its own when \( x_* \) is a local cut point.

Theorem 3.5. Let \( f : (X, x_*) \to (X, x_*) \) be a based map of a compact connected polyhedron \( X \). Then

\[
N_*(f) \geq N(f; x_*, x_*) \geq N(f).
\]

If the base point \( x_* \) is not a local cut point, then \( N_*(f) = N(f; X, x_*) \), that is

\[
N_*(f) = \begin{cases} N(f) & \text{if } F_* \text{ is essential,} \\ N(f) + 1 & \text{if } F_* \text{ is inessential.} \end{cases}
\]

Note that both inequalities in Theorem 3.5 are strict for Example 3.2. Note also that since the base point \( x_* \) may or may not appear in an essential fixed point class of \( f \), it is evident that \( N_*(f) \) is not independent of the base point.

The proof of Theorem 3.5 requires the next two propositions which will be useful in the next section too. Recall from [6] that a subspace \( A \) of \( X \) can be bypassed if any path in \( X \) with endpoints in \( X - A \) is homotopic to a path in \( X - A \) keeping end points fixed.

Proposition 3.6. Let \( X \) be a compact polyhedron and \( x_* \) a base point, then \( x_* \) can be bypassed if and only if \( x_* \) is not a local cut point.

Proof. Suppose that \( x_* \) is not a local cut point, and let \( c : I \to X \) be a path with end points in \( X - x_* \). We may assume without loss of generality that \( x_* \) is a vertex and that both the unit interval \( I \), and the space \( X \) come equipped with a triangulation, and that with respect to these triangulations \( c \) is a simplicial map. In particular \( c^{-1}(x_*) \) is a finite set \( \{t_1, t_2, \ldots, t_k\} \) of vertices. All this being the case there exist an \( \varepsilon > 0 \) such that the "ball" \( B^\varepsilon = \{x \in X \mid d(x, x_*) < \varepsilon\} \) is contained in the star \( St_{x_*} \) of \( x_* \), and has the property that the set \( c^{-1}(B^\varepsilon) \) is the disjoint union of \( k \) intervals \( (t'_1, t''_1), (t'_2, t''_2), \ldots, (t'_k, t''_k) \) where for \( 1 \leq j \leq k \), we have that \( t'_j < t_0 < t''_j \). Since \( x_* \) is not a local cut point we may assume furthermore that the boundary \( \text{Bd}(B^\varepsilon) \) of \( B^\varepsilon \) is connected. Let \( B \) denote \( \text{Bd}(B^\varepsilon) \cup B^\varepsilon \). Under these constraints we may think of the set \( c(I) \cap B \) as consisting of \( k \) "angles" subtended at \( x_* \). Since \( B \) is clearly simply connected we can (up to homotopy) change the path \( c \) so that its image on the union of the intervals \( [t'_1, t''_1], [t'_2, t''_2], \ldots, [t'_k, t''_k] \) is contained in \( \text{Bd}(B^\varepsilon) \). Clearly this changed path is homotopic rel end points to the original \( c \) but contained in \( X - x_* \) as required.

The converse is intuitively obvious, however we sketch a proof. Note that if \( x_* \) is a local cut point then there is an \( \varepsilon > 0 \) such that \( B^\varepsilon = \{x \in X \mid d(x, x_*) < \varepsilon\} \) has a
disconnected boundary. Let \( c \) be a path in \( B^\circ \) connecting different components of the boundary. Clearly \( c \) passes through the point \( x_\ast \). We assume (to obtain a contradiction) that \( c \) is homotopic (keeping end points fixed) to a path \( d \) in \( X - x_\ast \). Since any ball of smaller radius than \( \varepsilon \) also has disconnected boundary we may assume without loss of generality that \( d \) is contained in \( X - B^\circ \). Consider the composite

\[
\pi_1 X \to \pi_1 \left( \frac{X}{X - B^\circ} \right) \to \pi_1 \left( \frac{B}{Bd(B)} \right) \to H_1 \left( \frac{B}{Bd(B)} \right) \to H_1 (B, Bd(B))
\]

where the first homomorphism is induced by projection, and the second by the obvious homeomorphism at the space level. The image of the trivial loop \( d^{-1}c \in H_1 (B, Bd(B)) \) is clearly \( c \), which is obviously a generator of \( H_1 (B, Bd(B)) \). On the other hand since the image of a trivial loop under a homomorphism is clearly trivial we see that this same \( c \) is also trivial, giving our contradiction. Thus \( x_\ast \) cannot be bypassed.  

We use Proposition 3.6 in the next proposition.

**Proposition 3.7.** Let \( f : (X, x_\ast) \to (X, x_\ast) \) be a based map, in which \( x_\ast \) is not a local cut point of \( X \). Then

\[
SN(f; X - x_\ast) = \begin{cases} 
N(f) - 1 & \text{if } F_\ast \text{ is essential,} \\
N(f) & \text{if } F_\ast \text{ is inessential.}
\end{cases}
\]

**Proof.** Note for \( A = \{x_\ast\} \) that \( N(f, \{f\} A) = E(f, \{f\} A) \), where \( E(f, \{f\} A) \) (defined in [9]) is the number of essential classes of \( f \) that contain a fixed point class of \( \{f\} A \). Since \( N(f; X - A) \) in [9, 2.7] is defined to be the difference \( N(f) - E(f, \{f\} A) \), and since \( x_\ast \) is not a local cut point then by Proposition 3.6 and [10, 3.8], we have that

\[
SN(f; X - A) = N(f) - N(f, \{f\} A).
\]

If \( F_\ast \) is essential then \( N(f, \{f\} A) = 1 \), otherwise \( N(f, \{f\} A) = 0 \). The result follows.  

**Proof of Theorem 3.5.** That \( N(f, X, x_\ast) \geq N(f) \) is standard relative theory (see also Proposition 2.3). Note by Proposition 2.1 that the nonsurplus fixed point classes are all contained in \( F_\ast \), the class of \( x_\ast \) with respect to \( f \). In the case that \( F_\ast \) is essential, by Proposition 2.3 we have that \( N(f; X, x_\ast) = N(f) \), so we need only show that \( N_\ast(f) \geq N(f) \). By the additivity of the fixed point index, each of the \( N(f) - 1 \) fixed point classes of \( \mathcal{E}(f) - \{F_\ast\} \) contains at least one essential surplus fixed point class of \( f \) on \( X - x_\ast \), so \( SN(f; X - x_\ast) \geq N(f) - 1 \). Thus for \( F_\ast \) essential we have

\[
N_\ast(f) = SN(f; X - x_\ast) + 1 \geq N(f) = N(f; X, x_\ast)
\]

as required. Similarly when \( F_\ast \) is inessential then \( SN(f; X - x_\ast) \geq N(f) \), and so \( N_\ast(f) \geq N(f) + 1 = N(f; X, x_\ast) \). The last part of the theorem follows from Propositions 2.3, 3.6, 3.7 and the definitions.  

Since \( SN(f : X - A) \) is not a homotopy type invariant, it should be no surprise to discover that \( N_*(f) \) is not either. We need to include a counterexample here however, since the counterexample in [10] is not for \( A \) a singleton.

**Example 3.8.** Let \( X \) be the figure eight as in Example 3.2 and let \( x_* \) be the wedge point. Let \( P \) be the “pants” that is the disc with two holes (see [5]) and let \( i : X \to P \) be an imbedding for which there exists a retraction \( r : P \to X \). Let \( y_* = i(x_*) \), and \( h = i \circ f \circ r \) where \( f \) is given as in Example 3.2. Then \( h \) and \( f \) have the same homotopy type as maps, but \( y_* \) is not a local cut point in \( P \), so by Theorem 3.5

\[
N_*(h) = N(h; P, y_*) = 1 \neq N_*(f) = 3.
\]

**Remarks 3.9.** As we have seen in the previous example \( N_*(h) = 1 \). However since \( h \) is a deformation retract of \( f \) as a map (i.e., \( h = i \circ f \circ r \) as above), then fixed points of \( f \) are also fixed points of \( h \), so we can also deduce that \( \#(\Phi(h)) \geq \#(\Phi(f)) \geq N_*(f) - 3 \).

Since \( N_*(f) \) is not a homotopy type invariant neither does it satisfy the commutative property (take \( g = fr \), and \( k = i \) in Example 3.8 then \( N_*(gk) = N_*(f) \neq N_*(g) = N_*(kg) \)). However as our final property of \( N_*(f) \) shows it is invariant under conjugation.

**Theorem 3.10.** Let \( f : (X, x_0) \to (X, x_0) \) be a based map of a compact connected polyhedron \( X \), and \( k : X \to X \) a homeomorphism with \( k(x_0) = x_0 \). Then \( N_*(f) = N_*(kfk^{-1}) \).

Theorem 3.10 follows easily from Proposition 3.11 which discusses the invariance of \( SN(f; X - A) \) under conjugation. This property which is not found in [10], is common in dynamical systems. We will use it in the next section, too.

A map \( k : (X, A) \to (X, A) \) of pairs is said to be a homeomorphism of pairs, if it is a map of pairs and a \( k \) is an ordinary homeomorphism of \( X \) such that its inverse \( k^{-1} \) is also a map of pairs. If \( k \) is a homeomorphism of pairs, then it is clear that its inverse \( k^{-1} \) is also a homeomorphism of pairs. If \( A \) is empty then \( k \) is an ordinary homeomorphism. Let \( f, g : (X, A) \to (X, A) \) be maps of pairs, then \( f \) and \( g \) are said to be conjugate if there is a homeomorphism \( k : (X, A) \to (X, A) \) of pairs such that \( k \cdot f \cdot k^{-1} = g \). It is obvious that \( k \) induces a 1-1 correspondence between the fixed point sets of \( f \) and \( g \) on \( X - A \). Moreover, we have

**Proposition 3.11.** Let \( f, g : (X, A) \to (X, A) \) be two maps of pairs which are conjugate, by a homeomorphism \( k : (X, A) \to (X, A) \) of pairs. Then, for any fixed point class \( F \) of \( f \) on \( X - A \), \( k(F) \) is a fixed point class of \( g \) on \( X - A \). If \( F \) is surplus, then \( k(F) \) is surplus, \( \text{ind}(f, F) = \text{ind}(g, k(F)) \), and so \( SN(f; X - A) = SN(g; X - A) \).

**Proof.** Let \( F \) be a fixed point class of \( f \) on \( X - A \), note that \( k(F) \) is contained in the fixed point set of \( g \). Let \( x_1 \) and \( x_2 \) be in \( F \). Then there is a path \( \alpha \) in \( X - A \) such that \( \alpha \sim f \cdot \alpha \) rel \( \{0, 1\} \) in \( X \). Since \( k \) is a homeomorphism and \( k(A) = A \), then \( k(X - A) \subseteq X - A \).
and so \( k \cdot \alpha \) is a path in \( X - A \), and we have \( k \cdot \alpha \sim k \cdot (f \cdot \alpha) = g \cdot (k \cdot \alpha) \) rel \( \{0,1\} \) in \( X \). This implies that any two points in \( k(F) \) are contained in the same fixed point class of \( g \) on \( X - A \). Since \( k \) is a homeomorphism, we have \( F = k^{-1} \cdot (k \cdot F) \). Thus, \( k(F) \) is a fixed point class of \( g \) on \( X - A \).

Consider the relative map \( g : (X, A) \rightarrow (X, A) \). If \( k(F) \) is nonsurplus (with respect to \( g \)), then there is a path \( \alpha \) in \( X - A \) from a point \( y_0 \) in \( k(F) \) to \( A \) such that

\[ \alpha : (I, 0, I - \{1\}, 1) \rightarrow (X, y_0, X - A, A), \]

and with the property that \( \alpha \simeq g(\alpha) : (I, \{0\}, \{1\}) \rightarrow (X, y_0, A) \). Write \( x_0 = k^{-1}(y_0) \). Then \( x_0 \) is a point in \( F \), and \( k^{-1} \cdot \alpha \) is path in \( X - A \) from \( x_0 \) to \( A \), and this shows that \( F \) is a nonsurplus fixed point class of \( f \) on \( X - A \).

We show next that if \( F \) is surplus then \( \text{ind} \left( f, F \right) = \text{ind} \left( g, k(F) \right) \). Since a surplus fixed point class is compact, we choose an open neighborhood \( U \) of \( F \) in \( X - A \) such that \( U \cap \text{Fix} \, f = F \). As \( k \) is a homeomorphism, \( k(U) \) is an open neighborhood of \( k(F) \) in \( X - A \), and since \( f \) and \( g \) are conjugate, \( k(U) \cap \text{Fix} \, g = k(F) \). We apply the commutativity of fixed point index to the maps \( k|U \) and \( f \cdot k^{-1}|k(U) \), to get that

\[ \text{ind} \left( (f \cdot k^{-1}) \cdot k|U, U \right) = \text{ind} \left( k \cdot (f \cdot k^{-1}|k(U)), k(U) \right), \]

i.e., \( \text{ind}(f|U, U) = \text{ind}(g|k(U), k(U)) \). It follows that \( \text{ind}(f, F) = \text{ind}(g, k(F)) \). Finally for the last part we merely observe from above that \( k \) induces a 1-1 correspondence from the essential surplus classes of \( f \) on \( X - A \), to the essential surplus classes of \( g \) on \( X - A \).

3.2. The minimum theorem for \( N_*(f) \)

The minimum theorem for \( N_*(f) \) follows below. Note that unlike its counterpart for arbitrary \( A \) \cite[Theorem 4.2]{10} we do not require that \( X \) has no local cut point. Thus our minimum theory is in harmony with the fact that our number comes into its own when the basepoint is indeed a local cut point.

**Theorem 3.12.** Let \( X \) be a compact polyhedron, and \( x_* \) a base point. If every component of \( X - x_* \) has no local cut point, and is not a 2-manifold, then every map \( f : (X, x_*) \rightarrow (X, x_*) \) is based homotopic to a map that has exactly \( N_*(f) \) fixed points.

Theorem 3.12 applies, of course, to Jiang’s map of the figure eight (see Example 3.2). The following proof is a modification of \cite[4.2]{10}, and uses \cite[6.1]{6} instead of \cite[6.1]{10}. We are able to do this because the invariant subspace \( A = \{x_*\} \) being a singleton has every point a fixed point.

**Proof.** By \cite[3.1]{9} we may assume that \( f \) is fix finite, and that all fixed points of \( f \) on \( X - x_* \) lie in maximal simplexes. We unite fixed points in the same fixed point class of \( f \) on \( X - x_* \) using the technique of \cite[6.2]{6}. If \( x \) lies in a nonsurplus fixed point class of \( f \) on \( X - x_* \), then there is a path \( \alpha : (I, 0, I - \{1\}, 1) \rightarrow (X, x, X - x_*, x_*) \) from \( x \) to \( x_* \) such that \( \alpha \simeq f(\alpha) : (I, \{0\}, \{1\}) \rightarrow (X, x, x_*). \) Note that (unlike the
proof of [10, 4.2]), we can deduce that \( \alpha(1) = x_1 \) is a fixed point, and we can now use [6, 6.1] to move \( x \) to \( x_1 \). As in [10, 4.2], the proof is completed by deleting fixed point classes of \( f \) which consist of a single fixed point of zero index in the usual way [1, p. 123]. \( \square \)

4. A Nielsen number for noncompact spaces

In the classical definition of the Nielsen number some form of compactness is essential (see, for example, [1], or [7]). For example let \( f: [0, 1] \to [0, 1] \) be any map, then \( f \) has Nielsen number \( N(f) = 1 \), so any map homotopic to \( f \) has at least one fixed point. But, if we remove the end points of \([0, 1]\) and consider the noncompact space \((0, 1)\), then the map \( g: (0, 1) \to (0, 1) \) defined by \( g(x) = x^2 \), has no points. On the other hand if we restrict ourselves to homeomorphisms and isotopies of homeomorphisms, we may get a different story. For example if one considers the homeomorphism \( g: (0, 1) \to (0, 1) \) defined by \( g(x) = 1 - x \), then it is intuitively clear that any homeomorphism \( g \), for which there is an isotopy \( H: f \cong g: Y \to Y \), has at least one fixed point. It is the purpose of this part of the paper to sketch out a Nielsen type number that will allow a more rigorous approach to this phenomenon. Our primary interest is in homeomorphisms, but we develop the theory in slightly more generality.

4.1. One point compactification and proper maps

For a noncompact space \( Y \), let \( Y^\infty \) be the Alexandrov one point compactification, that is \( Y^\infty = Y \cup \{ \infty \} \) with the topology that includes the open sets of \( Y \), together with the complement of closed and compact subsets \( K \) of \( Y \). Since \( Y \) is not compact \( K \) can never be \( Y \). In particular \( Y \) is not closed in \( Y^\infty \), and the set \( \{ \infty \} \) is not open in \( Y^\infty \).

For any map \( f: Y \to Y \), let \( f^\infty: Y^\infty \to Y^\infty \) be the function defined on \( Y^\infty \) by \( f^\infty(y) = f(y) \), if \( y \neq \infty \), and \( f^\infty(\infty) = -\infty \). We say that the map \( f \) extends to \( Y^\infty \) if \( f^\infty \) is continuous. We are interested in conditions that will ensure that \( f \) extends to \( Y^\infty \). This is not always the case, as the following example shows.

**Example 4.1.** Let \( Y = (0,2\pi) \), then \( Y^\infty = S^1 \), the unit circle with the usual topology. Define \( f \) to be the constant map at \( \pi \), then \( f^\infty \) is not continuous since \( (f^\infty)^{-1}(\{\pi\}) = Y \) is not closed in \( Y^\infty \).

We examine exactly where the continuity may fail. If \( U \subset Y \) is open then \( (f^\infty)^{-1}(U) = f^{-1}(U) \) is open in \( Y \), and so open in \( Y^\infty \). Let \( U \subset Y^\infty \) be subset of \( Y^\infty \) with \( \infty \in U \). Then \( U \) is open in \( Y^\infty \) if and only if \( K = (Y - U) \) is compact. Now \( (f^\infty)^{-1}(U) = Y^\infty - f^{-1}(K) \) and is open in \( Y^\infty \) if and only if \( f^{-1}(K) \) is compact. So \( f^\infty \) is continuous if and only if for any compact \( K \subset Y \), \( f^{-1}(K) \) is compact, but this is just the definition that \( f \) is a proper map (see, for example, [8, p. 319]). Clearly we have:
**Proposition 4.2.** Let \( f : Y \to Y \) be continuous where \( Y \) is noncompact. Then \( f \) is extends to \( Y^\infty \) if and only if \( f \) is proper.

Since the continuous image of a compact set is compact, the following corollary of Proposition 4.2 is clear.

**Corollary 4.3.** Let \( f : Y \to Y \) be a map of a noncompact space \( Y \), if \( f \) is a homeomorphism then \( f \) is proper, and moreover the extension \( f^\infty : Y^\infty \to Y^\infty \) is a homeomorphism.

Homeomorphisms are not the only proper maps.

**Example 4.4.** Let \( Y = Y_1 \cup Y_2 \), where \( Y_1 \) is the unit circle \( \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \), and \( Y_2 = \{1\} \times (-\pi, \pi) \). Let \( f : Y \to Y \) be the standard map of degree 3 on \( Y_1 \), and the map which takes \((1, x)\) to \((1, -x)\) on \( Y_2 \). Then \( Y^\infty = S^1 \vee S^1 \) the figure eight. Clearly, \( f \) extends to \( Y^\infty \), and so is proper. Note that \( f^\infty \) is a map of degree minus one on the right hand circle (and of course degree three on the left hand circle).

Our definition of the noncompact Nielsen number will utilize the surplus number on \( Y^\infty \), the subspace consisting of the singleton \( \infty \). Since the surplus number is defined for compact connected polyhedra, we will therefore naturally require that \( Y^\infty \) is a compact connected polyhedron. We spend a little while discussing when this might be the case.

**Proposition 4.5.** Let \((X, A)\) be a compact polyhedral pair, then \((X - A)^\infty\) is homeomorphic to \(X/A\).

**Proof.** We define a map \( g : (X - A)^\infty \to X/A \) to be the identity on \( X - A \), and we define \( g(\infty) = q(A) \), where \( q : X \to X/A \) is the quotient map. If \( i : (X - A) \to X \), and \( j : (X - A) \to (X - A)^\infty \) are the inclusions,

\[
\begin{array}{ccc}
X - A & \xrightarrow{i} & X \\
\downarrow j & & \downarrow q \\
(X - A)^\infty & \xrightarrow{g} & X/A
\end{array}
\]

Clearly \( g \) is a local homeomorphism on the neighborhoods that exclude \( \infty \) and \( A \). To see that \( g \) is continuous let \( U \) be an open set containing \( A \) in \( X/A \). Then \( X - q^{-1}(U) \) is closed in \( X \), and therefore compact. So \( q^{-1}((X/A) - U) = j(i^{-1}(X - q^{-1}(U))) \) is also compact, and \( g \) is continuous as required. We complete the proof by showing that \( g \) is open. We need only consider open sets containing \( \infty \). Let \( U \) be such a set, then by definition \((X - A)^\infty - U \) is compact. Since \( g \) is continuous \( g((X - A)^\infty - U) \) is compact, so \( g(U) \) is open in \( X/A \) as required. \( \Box \)

It is often not difficult to see that \( X/A \) is a polyhedron, this is certainly the case if \( X = I \) and \( A = \{0, 1\} \). The next proposition gives a method of proving that this is
the case when \( A \) has empty interior (i.e., \( \text{Bd}(A) = A \)). Recall first that a collar of \( A \) in \( X \) is an embedding \( c : A \times I \to X \) with the property that \( c(x, 0) = x \), and such that \( c(A \times [0, 1)) \) is an open neighborhood of \( A \) in \( X \).

**Proposition 4.6.** Let \((X, A)\) be a polyhedral pair with \( \text{Bd}(A) = A \). If \( A \) has a collar in \( X \), then \((X - A)^\infty\) is a polyhedron.

**Proof.** We show that \( X/A \) is homeomorphic to \( X \cup CA \), where \( CA \) is the (unreduced) cone on \( A \). The proof will then follow from Proposition 4.5. Let \( c: A \times I \to X \) be a collar of \( A \) in \( X \), and let \( X \cup (A \times I) \) be the adjunction space with pairs \((a, 0)\) in \( A \times I \) identified with \( a \) in \( X \). We define a homeomorphism \( g: X \to (X \cup (A \times I)) \) as follows.

\[
g(x) = \begin{cases} 
(a, 1 - 4t) & \text{if } x \in c(A \times [0, \frac{1}{4}]) \text{ and } x = c(a, t), \\
(a, 2t - \frac{1}{2}) & \text{if } x \in c(A \times [\frac{1}{4}, \frac{3}{4}]) \text{ and } x = c(a, t), \\
x & \text{otherwise.}
\end{cases}
\]

It is not hard to see that \( g \) is a homeomorphism, and that \( g \) maps \( A \) homeomorphically onto \( A \times 1 \). Thus \( g \) induces the required homeomorphism between \( X/A \) and \( X \cup CA \). \( \square \)

The space \( X/A \) (and hence \((X - A)^\infty\)) is a polyhedron, under different constraints. For example if \( X \) is the figure eight, and \( A \) the left hand circle then \( X/A \cong (X - A)^\infty \cong S^1 \). Note however the method of the proof of Proposition 4.6 breaks down since \((X - A)^\infty\) only has the homotopy type of \( X \cup CA \) and, as we shall see, the noncompact Nielsen number \( N^\infty(f) \) is not a homotopy type invariant. It would be interesting to know if, under the conditions of Proposition 4.5, that \( X/A \) is always a polyhedron.

**4.2. The definition and properties of \( N^\infty(f) \)**

If \( Y^\infty \) is a compact polyhedron, and the map \( f: Y \to Y \) is proper, of the relative map \( f^\infty: (Y^\infty, \infty) \to (Y^\infty, \infty) \) is defined.

**Definition 4.7.** Let \( Y \) be a noncompact space, with \( Y^\infty \) a polyhedron, and let \( f: Y \to Y \) be a proper map, then the noncompact Nielsen number \( N^\infty(f) \) is defined by the formula

\[
N^\infty(f) = SN(f^\infty; Y^\infty - \{\infty\}).
\]

We invite the reader to distinguish carefully between \( N^\infty(f) \) and \( N(f^\infty) \). It is not hard to see for \( f \) in Example 4.4 that \( N^\infty(f) = 2 \). Combining Proposition 3.11 with Corollary 4.3 we have immediately the following invariance of \( N^\infty(f) \).

**Theorem 4.8.** Let \( Y \) be a noncompact space, with \( Y^\infty \) a polyhedron, and let \( f: Y \to Y \) be a proper map, if \( k: Y \to Y \) is a homeomorphism, then \( kf k^{-1} \) is a proper map, and \( N^\infty(f) = N^\infty(kfk^{-1}) \).

Since we are constructing a Nielsen theory, we need to know in what sense we have homotopy invariance. As with other Nielsen theories for restricted classes of maps we
cannot expect to have a full homotopy invariance (e.g., the homotopy invariance of compactly fixed maps is restricted to compactly fixed homotopies [7]). Clearly then we need to know not only what maps extend to $Y^\infty$, but also what homotopies extend. For isotopies our work has been done for us in [2].

Recall (i.e., [2]) that an isotopy of a topological space $X$ is a homotopy $H : X \times I \to X$ such that for each $t \in I$, the map $H_t : X \to X$ defined by $H_t(x) = H(x, t)$ is a homeomorphism. We have seen that a homeomorphism $f : Y \to Y$ of a noncompact space extends to homeomorphism $f^\infty$ (Corollary 4.3). For homotopies we have the following result of Crowell from [2]. We would like to thank R.F. Brown for pointing out this reference to us.

**Proposition 4.9** [2]. Let $H : Y \times I \to Y$ be an isotopy between homeomorphisms $f$ and $g$. If $Y$ is locally compact Hausdorff, then $H$ has a unique extension to isotopy $H^\infty$ of $f^\infty$ and $g^\infty$ defined in the obvious way.

Crowell shows that without the hypothesis in $Y$ the proposition is wrong! From the relative homotopy invariance of the surplus Nielsen number (Theorem 2.2) and Proposition 4.9 we have:

**Theorem 4.10.** Let $Y$ be a noncompact space, with $Y^\infty$ a polyhedron, and let

$$H : f \simeq g : Y \times I \to Y$$

extend to $Y^\infty$, then $N^\infty(f) = N^\infty(g)$.

In particular from Proposition 4.9 and Theorem 4.10 we have:

**Corollary 4.11.** Let $Y$ be a noncompact space, with $Y^\infty$ a polyhedron, and

$$H : f \simeq g : Y \times I \to Y$$

an isotopy of homeomorphisms $f$ and $g$, then $N^\infty(f) = N^\infty(g)$.

From Theorem 2.2 we have

**Theorem 4.12.** Let $Y$ be a noncompact space, with $Y^\infty$ a polyhedron, and let $f : Y \to Y$ be a proper map. Then $N^\infty(f)$ is defined, is a nonnegative integer, and $f$ has at least $N^\infty(f)$ fixed points on $Y$.

**Proof.** Notice that the fixed point set of $f^\infty$ on $Y^\infty - \{\infty\}$ is just the fixed point set of $f$. Thus, by definition of $N^\infty(f)$ and standard surplus theory (see Theorem 2.2) $f$ has at least $N^\infty(f)$ fixed points on $Y$ ($= Y^\infty - \{\infty\}$).

From this theorem and Proposition 4.9 we get

**Corollary 4.13.** Let $Y$ be a noncompact space, with $Y^\infty$ a polyhedron, and let $f : Y \to Y$ be a homeomorphism. Then $N^\infty(f)$ is defined, is a nonnegative integer, and any map
g, for which there is an isotopy $H : f \cong g : Y \to Y$, has at least $N^\infty(f)$ fixed points on $Y$.

The following example formalizes the intuition discussed in the introduction to this part of the paper.

**Example 4.14.** Let $Y = (0, 2\pi)$, and let $f, g : Y \to Y$ be two homeomorphisms defined by $f(x) = 2\pi - x$ and $g(x) = x$ respectively. Then $Y^\infty = S^1$, which we represent by $\{e^{i\theta}, 0 \leq \theta < 2\pi\}$. Now $f$ and $g$ extend to $f^\infty$ and $g^\infty$ defined respectively by $f^\infty(e^{i\theta}) = e^{i(2\pi - \theta)}$ and $g^\infty(e^{i\theta}) = e^{i\theta}$. Note that $f^\infty$ is a reflection, and has two fixed points. One of these is $\infty$ so there is only one fixed point on $Y^\infty \setminus \{\infty\}$. This point belongs to an essential surplus fixed point class of $f$ on $Y^\infty \setminus \{\infty\}$, so $N^\infty(f) = 1$. By Corollary 4.13 any map $h$ for which there is an isotopy $H : f \cong h : Y \to Y$ has at least one fixed point.

On the other hand to calculate $N^\infty(g)$ we note that the set $Y^\infty \setminus \{\infty\}$ is a single fixed point class of $g$ on $Y^\infty \setminus \{\infty\}$. This class is nonsurplus so

$$N^\infty(g) = SN(g^\infty; S^1 \setminus \{\infty\}) = 0.$$  

We remark that the map $g$ has a noncompact fixed point set. This shows an improvement in methodology over the work of Scholz [7], who required that his maps and homotopies have compact support.

We can also use Example 4.14 to show that $N^\infty(f)$ is not a full homotopy invariant. Let $H : Y \times I \to Y$ be defined by $H(x, t) = 2\pi(1 - t) + x(2t - 1)$, then $H$ is a homotopy $f \simeq g$, but $N^\infty(f) \neq N^\infty(g)$. Note however that $H$ clearly does not extend to $Y^\infty \times I$ since $H(\cdot, \frac{1}{2})$ does not extend to $Y^\infty$ (see Example 4.1).

The reader is invited to compare how intuition and theory blend in the next example.

**Example 4.15.** Let $Y = Y_1 \cup Y_2$, where $Y_1 = \{0\} \times (-\pi, \pi)$ and $Y_2 = \{1\} \times (-\pi, \pi)$. Let $f : Y \to Y$ be the homeomorphism which takes $(i, x)$ to $(i, -x)$ for $i = 0, 1$. Then $Y^\infty = S^1 \vee S^1$ the figure eight, $f$ extends to $Y^\infty$, and $N^\infty(f) = 2$. So any map $g$ for which there is an isotopy $H : f \cong g : Y \to Y$ will have at least two fixed points.

We close this subsection with two results that can help in computation. From Theorem 3.5 and Proposition 3.7, using the fact that $N^\infty(f) = SN(f^\infty, Y^\infty \setminus \{\infty\})$, we have the first of these results (we use $F_\infty$ to denote the class of $\infty$ in $Y^\infty$):

**Theorem 4.16.** Let $Y$ be a noncompact space, $f : Y \to Y$ a proper map, and $Y^\infty$ a polyhedron, then

$$N^\infty(f) \geq N(f^\infty; Y^\infty, \infty) - 1 \geq N(f^\infty) - 1.$$  

Suppose that $\infty$ is not local cut point, then $N^\infty(f) = N(f^\infty; Y^\infty, \infty) - 1$, and

$$N^\infty(f) = \begin{cases} N(f^\infty) - 1 & \text{if } F_\infty \text{ is essential}, \\ N(f^\infty) & \text{if } F_\infty \text{ is inessential}. \end{cases}$$
The next example serves not only as an illustration of Theorem 4.16, but gives the basis for a relative example in the next section. It also forms a basis for some concluding remarks for this section.

**Example 4.17** (Bells and whistles). For each \( z \) in \( \mathbb{C} \) the complex plane let \( D_z^2 \) denote the unit disc with centre \( z \), and let \( S_z^1 \) denote its subspace the unit circle. Consider the space \( Y \) which is the union \( D_2^2 \cup S_0^1 \cup S_{2i}^1 \cup D_{-2}^2 \), with the point \( \{2\} \) removed. Then \( Y \) is connected by the points \( -1, i \) and \( +1, \) and \( Y^\infty \cong Y \cup \{2\} \) (the point \( \{2\} \) plays the role of the point at infinity). We define a map \( f \) of \( Y \) as follows. On \( S_0^1 \) it is the standard map of degree 5 (i.e., \( z \mapsto z^5 \)), so the points \( \pm 1, \) and \( \pm i \) are fixed points. On \( D_2^2 \) the map \( f \) is reflection about the real axis, on \( D_{-2}^2 \) it is the identity, while on \( S_{2i}^1 \) it is reflection in the imaginary axis. Clearly \( f \) extends to \( Y^\infty \) and so is proper. There are 5 fixed point classes of \( f^\infty \) namely the classes of the sets \( \{-3 \leq x \leq -1\}, \{i\}, \{-i\}, \{3i\} \) and \( D_{-2}^2 \). Each fixed point class is essential (an easy way to see this is to use [3, Lemma A1]). Thus \( N(f^\infty) = 5 \), and since \( 2 = \infty \) is not a local cut point and \( F_\infty \) is essential, we have by Theorem 4.16, that \( N^\infty(f) = 5 - 1 = 4 \).

If one prefers examples that are homeomorphisms one can modify Example 4.17 as follows:

**Example 4.18.** Let \( Y = D_2^2 \cup S_0^1 \cup (D_{-2}^2 \setminus \{2\}) \), and \( f \) defined as in Example 4.17 on \( D_{-2}^2 \) and \( D_2^2 \setminus \{2\} \), but reflection in the real axis on \( S_0^1 \). Then any map \( g \) for which there is an isotopy \( H : f \cong g : Y \to Y \) has at least \( N^\infty(f) = 1 \) fixed point.

The final result of this subsection shows how to relate the computation of \( N^\infty(f) \) to the induced map on homology.

**Theorem 4.19.** Let \( Y \) be a noncompact space in which \( Y^\infty \) is a Jiang space, and a polyhedron in which \( \infty \) is not a local cut point. Let \( f : Y \to Y \) be a proper map, then

\[
N^\infty(f) = \begin{cases} 
0 & \text{if } L(f^\infty) = 0, \\ 
\#\text{Coker}(1 - f^\infty) - 1 & \text{if } L(f^\infty) \neq 0,
\end{cases}
\]

where \( f^\infty : H_1(Y^\infty) \to H_1(Y^\infty) \) is the homomorphism on the one-dimensional homology group.

Example 4.14 can be used to illustrate Theorem 4.19.

**Proof.** Under the conditions of the theorem if \( L(f^\infty) = 0 \), then \( N(f^\infty) = 0 \). The class \( F_\infty \) of \( \infty \) in \( Y^\infty \) is clearly inessential, so by the second part of Theorem 4.16, \( N^\infty(f) = N(f^\infty) = 0 \). If \( L(f^\infty) \neq 0 \), since \( \pi_1(Y^\infty, \infty) \) is abelian and \( Y^\infty \) is a Jiang space, then \( N(f^\infty) = \#\text{Coker}(1 - f^\infty) \) by standard Nielsen theory, and so again by Theorem 4.16 \( N^\infty(f) = \#\text{Coker}(1 - f^\infty) - 1 \).

**Remarks 4.20.** The way that we have chosen to define a Nielsen number for a non-compact space \( Y \) can at times be rather clumsy. Consider for example the space \( Y \) in
Example 4.17 with the point $-2$ removed, but the same map. Then $Y^\infty$ is homeomorphic to the union of $S^1_0$ and the cones on $S^1_1$ and $S^1_{-2}$ joined at the top. Now the point $\infty$ is a local cut point and so in light of Theorem 4.16 one might expect $N^\infty(f)$ in this example to be greater than $N^\infty(f)$ in Example 4.17. In fact it is less, since we lose the class of $\{-3 \leq z \leq -1\}$. In fact the fixed point classes $\{-3 \leq z < -2\}$, $\{-2 < z \leq -1\}$ and $D^2_2 - \{0\}$ of the new example are now all nonsurplus so the new $N^\infty(f) = 3$. As with ordinary Nielsen theory there are various ways of getting round the inadequacy in particular situations. We will discuss one such way in the last subsection where we introduce a relative Nielsen theory for periodic points.

4.3. Relative Nielsen theory for noncompact spaces

In this section, we show how Nielsen theory for noncompact spaces may be extended to the relative situation. In keeping with the motivation for ordinary relative Nielsen theory, we have a special interest in manifolds with nonempty boundary, and in particular with homeomorphisms of such spaces since they are naturally boundary preserving.

Throughout this section $f : (Y, A) \to (Y, A)$ will be a map of a pair of polyhedra $Y$ and $A$, with $Y$ noncompact, and $A$ compact. If $Y^\infty$ is a polyhedron, it is clear that $(Y^\infty, A)$ is a pair of compact polyhedra, and $f$ extends to $(Y^\infty, A)$ if and only if the underlying map $f : A \to A$ extends to $Y^\infty$, that is if and only if $f$ is proper. We will abuse notation in this case, and say that the map $f : (Y, A) \to (Y, A)$ itself is proper.

**Definition 4.21.** Let $f : (Y, A) \to (Y, A)$ be a proper map $A$, with with $Y$ a noncompact ANR, $A$ a compact polyhedron, and such that $Y^\infty$ is a polyhedron. A surplus fixed point class $F$ of $f$ on $Y^\infty - \{\infty\}$ is said to be a common surplus fixed point class of $f$ and its restriction $f|A : A \to A$, if $F$ contains a fixed point $x$ of $f|A$ which lies in an essential class of $f|A$. If in addition $F$ is essential then $F$ is said to be an essential common surplus fixed point class of $f$ and $f|A$.

The corollary to the next lemma shows we might just as well deal entirely with fixed point classes.

**Lemma 4.22.** Let $f : (Y, A) \to (Y, A)$ be as above, let $F$ be a surplus fixed point class of $f$ on $Y^\infty - \{\infty\}$, and let $F_A$ be a fixed point class of $f|A$. If $F \cap F_A \neq \emptyset$, then $F_A \subseteq F$.

The proof of Lemma 4.22 is identical to the proof of Lemma 2.2 of [6] once fixed point classes of $f$ are replaced by surplus fixed point classes of $f$ on $Y^\infty - \{\infty\}$. Similarly we have

**Corollary 4.23.** Let $f : (Y, A) \to (Y, A)$ be as above, an essential surplus fixed point class is a common essential surplus fixed point class if and only if it contains an essential fixed point class of $f|A : A \to A$. 

We define $\text{NS}(f^\infty, f | A)$ to be the number of common essential surplus fixed point classes of $f^\infty$ and $f | A$. Note that $\text{NS}(f^\infty, f | A) \leq N^\infty(f)$, and $\text{NS}(f^\infty, f | A) \leq N(f | A)$.

**Definition 4.24.** Let $f : (Y, A) \rightarrow (Y, A)$ be a proper map of polyhedra $Y$ and $A$, with $Y$ noncompact, $A$ compact and such that $Y^\infty$ is a polyhedron. We define $N^\infty(f; Y, A)$ to be

$$N^\infty(f; Y, A) = N^\infty(f) + N(f | A) - \text{NS}(f^\infty, f | A).$$

Note that $N^\infty(f; Y, A)$ is an integer, and that $N^\infty(f; Y, 0) = N^\infty(f)$.

**Example 4.25** (Relative bells and whistles). Let $Y$ and $f$ be as in Example 4.17, then $N^\infty(f) = 4$. Let $A$ be the union of the subsets $S_1$ and $S_{-2}$ of $D_2^1$ and $D_2^{1-2}$ respectively. There are 3 fixed point classes of $f | A$ on $A$, they are the classes $\{-3\}$, $\{-1\}$ and the class of $S_1$. The first two classes are essential, while the third is inessential so $N(f | A) = 2$. It is easy to see that $\text{NS}(f^\infty, f | A) = 1$, so $N^\infty(f; Y, A) = 4 + 2 - 1 = 5$.

As in Section 5 we can combine Proposition 3.11 with an easy extension of Corollary 4.3 to give the following invariance of $N^\infty(f; Y, A)$:

**Theorem 4.26.** Let $f : (Y, A) \rightarrow (Y, A)$ be a proper map of polyhedra $Y$ and $A$, with $Y$ noncompact, $A$ compact, and $Y^\infty$ a polyhedron. If $k : (Y, A) \rightarrow (Y, A)$ is a homeomorphism, then $N^\infty(f; Y, A) = N^\infty(kfk^{-1}; Y, A)$.

Using the relative homotopy invariance of the surplus Nielsen number (Theorem 2.2), we are now able to mimic the proof of [6, 3.3] to obtain:

**Theorem 4.27.** Let $f : (Y, A) \rightarrow (Y, A)$ be a proper map of polyhedra, with $Y$ noncompact, $A$ compact, and $Y^\infty$ a polyhedron. If $H : f \simeq g : (Y, A) \times I \rightarrow (Y, A)$ is a homotopy in which the underlying homotopy $H : f \simeq g : Y \times I \rightarrow Y$, extends to $Y^\infty$, then $N^\infty(f; Y, A) = N^\infty(g; Y, A)$.

From Proposition 4.9 and Theorem 4.27 we have

**Corollary 4.28.** Let $Y$ be a noncompact space, with $Y^\infty$ a polyhedron, $A \subset Y$ compact, and $H : f \simeq g : (Y, A) \times I \rightarrow (Y, A)$ a pairwise isotopy of homeomorphisms $f$ and $g$, then $N^\infty(f; Y, A) = N^\infty(g; Y, A)$.

The proof of the next theorem is an easy modification of [6, 3.1], we refer the reader to this reference for more details.

**Theorem 4.29.** Let $f : (Y, A) \rightarrow (Y, A)$ be a proper map of polyhedra, with $Y$ noncompact, $A$ compact, and $Y^\infty$ a polyhedron. Then $N^\infty(f; Y, A)$ is a nonnegative integer, and $f$ has at least $N^\infty(f; Y, A)$ fixed points.
Putting Corollary 4.28 and Theorem 4.29 together we get:

**Corollary 4.30.** Let $Y$ be a noncompact space, with $Y^\infty$ a polyhedron, $A \subset Y$ compact, and let $f : (Y, A) \to (Y, A)$ be a homeomorphism of pairs. Then $N^\infty(f; Y, A)$ is defined, is a nonnegative integer, and any map $g$ for which there is a pairwise isotopy

$$H : f \cong g : (Y, A) \to (Y, A)$$

has at least $N^\infty(f; Y, A)$ fixed points.

**Example 4.31.** Let $Y = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 < 2\}$, and let $f : Y \to Y$ be reflection about the $x$ axis. Let $A$ be the unit circle centered at the origin, which is contained in $Y$, then $f$ is a self homeomorphism of the pair $(Y, A)$. Now $Y^\infty$ is homeomorphic to $S^2$ with an open disc removed, that is $Y^\infty$ is of the homotopy type of a two disc. There are two fixed point classes of $f^\infty$ on $Y^\infty - \{0\}$, and both are nonsurplus, so $N^\infty(f; Y, A) = N(f|S^1) = 2$. Thus any homeomorphism $g$ for which there is an isotopy $H : f \cong g : Y \to Y$ has at least $N^\infty(f; Y, A) = 2$ fixed points. Note that $N^\infty(f) = 0$.

We would expect the next theorem by analogy with relative Nielsen theory. It follows trivially from Definition 4.24 and the remarks preceding it.

**Theorem 4.32.** Let $Y$ be a noncompact space, with $Y^\infty$ a polyhedron, and $A \subset Y$ compact. Let $f : (Y, A) \to (Y, A)$ be a proper map, then

(i) $N^\infty(f; Y, A) \geq N(f|A)$,

(ii) $N^\infty(f; Y, A) \geq N^\infty(f)$.

We observe that in Example 4.25 both inequalities in Theorem 4.32 are strict. We close with the following theorem which gives a simple condition which allows us to calculate $N^\infty(f; Y, A)$ directly from $N(f^\infty; Y^\infty, A)$.

**Theorem 4.33.** Let $Y$ be a noncompact space, with $Y^\infty$ a polyhedron, and $A \subset Y$ compact. Let $f : (Y, A) \to (Y, A)$ be a proper map, and suppose furthermore that the point $\infty$ is not a local cut point. If $F_\infty$ is inessential, or if it is an (ordinary) essential common fixed point class of $f^\infty$ and $f|A$, then

$$N^\infty(f; Y, A) = N(f^\infty; Y^\infty, A).$$

If $F_\infty$ is essential, but not an (ordinary) common fixed point class of $f^\infty$ and $f|A$, then

$$N^\infty(f; Y, A) = N(f^\infty; Y^\infty, A) - 1.$$ 

Examples 4.25 and 4.31 furnish us with examples of these equalities.

**Proof.** Note first that

$$N^\infty(f; Y, A) - N(f^\infty; Y, A)$$

$$= (N^\infty(f) - N(f^\infty)) + (N(f^\infty, f|A) - NS(f^\infty, f|A)).$$
Since \( \infty \) is not a local cut point we can use Theorem 4.16 to help determine the first of these two differences. In particular if \( F_\infty \) is inessential, then the first difference on the right hand side of the equation is zero. Also since \( \infty \) is not a local cut then it satisfies the bypassing condition, and so each essential surplus class of \( f^\infty \) (with respect to the subspace \( \{\infty\} \)), coincides with an ordinary Nielsen class of \( f^\infty \) by [10, 3.8]. Thus from the definitions, the second difference is also zero. In a similar way if \( F_\infty \) is an (ordinary) essential common fixed point class of \( f^\infty \) and \( f|A \), the first difference is \(-1\), while the second difference is \(+1\) (here \( N(f^\infty, f|A) \) counts the class \( F_\infty \), but \( NS(f^\infty, f|A) \) does not).

In the case that \( F_\infty \) is essential, but not an (ordinary) essential common fixed point class of \( f^\infty \) and \( f|A \), the first difference is \(-1\) (Theorem 4.16), but the second is zero as before. \( \Box \)

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References