Rational approximation schemes for bi-continuous semigroups

Patricio Jara

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA

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Abstract

This paper extends the Hille–Phillips functional calculus and rational approximations results due to R. Hersh, T. Kato, P. Brenner, and V. Thomée to generators of bi-continuous semigroups. The method yields error estimates for rational time-discretization schemes for such semigroups, in particular for dual semigroups, Feller semigroups such as the Ornstein–Uhlenbeck semigroup, the heat semigroup, semigroups induced by nonlinear flows, implemented semigroups, and evolution semigroups. Furthermore, the results provide error estimates for a new class of inversion formulas for the Laplace transform.

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Introduction

Around 1940, E. Hille [21,22] and K. Yosida [44] started to study what is now known as the theory of strongly continuous semigroups \( T : t \mapsto e^{tA} \) on Banach spaces \( X \), where \( A \) is a linear, in general unbounded, operator with domain \( D(A) \) and range in \( X \). They were pursuing a qualitative and a quantitative analysis of solutions \( u(t) = T(t)x \) of initial value problems \( u'(t) = Au(t) \) with \( u(0) = x \in D(A) \). Since then on, the theory of strongly continuous semigroups matured and found many applications in different areas of science. However, it was clear from the very beginning that not every semigroup is strongly continuous and that a comprehensive theory will require a more general set-up. For this reason, many other classes of semigroups were studied, e.g., strongly continuous semigroups with discontinuities for \( t = 0 \) [23], distribution semigroups [33,34], semigroups of growth of order \( \alpha \) [40], integrated semigroups [2,27,39], and convolution semigroups [6], among others. The main difficulty with these theories is the lack of a substantial body of results concerning the qualitative properties of these semigroups, e.g., perturbations, asymptotic behavior, and approximations, among others.

In [30,31], F. Kühnemund introduced a new class of semigroups called bi-continuous semigroups. This class has two remarkable properties. It allows for a rich qualitative theory since bi-continuous semigroups are still sufficiently regular to be treated by classical Laplace transform methods, e.g., see [1,14–16,30,31]. Second, there is a variety of interesting semigroups belonging this class, e.g., Feller semigroups [1,35], the heat semigroup [14], adjoint semigroups [3,36], implemented semigroups [3], weakly continuous semigroups [5] including the Ornstein–Uhlenbeck semigroup.
[9,32], as well as certain evolutions semigroups [7] and semigroups induced by nonlinear flows studied by J.R. Dorroh and J.W. Neuberger [10–12]. It is the purpose of this paper to extend two additional Laplace transform based results to the bi-continuous setting: the Hille–Phillips functional calculus and stability results for rational approximation of semigroups. In this way, the results obtained by R. Hersh and T. Kato [18] and by P. Brenner and V. Thomée [4] can be extended to the bi-continuous case.

The first section contains an introduction to bi-continuous semigroups and some remarks on the Riemann–Stieltjes integral. The second section discusses the Hille–Phillips functional calculus for generators of bi-continuous semigroups. The third section contains stability and norm convergence results for rational approximations of bi-continuous semigroups. The last section contains stability and norm convergence results for rational approximations of bi-continuous semigroups (with error estimates). Applications to the semigroups mentioned above as well as to the inversion of the Laplace transform are included in section four.

1. Preliminaries

Let \( \mathcal{L}(X) \) denote the Banach space of bounded operators on a Banach space \( X \). A semigroup \( T \) is a map from \([0, \infty)\) into \( \mathcal{L}(X) \) that satisfies the functional equation

\[
\begin{align*}
(T(t+s) & = T(t)T(s), \\
T(0) & = \text{Id}
\end{align*}
\]

for all \( t, s \geq 0 \). The semigroup \( T \) is strongly continuous if the maps \( t \mapsto T(t)x \) are continuous on \( \mathbb{R}_0^+ := [0, \infty) \) for each \( x \in X \), and it is of type \((M, \omega)\) if there exist \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that \( \|T(t)\| \leq Me^{\omega t} \) for all \( t \in \mathbb{R}_0^+ \). The generator of a strongly continuous semigroup \( T \) is defined by \( \lambda X := \lim_{h \to 0^+} \frac{T(h)X - X}{h} \) with domain \( D(A) \) consisting of those \( x \in X \) for which the limit exists. One can think of \( \lambda X \) as \( e^{\lambda x} := \lim_{n \to \infty} (I + \lambda n)^{-n} x \), where \( A \) is closed, densely defined, linear operator with a “sufficiently nice” resolvent \( \lambda \mapsto R(\lambda, A) := (\lambda I - A)^{-1} \). In fact, the celebrated Hille–Yosida Theorem (see [13, p. 77] or [17, p. 20]) states that \( T(t)x = e^{\lambda x} := \lim_{n \to \infty} (I + \lambda n)^{-n} x = \lim_{n \to \infty} (\frac{\lambda}{n})^n \lambda^n R(\lambda, A) \) exists for all \( x \in X \) if and only if \( D(A) \) is dense in \( X \) and there exist \( M, \omega > 0 \) such that \( R(\lambda, A) \) exists and \( \| (\lambda - \omega)^n R(\lambda, A) \| \leq M \) for all \( \lambda > \omega \).

Remark 1.1. The algebraic property \( T(t+s) = T(t)T(s) \) amplifies many topological properties of the semigroup \( T \). For instance, if \( T \) is a strongly measurable semigroup on \((0, \infty)\), then it is strongly continuous on \((0, \infty)\) (not necessarily on \( \mathbb{R}_0^+ \)). Moreover, \( T \) is strongly continuous if and only if it is weakly continuous at \( t = 0 \), see [41].

If \( X := C_{ub}(\mathbb{R}_0^+) \) is the Banach space of bounded, uniformly continuous, and complex-valued functions with \( \|x\|_\infty := \sup_{x \in \mathbb{R}_0^+} |x(s)| \), then the shift semigroup \( T(t)x : s \mapsto x(s + t) \) is a strongly continuous semigroup with generator \( A = \frac{d}{ds} \) and \( D(A) = \{ x \in C_{ub}(\mathbb{R}_0^+) : x' \in C_{ub}(\mathbb{R}_0^+) \} \). However, if one chooses \( X \) to be \( C_b(\mathbb{R}_0^+) \), the Banach space of bounded, continuous, complex-valued functions with norm \( \|x\|_\infty \), then the shift semigroup fails to be strongly continuous. For example, by shifting rapidly oscillating functions \( x \) with \( \|x\|_\infty = 1 \) (like \( x : s \mapsto e^{is^2} \)) one obtains that \( \|T(t)x - T(s)x\| = 2 \) for all \( t, s \geq 0 \) with \( t \neq s \). Therefore, for certain \( x \in X \), the map \( t \mapsto T(t)x \) is nowhere continuous (and hence, by Remark 1.1, the shift semigroup is also not strongly measurable on \((0, \infty)\)). Notice that \( D(A) := \{ x \in C_b^1(\mathbb{R}_0^+) : x' \in C_b(\mathbb{R}_0^+) \} \) is not dense in \( C_b(\mathbb{R}_0^+) \), and that the part of \( A \) in \( D(A) = C_b^1(\mathbb{R}_0^+) \) generates a strongly continuous semigroup on \( D(A) \), see also Corollary 13 of [31].

This example shows that in order to discuss the shift semigroup on spaces such as \( C_b \) or \( L^\infty \) one needs a more general framework than the one provided by strongly continuous semigroups. Because of the applications of the shift semigroup in Laplace transform theory given in Section 4, special attention will be given to it on the Banach spaces \( C_b(\mathbb{R}_0^+) \) and \( L^\infty(\mathbb{R}_0^+) \).

Now, despite the fact that the shift semigroup is not strongly continuous on \( L^\infty(\mathbb{R}_0^+) \) or \( C_b(\mathbb{R}_0^+) \), it has still enough regularity to be integrated in a classical sense. The basic idea behind bi-continuous semigroups is to consider a suitable coarser topology \( \tau \) on the space \( X \) such that the maps \( t \mapsto T(t)x \) are \( \tau \)-continuous on \( \mathbb{R}_0^+ \) for each \( x \in X \); i.e., the semigroup is \( \tau \)-strongly continuous but not necessarily strongly continuous on \( X \).

Definition 1.2. A locally convex topology \( \tau \) defined via a family \( \mathcal{P}_\tau \) of seminorms (cf. [42, p. 48]) on a Banach space \( X \) is said to be coherent (with the norm topology) if
(i) the space \((X, \tau)\) is sequentially complete on \(\| \cdot \|\)-bounded sets; i.e., every \(\| \cdot \|\)-Cauchy sequence converges in \((X, \tau)\);

(ii) the topology \(\tau\) is Hausdorff and \(p(x) \leq \|x\|\) for all \(x \in X\) and \(p \in \mathcal{P}_\tau\); and

(iii) the topological dual \((X, \tau)'\) is norming for \((X, \| \cdot \|)\); i.e., for all \(x \in X\),

\[
\|x\| = \sup \{ |(x, \phi)| : \phi \in (X, \tau)' \text{ and } \|\phi\|_{(X, \| \cdot \|)'} \leq 1 \}.
\]

The condition (ii) implies that \((X, \tau)' \subseteq (X, \| \cdot \|)'\). Moreover, if \(\Phi := \{ \phi \in (X, \tau)' : \|\phi\|_{(X, \| \cdot \|)'} \leq 1 \}\), then

\[
\sup_{\phi \in \Phi} |(x, \phi)| \leq \|x\| \tag{1.1}
\]

for all \(x \in X\). In this way, the dual space \((X, \tau)\) is norming for \((X, \| \cdot \|)\) if the supremum reaches the norm in (1.1).

A subset \(\Phi_0\) of \(\Phi\) for which \(\sup_{\phi \in \Phi_0} |(x, \phi)| = \|x\|\) for all \(x \in X\) is called a coherent norming set for \(X\).

For example, a coherent topology \(\tau\) for \(X = C_b(\mathbb{R}^+_0)\) is the topology of uniform convergence on compact subsets \(K\) of \(\mathbb{R}^+_0\); i.e., the topology given by the family of seminorms \(p_K(x) := \sup_{s \in K} |x(s)|\), \(K \subset \mathbb{R}^+_0\) is compact. Furthermore, a coherent norming set \(\Phi_0 \subset \Phi\) is given by the point measures \(\phi_0 : X \mapsto \mathbb{C}\) defined by \(\phi_0(x) = x(a)\).

From now on, and unless stated otherwise, it will be always assumed that a \(\tau\) topology is coherent with the norm topology of the Banach space \(X\).

**Definition 1.3.** A semigroup \(T\) of type \((M, \omega)\) is bi-continuous (with respect to \(\tau\)) if

(i) \(T\) is \(\tau\)-strongly continuous; i.e., \(t \mapsto T(t)x\) is \(\tau\)-continuous on \(\mathbb{R}^+_0\) for all \(x \in X\), and

(ii) \(T\) is locally bi-equicontinuous; i.e., for every \(t_0 \geq 0\), \(\varepsilon > 0\), \(p \in \mathcal{P}_\tau\) and for every \(\| \cdot \|\)-bounded sequence \(\{x_n\}_{n \in \mathbb{N}}\)

with \(\tau\)-\(\text{lim}_{n \to \infty} x_n = x\) there exists \(N \in \mathbb{N}\) such that \(\sup_{0 \leq t \leq t_0} p(T(t)(x_n - x)) < \varepsilon\) for all \(n \geq N\).

Notice that every strongly continuous semigroup on a Banach space is a bi-continuous semigroup by taking the locally convex topology \(\tau\) as the topology given by the norm. However, not every bi-continuous semigroup is strongly continuous. For example, the shift semigroup is not strongly continuous but bi-continuous with respect to \(\tau\) on \(C_b(\mathbb{R}^+_0)\) where \(\tau\) is the topology of uniform convergence on compact sets. To see this, consider \(K \subset \mathbb{R}^+_0\) compact and \(x \in C_b(\mathbb{R}^+_0)\). By the uniform continuity of \(x\) on compact subsets, it follows that \(p_K(T(t_n)x - T(t)x) = \sup_{s \in K} |x(s + t_n) - x(s + t)| \to 0\) as \(t_n \to t\). Thus, \(T\) is \(\tau\)-strongly continuous on \(C_b(\mathbb{R}^+_0)\). In order to show bi-equicontinuity, let \(t_0 \in \mathbb{R}^+_0\) and \(K \subset \mathbb{R}^+_0\) be compact. Define \(W := \bigcup_{t \in [0, t_0]} K + t\) and let \(\{x_n\}_{n \in \mathbb{N}}\) be a norm-bounded sequence with \(\tau\)-\(\text{lim}_{n \to \infty} x_n = x\). For \(\varepsilon > 0\) choose \(N \in \mathbb{N}\) such that \(p_W(x_n - x) = \sup_{s \in W} |x_n(s) - x(s)| \leq \varepsilon\) for \(n \geq N\). Then \(\sup_{t \in [0, t_0]} p_K(T(t)(x_n - x)) = \sup_{t \in [0, t_0]} \sup_{s \in K} |x_n(s) - x(s)| \leq p_W(x_n - x) \leq \varepsilon\). Thus, the shift semigroup is bi-continuous on \((C_b(\mathbb{R}^+_0), \| \cdot \|, \tau)\).

Finally, some basic properties of \(\tau\)-integrals are needed. Let \(\alpha : [0, R] \mapsto \mathbb{C}\) be a function of total variation \(\text{Var}_{[0,R]}(\alpha)\). The space of normalized functions of bounded variation is denoted by \(\text{NBV}[0, R]\) and \(\text{NBV}_{\text{loc}} := \bigcap_{R > 0} \text{NBV}(0, R)\). If \(\alpha \in \text{NBV}_{\text{loc}}\), then \(V_\alpha : t \mapsto \text{Var}_{[0,t]}(\alpha) \in \text{NBV}_{\text{loc}}\) and \(\text{Var}(\alpha)\) denotes \(\text{Var}_{[0,\infty]}(\alpha)\). Moreover, the Riemann–Stieltjes convolution on \(\text{NBV}_{\text{loc}}\) defined by \(\gamma(t) := (\alpha * \beta)(t) := \int_0^t \alpha(t - s) \, d\beta(s)\) satisfies that \(\gamma(t) \leq V_\alpha(\alpha)V_\beta(\beta)\). For proofs, see [43].

Let \(X\) be a Banach space with a coherent topology \(\tau\) and let \(\alpha \in \text{NBV}[0, R]\). A function \(f : [0, R] \mapsto X\) is \(\tau\)-Riemann–Stieltjes integrable with respect to \(\alpha\) if

\[
\int_0^R f(s) \, d\alpha(s) := \tau\lim_{|\pi| \to 0} \sum_{i=1}^n (\alpha(s_i) - \alpha(s_{i-1})) f(\xi_i)
\]

exists for every sequence of (refining) partitions \(\pi := \{0 = s_0 < \cdots < s_n = R\}\) of \([0, R]\), where \(\xi_i \in (s_i, s_{i-1})\) and \(|\pi| = \max_{1 \leq i \leq n} \{|s_i - s_{i-1}|\}\). The proof of the following lemma follows directly from the definitions.
Lemma 1.4. Let \( f : [0, R] \mapsto X \) be \( \tau \)-continuous, \( \alpha \in \text{NBV}[0, R] \), and \( B : (X, \tau) \mapsto (X, \tau) \) be linear and sequentially \( \tau \)-continuous on norm-bounded sets. Then \( \int_0^R f(t) \, d\alpha(t) \) exists, the map \( t \mapsto Bf(t) \) is \( \tau \)-continuous, and \( \int_0^R Bf(t) \, d\alpha(t) = B\int_0^R f(t) \, d\alpha(t) \). Moreover, if \( \phi \in (X, \tau)' \) then the map \( t \mapsto \langle f(t), \phi \rangle \) is continuous and

\[
\left\{ \int_0^R f(t) \, d\alpha(t), \phi \right\} = \int_0^R \langle f(t), \phi \rangle \, d\alpha(t).
\]

(1.2)

2. The operational calculus

In [23], E. Hille and R. Phillips developed a functional calculus for generators \( A \) of strongly continuous semigroups \( T \) of type \( (M, \omega) \) on a Banach space \( X \). The functional calculus applies to analytic functions \( f \) with a Laplace–Stieltjes representation \( f(z) = \int_0^\infty e^{zt} \, d\mu(t) \) for \( \text{Re}(z) \leq \omega \), where \( \mu \) belongs to the convolution Banach algebra of weighted bounded regular complex Borel measures on \( \mathbb{R}_0^+ \) of the form

\[
\mu(E) = \int_E e^{-\omega s} \, d\mu_0(s),
\]

(2.1)

where \( \mu_0 \) is a bounded regular complex Borel measure on \( \mathbb{R}_0^+ \). In this way,

\[
f(A)x := \int_0^\infty T(t)x \, d\mu(t)
\]

(2.2)
defines a bounded linear operator on \( X \), and the map \( \Phi : f \mapsto f(A) \) defines an algebra homomorphism from the algebra of Laplace–Stieltjes transforms of \( \mu \) of the form (2.1) into the algebra \( \mathcal{L}(X) \) of bounded linear operators on \( X \).

On the other hand, the Riesz representation theorem asserts that an element of the dual space of \( C_{0,\omega}(\mathbb{R}_0^+) := \{ x : \mathbb{R}_0^+ \mapsto \mathbb{C} \text{ continuous and } \lim_{t \to 0} e^{\omega t} x(t) = 0 \} \) with \( ||x||_{\infty,\omega} := \sup_{t \in \mathbb{R}_0^+} |e^{\omega t} x(t)| \) can be uniquely represented by a weighted bounded regular complex Borel measure on \( \mathbb{R}_0^+ \) of the form (2.1). Also, it is well known (see, e.g., [25]) that \( (C_{0,\omega}(\mathbb{R}_0^+))' \) is isometrically isomorphic, as Banach algebra, to \( \text{NBV}^\omega := \{ \alpha \in \text{NBV}_{\text{loc}} : ||\alpha||_{\omega} := \int_0^\infty e^{\omega t} \, dV_\alpha(t) < \infty \} \) with the Stieltjes-convolution as product. In other words, the Hille–Phillips functional calculus can be stated in terms of measures or in terms of normalized functions of bounded variation (see also [28, Theorem 6] or [38]). One reason for working with the Hille–Phillips functional calculus in terms of functions of bounded variation is simple, but strong: the integration by parts formula holds for functions of bounded variation. Define

\[
F_\omega := \left\{ f_\alpha : f_\alpha(z) := \int_0^\infty e^{zt} \, d\alpha(t) \text{ for } \text{Re}(z) \leq \omega, \text{ and some } \alpha \in \text{NBV}^\omega \right\}.
\]

Then \( \Phi : \text{NBV}^\omega \mapsto F_\omega \) defined by \( \Phi(\alpha) = f_\alpha \) is an algebra isomorphism between \( (\text{NBV}^\omega, +, \cdot) \) and \( (F_\omega, +, \cdot) \). Moreover, if \( \|f_\alpha\| := ||\alpha||_{\omega} \), then \( F_\alpha \) is a Banach algebra and the inclusion \( F_\omega \subset F_\kappa \) holds for \( \omega \geq \kappa \).

Proposition 2.1. Let \( T \) be a bi-continuous semigroup of type \( (M, \omega) \) on \( X \). If \( \alpha \in \text{NBV}^\omega \), then \( \lim_{R \to \infty} \int_0^R T(s)x \, d\alpha(s) \) exists and the map

\[
x \mapsto \int_0^\infty T(s)x \, d\alpha(s) := \lim_{R \to \infty} \int_0^R T(s)x \, d\alpha(s)
\]

is a bounded linear operator on \( (X, \| \cdot \|) \). Furthermore, if \( B \) is a continuous linear operator from \( (X, \tau) \) into \( (X, \tau) \), then \( B\int_0^\infty T(s)x \, d\alpha(s) = \int_0^\infty BT(s)x \, d\alpha(s) \).

Proof. Consider \( \alpha \in \text{NBV}^\omega \). By Lemma 1.4, \( x_\alpha := \int_0^R T(s)x \, d\alpha(s) \) exists for each \( R > 0 \). Applying Lemma 1.4 once again yields
\[ \|x_R - x_S\| = \left\| \int_R^S T(s)x \, d\alpha(s) \right\| = \sup_{\phi \in \Phi} \left\| \int_R^S T(s)x \, d\alpha(s), \phi \right\| = \sup_{\phi \in \Phi} \left\| \int_R^S \langle T(s)x, \phi \rangle \, d\alpha(s) \right\| \]

\[ \leq \sup_{\phi \in \Phi} \int_R^S \left\| T(s)x, \phi \right\| \, dV_\alpha(s) \leq \int_R^S \| T(s) \| \| x \| \, dV_\alpha(s) \leq M \| x \| \int_R^S e^{\alpha s} \, dV_\alpha(s) \to 0 \]

as \( R, S \to \infty \). Therefore, the net \( \{x_R\} \) is \( \| \cdot \| \)-Cauchy and thus norm-convergent. Since \( (X, \tau)' \subseteq (X, \| \cdot \|)' \), it follows that

\[ \left\langle \int_0^\infty T(s)x \, d\alpha(s), \phi \right\rangle = \int_0^\infty \langle T(s)x, \phi \rangle \, d\alpha(s) \quad (2.3) \]

for all \( \phi \in (X, \tau)' \). By using the norming property of \( (X, \tau)' \) and (2.3), the boundedness of the operator \( x \mapsto \int_0^\infty T(s)x \, d\alpha(s) \) follows from

\[ \left\| \int_0^\infty T(s)x \, d\alpha(s) \right\| = \sup_{\phi \in \Phi} \left\| \int_0^\infty T(s)x \, d\alpha(s), \phi \right\| = \sup_{\phi \in \Phi} \left\| \int_0^\infty \langle T(s)x, \phi \rangle \, d\alpha(s) \right\| \]

\[ \leq M \| x \| \int_0^\infty e^{\alpha s} \, dV_\alpha(s) = M \| x \| \| \alpha \|_{\omega}. \]

The second part of the statement is a consequence of Lemma 1.4. \( \square \)

**Theorem 2.2 (Hille–Phillips functional calculus).** If \( A \) generates a bi-continuous semigroup \( T \) of type \((M, \omega)\) on \( X \), then \( \Psi : F_\omega \to L(X) \) defined by

\[ \Psi(f)x := \int_0^\infty T(t)x \, d\alpha(t) \quad (2.4) \]

is an algebra homomorphism. Moreover, if \( f(A) := \Psi(f) \) then \( \| f(A) \| \leq M \| \alpha \|_{\omega} \), where \( \alpha \in NBV^\omega \) is such that \( f(z) = \int_0^\infty e^{zt} \, d\alpha(t) \) for \( \text{Re}(z) \leq \omega \).

The proof of Theorem 2.2 is analogous to the one given in [28, Theorem 6] by using Lemma 1.4, Proposition 2.1, and the norming property of the topology \( \tau \) (for details, see [26]).

### 3. Rational approximation of bi-continuous semigroups

As observed first by R. Hersh and T. Kato [18], the Hille–Phillips functional calculus is a powerful tool to study rational approximations of strongly continuous semigroups. This section shows that their results, as well as the results of P. Brenner and V. Thomée [4], extend fully to bi-continuous semigroups. Let

\[ H_0(s) := \begin{cases} 0 & \text{if } s = 0, \\ 1 & \text{if } s > 0, \end{cases} \quad \text{and} \quad H_t(s) := \begin{cases} 0 & \text{if } 0 \leq s < t, \\ \frac{1}{2} & \text{if } s = t, \\ 1 & \text{if } s > t. \end{cases} \quad (3.1) \]

Since \( H_t \in NBV^\omega \) for all \( \omega \in \mathbb{R} \) and \( t \geq 0 \), the functions \( z \mapsto e^{zt} = \int_0^\infty e^{zt} \, dH_t(s) \) are in \( F_\omega \) \((\omega \in \mathbb{R}, t \geq 0)\). In particular, the constant functions are in \( F_\omega \) for all \( \omega \in \mathbb{R} \). Also, rational functions that are bounded for \( \text{Re}(z) \leq \omega \) are in \( F_\omega \). In order to see this, notice that there are \( b_j \in \mathbb{C} \) with \( \text{Re}(b_j) > \omega \) and \( B_{i,j} \in \mathbb{C} \) such that

\[ r(z) = B_{0,0} + \sum_{1 \leq i,j} \frac{B_{i,j}}{(b_i - z)^j} \quad (\text{Re}(z) \leq \omega). \]
Since $F_\omega$ is an algebra and since $z \mapsto \frac{1}{n-z} = \int_0^\infty e^{zs} d\alpha_i(s) \in F_\omega$, where $\alpha_i(s) = \int_0^s e^{-h_s} ds$, it follows that $r \in F_\omega$.

Moreover, if $t > 0$ and $n \in \mathbb{N}$, then

$$r^n \left( \frac{1}{n} \right) = \int_0^\infty e^{zs} d\alpha_{n,t}(s),$$

(3.2)

where $\alpha_{n,t}(s) := \alpha^{*n} (\frac{s}{t})$ and $\alpha^{*n}$ denotes the $n$th convolution product $\alpha \ast \cdots \ast \alpha$.

Recall that a rational function $r$ is called $\mathcal{A}$-stable if $|r(z)| \leq 1$ for $\text{Re}(z) \leq 0$. If, in addition, $r(z) = e^z + o(z)$ as $z \to 0$, then $r$ is said to be $\mathcal{A}$-acceptable. Moreover, $r$ is an approximation to the exponential function of order $q \geq 1$ if

$$r(z) = e^z + O \left( |z|^{q+1} \right)$$

(3.3)

as $z \to 0$. Notice that an $\mathcal{A}$-stable rational approximation to the exponential function of order $q \geq 1$ is $\mathcal{A}$-acceptable.

By the above, if $r$ is an $\mathcal{A}$-stable rational function that approximates the exponential function of order $q$, then

$$f : z \mapsto \frac{r^n \left( \frac{1}{n} \right) - e^{zt}}{z^{k+1}} \in F_0$$

(3.4)

for every $k \in \{0, \ldots, q\}$, $n \in \mathbb{N}$ and $t \geq 0$. Finally, an $\mathcal{A}$-acceptable rational function $r$ is said to satisfy the condition $(\ast)$ if the following two conditions hold:

(a) $|r(i\xi)| < 1$ for $0 \neq \xi \in \mathbb{R}$ and $|r(\infty)| < 1$.

(b) There exist positive integers $p, q$, where $p$ is even, $p \geq q + 1$, and a positive number $\gamma$ such that $r(i\xi) = e^{i\xi + \psi(\xi)}$ with $\psi(\xi) = O(|\xi|^{q+1})$ as $\xi \to 0$; and $\text{Re}(\psi(\xi)) \leq -\gamma \xi^p$ for $|\xi| \leq 1$.

It is known that condition (3.3) together with condition (a) imply that (b) is satisfied; in this case, the order of approximation $q$ given by (3.3) coincide with the order estimate for $\psi$ near zero, see [4].

Let $I^{0}$ be the identity on $\text{NBV}_{\text{loc}}$ and let $I^{k}$ denotes $k$th-antiderivative; i.e., $I[\alpha](t) = \int_0^t \alpha(\xi) \, d\xi$. The following result summarizes crucial estimates obtained in [4] in terms of functions of bounded variation (compare with [29, Theorem 16]).

**Theorem 3.1 (Brenner–Thomée).** If $r(z) = \int_0^\infty e^{zs} d\alpha(s)$ (Re$(z) \leq 0$) is an $\mathcal{A}$-stable rational function, then there exists $K > 0$ such that

$$\|\alpha^n\|_0 \leq K \sqrt{n} \quad \text{for all } n \in \mathbb{N}. \quad (3.5)$$

If, in addition, $r$ is $\mathcal{A}$-acceptable and satisfies the condition $(\ast)$ then there is a constant $K$ such that

$$\|\alpha^n\|_0 \leq K n^{\frac{1}{2} \left( 1 - \frac{q+1}{p} \right)} \quad \text{for all } n \in \mathbb{N}. \quad (3.6)$$

Furthermore, if $f$ is as in (3.4), then there exist $\beta_{k,n,t} \in \text{NBV}_{\text{loc}}$ and $K > 0$ such that

$$f(z) = \int_0^\infty e^{zs} d\beta_{k,n,t}(s) \quad \text{for } \text{Re}(z) \leq 0$$

and

$$\text{Var}(\beta_{k,n,t}) = \left\| I^{k-1}[\alpha_{n,t} - H_t] \right\|_{L^1(\mathbb{R}^+)} \leq K t^k \left( \frac{1}{n} \right)^{\theta_q(k)} \quad (3.7)$$

for every $k \in \{1, \ldots, q+1\}$, except for $k = \frac{q-1}{2}$ in which case

$$\text{Var}(\beta_{k,n,t}) = \left\| I^{k-1}[\alpha_{n,t} - H_t] \right\|_{L^1(\mathbb{R}^+)} \leq K t^{k} \left( \frac{1}{n} \right)^{\theta_q(k)} \ln(n+1), \quad (3.8)$$

where $\theta_q(k) = k \frac{q}{q+1} + \min \left\{ 0, k - \frac{1}{2} \right\}$. If, in addition, $r$ satisfies the condition $(\ast)$ then $\theta_q$ can be replaced by $\theta_q^*(k) = k \frac{q}{q+1} + \min \left\{ 0, (k - \frac{1}{2})(q+1)(\frac{1}{q+1} - \frac{1}{p}) \right\}$. 


Proof. Let \( r(z) = \int_0^\infty e^{z \sigma} d\sigma(s) \) (Re(z) \( \leq 0 \)) for some \( \alpha \in \text{NBV}_{\text{loc}} \) and let \( \mu \) be the bounded regular complex Borel measure associated with
\[
\alpha_0(t) := \begin{cases} 
\alpha(t), & t \geq 0, \\
0, & t < 0. 
\end{cases}
\] (3.9)

Define \( r(0)(t) := \tilde{\mu}(t) \) (the Fourier transform of \( \mu \)). If \( m(\tilde{\mu}) := \int_0^\infty d\mu(t) \), then \( m(r(0)) = m(\tilde{\mu}) = \|\alpha\|_0 \). Thus, \( m(r^n_0) = \|\alpha^n\|_0 \) and (3.5), (3.6) follow from the proof of Theorems 1 and 2 of [4] with \( \omega = 0 \).

On the other hand, from (3.4) it follows that there exists \( \beta_{k,n,t} \in \text{NBV}_{\text{loc}} \) such that \( f(z) = \int_0^\infty e^{z \sigma} d\beta_{k,n,t}(s) \) (Re(z) \( \leq 0 \)). As above, let \( \mu \) be the bounded regular complex Borel measure associated with \( \beta_{k,n,t} \) and let \( f(0) = \tilde{\mu} \) (the Fourier transform of \( \mu \)). Then, it follows from [4] that \( m(f(0)) = m(\tilde{\mu}) = \|\beta_{k,n,t}\|_0 \). Therefore, (3.7) and (3.8) follow from the proof of Theorem 4 and Remark 3 of [4] with \( \omega = 0 \). Finally, Theorem 1.4 of [25] shows that
\[
\text{Var}(\beta_{k,n,t}) = \|I^{k-1}[\alpha_{n,t} - H_I]\|_{L^1(\mathbb{R}^d)}. \quad \Box
\]

The estimation (3.5) is sharp. For instance, if \( r(z) = \frac{2z^2}{z^2+1} = -1 + \frac{4}{z-\frac{1}{2}} \in F_0 \) (Crank–Nicolson), then it follows from [8, (2.2)] that there are \( k_1, k_2 > 0 \) such that
\[
k_1\sqrt{n} \leq \alpha_{CN}^{\infty,0}_n \leq k_2\sqrt{n} \quad \text{for all } n \in \mathbb{N},
\] (3.10)

where \( \alpha_{CN}(s) = -H_0(s) + 4\int_0^1 e^{-2\xi} d\xi \).

Theorem 3.2. Let \( T \) be a bi-continuous semigroup of type \((M, 0)\) with generator \( A \) and let \( r \) be an \( \mathcal{A}\)-stable rational function. Then there exists a constant \( K > 0 \) such that
\[
\|r^n(\rho A)\|_{\mathcal{L}^1(X)} \leq KMn^{\frac{1}{2}}, \quad \rho > 0, \ n \in \mathbb{N}.
\]

If, in addition, \( r \) is \( \mathcal{A}\)-acceptable and satisfies (⋆), then
\[
\|r^n(\rho A)\|_{\mathcal{L}^1(X)} \leq KMn^{\frac{1}{2}(1 - \frac{4q+1}{p})}, \quad \rho > 0, \ n \in \mathbb{N}.
\]

Proof. Let \( n \in \mathbb{N}, \rho > 0 \) and \( x \in X \). Theorem 2.2 and (2.3) yield
\[
\|r^n(\rho A)x\| = \sup_{\phi \in \Phi} \left| \|r^n(\rho A)x, \phi\| \right| = \sup_{\phi \in \Phi} \left| \int_0^\infty T(\rho s)x \, d\alpha^n(s, \phi) \right| = \sup_{\phi \in \Phi} \left| \int_0^\infty \langle T(\rho s)x, \phi \rangle \, d\alpha^n(s) \right| \leq \sup_{\phi \in \Phi} \int_0^\infty \left| \langle T(\rho s)x, \phi \rangle \right| \, d\alpha^n(s) \leq \int_0^\infty \|T(\rho s)x\| \, d\alpha^n(s) \leq M\|\alpha^n\|_0\|x\|.
\]

Now, the statement follows from Theorem 3.1. \( \Box \)

For instance, one can consider the Padé approximants of the exponential function of order \( j + l \) given by \( r_{j,l}(z) = \frac{P_j(z)}{Q_l(z)} \), where \( \deg(P_j) = l \) and \( \deg(Q_l) = j \). G. Wanner, E. Hairer and S.P. Nørsett showed in [19] that \( r_{j,l} \) is an \( \mathcal{A}\)-stable approximation of the exponential function of order \( q = j + l \) if and only if \( 0 \leq j - l \leq 2 \). In this case, condition (⋆)(b) holds with \( p = 2j \), see [20]. Therefore, Theorem 3.2 implies that if \( A \) is the generator of a bi-continuous semigroup of type \((M, 0)\), then
\[
\|r^n_{j,l-1}(\rho A)\|_{\mathcal{L}^1(X)} \leq C \quad \text{and} \quad \|r^n_{j,l-2}(\rho A)\|_{\mathcal{L}^1(X)} \leq Cn^{\frac{1}{2}}
\] (3.11)

for \( j \in \mathbb{N}_0 \) and \( \rho > 0 \). Similarly, in the case of the restricted Padé approximants
\[
r_j(z) := (-1)^j(1 - \gamma z)^{-j} \sum_{m=0}^j L_j^{(j-m)}(\gamma^{-1})(\gamma z)^m,
\]

where \( L_j \) is the Laguerre polynomial of degree \( j \), and \( \gamma \) is chosen such that \( r_j \) is an approximation of order \( q = j + 1 \), one obtains that the Calahan scheme given by \( r_2(\rho A) \) is stable, and the norm of \( r_3^j(\rho A) \) grows as \( O(n^{\frac{1}{2}}) \), see [26]. The next result shows the sharpness of the first estimate of Theorem 3.2.
Theorem 3.3. Let $X = C_{b}(\mathbb{R}_0^+)$ and let $T$ be the shift (bi-continuous) semigroup with generator $A = \frac{d}{dz}$. If $r(z) = \frac{2+z}{2-z}$ then there exists $K > 0$ such that

$$\| r^n(\rho A) \|_{\mathcal{L}(X)} \geq K \sqrt{n}, \quad \rho > 0, \ n \in \mathbb{N}.$$ 

Proof. Let $\Phi$ be the set of linear functionals $\phi_a : x \mapsto x(a) (a \in \mathbb{R}_0^+)$. Then $\| x \| = \sup_{\phi_a \in \Phi} | \langle x, \phi_a \rangle |$ for each $x \in X$. Now, let $\rho > 0$ and let $x \in X$ be fixed. From Theorem 2.2 and (2.3) it follows that

$$\| r^n(\rho A) x \| = \sup_{\phi_a \in \Phi} \left| \left( r^n(\rho A) x, \phi_a \right) \right| = \sup_{\phi_a \in \Phi} \left| \int_0^\infty T(\rho s) x d\alpha^n(s), \phi_a \right|$$

$$= \sup_{\phi_a \in \Phi} \left| \int_0^\infty T(\rho s) x d\alpha^n(s) \right| = \sup_{a \in \mathbb{R}_0^+} \left| \int_0^\infty x(a + \rho s) d\alpha^n(s) \right|.$$ 

Thus, $\| r^n(\rho A) \|_{\mathcal{L}(X)} \leq \| \alpha^n \|_0$. Notice that $| x_{a, \rho} | \leq \| x_0 \| = \| x \|$ for all $\rho > 0$ and $a \in \mathbb{R}_0^+$, where $x_{a, \rho} := s \mapsto x(a + \rho s)$. Since NBV$_{\text{loc}}$ and the dual of $C_0(\mathbb{R}_0^+)$ are isometric isomorphic Banach algebras (see, e.g., [26]), it follows that

$$\| \alpha^n \|_0 = \sup_{x \in \mathcal{C}_0(\mathbb{R}_0^+)} \| (x, \alpha^n) \| \leq \sup_{x \in \mathcal{C}_0(\mathbb{R}_0^+)} \sup_{\| x \| \leq 1} \| x, \alpha^n \| = \sup_{x \in \mathcal{C}_0(\mathbb{R}_0^+)} \sup_{\| x \| \leq 1} \| x_{a, \rho}, \alpha^n \|$$

$$\leq \sup_{x \in \mathcal{C}_0(\mathbb{R}_0^+)} \sup_{a \in \mathbb{R}_0^+} \| x_{a, \rho}, \alpha^n \| = \sup_{x \in \mathcal{C}_0(\mathbb{R}_0^+)} \sup_{\| x \| \leq 1} \| r^n(\rho A) x \| = \| r^n(\rho A) \|_{\mathcal{L}(X)}.$$ 

(3.13)

It follows from (3.12) and (3.13) that $\| r^n(\rho A) \|_{\mathcal{L}(X)} = \| \alpha^n \|_0$, and the theorem follows from (3.10).

Theorem 3.4. Let $T$ be a bi-continuous semigroup of type $(M, 0)$ generated by $A$. If $r(z) = \int_0^\infty e^{zs} da(s)$ ($\operatorname{Re}(z) \leq 0$) is an $\mathcal{A}$-stable rational approximation of the exponential function of order $q \geq 1$, then there exists $K > 0$ such that

$$\left\| r^n \left( \frac{t}{n} A \right) x - T(t)x \right\| \leq MK^k \left( \frac{1}{n} \right)^{\theta_q(k)} \| A^k x \|,$$

(3.14)

for every $k \in \{1, \ldots, q+1\}$, $k \neq \frac{q-1}{2}$, and $x \in D(A^k)$. If $k = \frac{q-1}{2}$ then

$$\left\| r^n \left( \frac{t}{n} A \right) x - T(t)x \right\| \leq MK^k \left( \frac{1}{n} \right)^{\theta_q(k)} \ln(n+1) \| A^k x \|.$$ 

(3.15)

If, in addition, $r$ satisfies the condition $(\ast)$, then $\theta_q$ can be replaced by $\theta_q^\ast$.

Proof. Let $x \in D(A^k)$ and let $\phi \in \Phi$. It follows that $t \mapsto \langle T(t)x, \phi \rangle$ is $k$-times differentiable and $\frac{d^k}{dt^k} \langle T(t)x, \phi \rangle = \langle T(t) A^k x, \phi \rangle$. Now, by Theorem 2.2, (2.3), and integrating by parts $k$-times,

$$\left\| r^n \left( \frac{t}{n} A \right) x - T(t)x \right\| = \sup_{\phi \in \Phi} \left\| r^n \left( \frac{t}{n} A \right) x - T(t)x, \phi \right\| = \sup_{\phi \in \Phi} \left\| \int_0^\infty T(s)x d\alpha_n(s) - \int_0^\infty T(s)x dH_t(s), \phi \right\|$$

$$= \sup_{\phi \in \Phi} \left\| \int_0^\infty T(s)x d[\alpha_n, H_t](s), \phi \right\| = \sup_{\phi \in \Phi} \int_0^\infty \langle T(s)x, \phi \rangle d[\alpha_n, H_t](s)$$

$$= \sup_{\phi \in \Phi} \int_0^\infty I^{(k-1)}(\alpha_n, H_t)(s) \frac{d^k}{ds^k} \langle T(s)x, \phi \rangle ds.$$
function of order $q$

By assumption, $\|an\|$ is an approximation to the exponential function of order $q$.


Let $\Theta$.

A. Albanese and E. Mangino in [1] provides convergence in the $\tau$ topology for all $x \in X$.

**Theorem 3.5.** Let $A$ be the generator of a bi-continuous semigroup $T$ of type $(M, 0)$. If $r(z) = \int_0^\infty e^{zt} d\alpha(s)$ $(\text{Re}(z) \leq 0)$ is an $A$-stable rational approximation to the exponential function of order $q \geq 1$ for which $\alpha^{m*}$ is uniformly bounded, then

$$T(t)x = \lim_{n \to \infty} r_n\left(\frac{t}{n} A\right)x,$$

(3.16)

for all $x \in X$, where the limit is uniform for compact intervals of $\mathbb{R}^+_{0}$.

**Proof.** It has to be shown that the conditions of the Chernoff product formula (see [1, Theorem 4.1]) hold. Since $r$ is an approximation to the exponential function of order $q \geq 1$, it follows that $r(0) = 1$. Thus, by Theorem 2.2, $r(0) = \text{Id}$. By assumption, $\|r(hA)^m\| \leq C$ ($h \geq 0$, $m \in \mathbb{N}$). In order to show uniform bi-equicontinuity of $\{r^m(hA): h \geq 0\}$ in $m \in \mathbb{N}$, consider a $\|\cdot\|$-bounded sequence $\{x_n\}_{n \in \mathbb{N}}$ which is $\tau$-convergent to $x \in X$. Let $\varepsilon > 0$, $p \in \mathcal{P}$, $m \in \mathbb{N}$, and choose $R > 0$ such that $p\left(\int_R^\infty T(s)(x_n - x) d\alpha_h^m(s)\right) < \varepsilon$. It follows from Theorem 2.2 and the local bi-equicontinuity of $T$ that

$$p\left(\int_0^R T(s)(x_n - x) d\alpha_h^m(s)\right) \leq M \varepsilon \|\alpha_h^{m*}\|_0 = M \varepsilon \|\alpha^{m*}\|_0,$$

and the uniform bi-equicontinuity in $m$ follows. Now, from Lemma 7(b) of [31] and by considering $x_n = n^2 R(n, A)^2 \in D(A^2)$ one obtains that $\lim_{n \to \infty} x_n = x$ for every $x \in X$; i.e., $D(A^2)$ is bi-dense in $X$. On the other hand, Theorem 3.4 with $k = 2$ and $n = 1$ shows that $\lim_{t \to 0^+} \frac{V(t)x - x}{t} = 0$, for all $x \in D(A^2)$. Thus, $\lim_{t \to 0^+} \frac{V(t)x - x}{t} = Ax$, for all $x \in D(A^2)$. Finally, since $\|\cdot\|$-convergence implies $\tau$-convergence, one obtains that for all $x \in D(A^2),

$$Ax = \lim_{t \to 0^+} \frac{V(t)x - x}{t}.$$

Therefore, all the conditions of [1, Theorem 4.1] are satisfied, and the result follows. 

In particular, Theorem 3.5 shows convergence for all subdiagonal Padé approximants $r_{j, j-1}$ to the exponential function in the $\tau$ topology.

Notice that if the bi-continuous semigroup is of type $(M, \omega)$ then all the previously obtained estimates will carry over the extra term $e^{\omega t}$.

**4. Applications**

The results of the previous sections can be applied to bi-continuous semigroups such as adjoint semigroups, jointly continuous semigroups generated by nonlinear flows, Feller semigroups such as the Ornstein–Uhlenbeck semigroup, the heat semigroup, induced semigroups, implemented semigroups, and certain evolution semigroups. This section illustrates the applications of the results of the previous sections by considering two examples.
4.1. Semigroups induced by flows

A jointly continuous flow on a metrizable (or locally compact) topological space $\Omega$ is a map $\phi : \mathbb{R}_0^+ \times \Omega \mapsto \Omega$, $\phi_t(v) := \phi(t,v)$, that is jointly continuous and satisfies the semigroup property $\phi_{t+s} = \phi_t \circ \phi_s$, $\phi_0 = \text{Id}$. If $X := C_b(\Omega)$, then the semigroup $S$ defined by $S(t) : x \mapsto x \circ \phi_t$ is called the semigroup induced by the flow $\phi$. F. Kühnemund shows in [30,31] that these semigroups, considered also by J.R. Dorroh and J.W. Neuberger in [10–12,37], are bi-continuous on $C_b(\Omega)$ and

$$
\tau \lim_{n \to \infty} \left[ \frac{n}{t} R \left( \frac{n}{t}, A \right) \right] x = S(t)x
$$

for all $x \in X$, where the limit of (4.1) is uniform for $t$ in compact sets of $\mathbb{R}_0^+$.

In the context of semigroups induced by flows, the backward-Euler formula (4.1) was obtained in [11, Theorem 3.2]. Now, if one considers $r_{BE}(z) = \frac{1}{1 - nz}$ for $\text{Re}(z) \leq 0$, then $r_{BE}$ is an $A$-stable rational approximation to the exponential function of order $q = 1$. Moreover, $\alpha_{BE}(s) = \int_0^s e^{-\xi} d\xi$, and from Theorem 3.5 one obtains that (3.16) coincides with the backward-Euler formula (4.1). Furthermore, if $x \in D(A)$ and $k = 1$, Theorem 3.4 provides error estimates in the norm sense for the backward-Euler approximation (4.1); i.e., the $\tau$-limit can be replaced by the $\| \cdot \|$-limit for $x \in D(A)$. Thus, convergence of the backward-Euler scheme is obtained in the original norm and the error is $O(\frac{\tau}{\sqrt{n}})$ as $n \to \infty$. However, Theorem 3.4 also shows that the backward-Euler scheme is the slowest, in terms of convergence, among the Padé approximants to the exponential function.

In order to illustrate this, consider the bi-continuous semigroup $S$ generated by the operator $Ax(s) = s^{\frac{3}{2}} x'(s)$ with maximal domain, cf. [11, Example 4.2]; i.e., $S(t)x(s) = x(\phi_t(s))$ is the semigroup induced by the flow $\phi_t(s) = (s^{\frac{1}{3}} + \frac{t}{3})^3$. The approximation of the flow $\phi_t$ through (3.16) on an interval $[a, b]$ can be done by considering any $x \in C_b(\Omega)$ such that $x(s) = s$ for $s \in [a, b]$. Fig. 1(a) shows that the backward-Euler scheme for $n = 10$ yields an approximation to $S(t)x$ that is accurate up to one decimal place for $t \in [0, 1]$ and $s \in [0, 4]$. Fig. 1(b) shows the approximation error of the scheme $r^n(\frac{L}{n} A)x$ defined via the subdiagonal Padé scheme

$$
r_{4,3}(z) = \frac{210 - 90z + 15z^2 - z^3}{840 + 480z + 120z^2 + 16z^3 + z^4}
$$

with $n = 3$. It follows that the scheme given by $r_{4,3}$ with $n = 3$ is accurate up to ten decimal places for $t \in [0, 1]$ and $s \in [0, 4]$.
4.2. Inversion of the vector-valued Laplace transform

Let $X$ be a Banach space and $X := C_{b}(\mathbb{R}_{0}^{+}; X)$. Then, the shift semigroup $T(t) f(s) := f(t + s)$ generated by $D = \frac{d}{dt}$ is bi-continuous with respect to the topology of uniform convergence on compact sets of $\mathbb{R}_{0}^{+}$. Notice that the shift semigroup is a semigroup induced by a flow. It follows from Theorem 3.4 that if $r$ is an $\mathcal{A}$-stable rational approximation to the exponential function of order $q \geq 1$, then for $f \in D(\frac{d}{dt})$ there exists $K > 0$ such that

$$
\| r^n \left( \frac{t}{n} D \right) f - T(t)f \| \leq K t^k \left( \frac{1}{n} \right)^{\theta_q(k)} \| f \|_{\infty}
$$

(4.2)

(with obvious modifications if $r$ satisfies the condition (⋆)). In particular,

$$
\| r^n \left( \frac{t}{n} D \right) f(0) - f(t) \| \leq K t^k \left( \frac{1}{n} \right)^{\theta_q(k)} \| f \|_{\infty}.
$$

(4.3)

Since $R(\lambda, D) f = \int_0^\infty e^{-\lambda t} T(t) f \, dt = \int_0^\infty e^{-\lambda t} f(t + \cdot) \, dt$, it follows that $R(\lambda, D) f(0) = \int_0^\infty e^{-\lambda t} f(t) \, dt = \hat{f}(\lambda)$ (the Laplace transform of $f$). Consequently,

$$
R(\lambda, D)^{n+1} f(0) = \frac{(-1)^n}{n!} R(\lambda, D)^{n} f(0) = \frac{(-1)^n}{n!} \int_0^\infty e^{-\lambda t} (-t)^n f(t) \, dt = \frac{(-1)^n}{n!} \hat{f}^{(n)}(\lambda).
$$

Now, let

$$
r(z) = B_{0,0} + \sum_{1 \leq i \leq s \atop 1 \leq j \leq r} \frac{B_{i,j}}{(b_i - z)^j}
$$

be an $\mathcal{A}$-stable rational approximation to the exponential function of order $q \geq 1$. Then, for each $n \in \mathbb{N}$, there exist constants $C_{i,j}^n$ such that

$$
r^n(z) = C^n_{0,0} + \sum_{1 \leq i \leq s \atop 1 \leq j \leq nr} \frac{C_{i,j}^n}{(b_i - z)^j}.
$$

In particular,

$$
r^n \left( \frac{t}{n} D \right) f(0) = C^n_{0,0} f(0) + \sum_{1 \leq i \leq s \atop 1 \leq j \leq nr} C_{i,j}^n R^j(b_i, \frac{t}{n} D) f(0)
$$

$$
= C^n_{0,0} f(0) + \sum_{1 \leq i \leq s \atop 1 \leq j \leq nr} C_{i,j}^n \left( \frac{n}{t} \right)^j R^j(b_i \frac{n}{t}, D) f(0)
$$

$$
= C^n_{0,0} f(0) + \sum_{1 \leq i \leq s \atop 1 \leq j \leq nr} C_{i,j}^n \left( \frac{n}{t} \right)^j \frac{(-1)^j}{(j-1)!} \hat{f}^{(j-1)} \left( \frac{n}{t} b_i \right).
$$

This yields the following novel inversion formulas for the Laplace transform which have, due to the rapid speed of convergence for large $n$ and the availability of error estimates, significant applications to evolution equations.

**Theorem 4.1 (Laplace Transform Inversion).** Let $\hat{f}$ be the Laplace transform of $f \in C_{b}(\mathbb{R}_{0}^{+}; X)$ and define $f_0 := \lim_{\lambda \to \infty} \lambda \hat{f}(\lambda)$. If $r$ is an $\mathcal{A}$-stable rational approximation to the exponential function of order $q \geq 1$, then there exist $K > 0$, $b_i \in \mathbb{C}$ with $\text{Re}(b_i) > 0$ and constants $C^n_{i,j} \in \mathbb{C}$ (independent of $f$) such that

$$
\left\| C^n_{0,0} f_0 + \sum_{1 \leq i \leq s \atop 1 \leq j \leq nr} C_{i,j}^n \left( \frac{n}{t} \right)^j \frac{(-1)^j}{(j-1)!} \hat{f}^{(j-1)} \left( \frac{n}{t} b_i \right) - f(t) \right\|_{\mathcal{X}} \leq K \frac{t^k}{n^{\theta_q(k)}} \| f \|_{\infty}.
$$

(4.4)
for every $k \in \{1, \ldots, q+1\}$, and $k \neq \frac{q-1}{2}$. If $k = \frac{q-1}{2}$ then

$$
\left\| C_{n,0}^n f_0 + \sum_{1 \leq i \leq s \atop 1 \leq j \leq nr} C_{i,j}^n \left( \frac{n}{t} \right)^{(j-1)(j-1)!} \hat{f}(t) \right\|_{X} \leq K \frac{t^k}{n^{\theta_q(k)}} \ln(n+1) \| f^{(k)} \|_{\infty}.
$$

(4.5)

If $r$ satisfies the condition (⋆), then $\theta_q$ can be replaced by $\theta^*_q$ (where $\theta_q$ and $\theta^*_q$ are as in Theorem 3.1).

The stiff ordinary differential equation of C.F. Curtiss and J.O. Hirschfelder given by $y' = -50(y - \cos(t))$, $y(0) = 1$, with solution $y(t) = \frac{2500}{2501} \cos(t) + \frac{50}{2501} \sin(t) + \frac{1}{2501} e^{-50t}$ ($t \geq 0$) is known to be a good test for computational algorithms, cf. [24]. Fig. 2(a) shows the error of the backward-Euler approximation to the solution of the Curtiss–Hirschfelder equation for $t \in [0, 3]$ and $n = 3$. The backward-Euler approximation to the solution of the Curtiss–Hirschfelder equation is accurate up to zero decimal places for $t \in [0, 3]$ and $n = 3$. Fig. 2(b) shows that the error of the subdiagonal Padé scheme provided by $r_{4,3}$ yields an accuracy of at least five decimal places for $t \in [0, 3]$ with $n = 3$. In other words, if the $b_i$’s are the four roots of $840 + 480z + 120z^2 + 16z^3 + z^4$, then by using $\hat{f}$, $\frac{\partial}{\partial t} \hat{f}$ and $\frac{d^2}{dt^2} \hat{f}$ in (4.4) by means of evaluating them at just 12 points of the form $\frac{n}{t} b_i$ for each $t \in [0, 3]$, one ensures an error less than $2.5 \times 10^{-6}$ on the interval $[0, 3]$.

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