JOURNAL OF DIFFERENTIAL EQUATIONS 15, 295–307 (1974)

Coincidence Degree and Periodic Solutions of Neutral Equations

JACK K. HALE*

Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912

AND

J. MAWHIN†

Université Catholique de Louvain, Institut Mathématique, B-1348 Louvain-la-Neuve, Belgium

Received April 18, 1973

1. INTRODUCTION

This paper is devoted to the problem of existence of periodic solutions for some nonautonomous neutral functional differential equations. It is essentially an application of a basic theorem on the Fredholm alternative for periodic solutions of some linear neutral equations recently obtained by one of the authors [2] and of a generalized Leray–Schauder theory developed by the second one [3, 4]. Although their proofs are surprisingly simple, the obtained results are nontrivial extensions to the neutral case of a number of recent existence theorems for periodic solutions of functional differential equations. In particular, Section 3 generalizes some existence criteria due to one of the authors [5] and a corresponding recent extension by J. Cronin [6], the example following Theorem 4.1 improves a condition for existence given by Lopes [14] for the equation of a transmission line problem, and Theorem 5.1 generalizes a result due to R. E. Fennell [7]. Lastly, criteria analogous to Theorem 5.2 for the retarded case can be found in [8]. For partly related results concerning periodic solutions of neutral functional differential equations, see [9].

* Research supported in part by the Air Force Office of Scientific Research under AF-AFOSR 71-2078, in part by the National Aeronautics and Space Administration under NGL 40-002-015 and in part by the Army Research Office under DA-ARO-D-31-124-71-G12S2.

† Work realized in part when the co-author was visiting the Division of Applied Mathematics at Brown University.

2. Fredholm Alternative for Linear Equations

Let $C([a, b], \mathbb{R}^n)$ be the space of continuous functions from $[a, b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. For a fixed $r \geqslant 0$, let $C = C([-r, 0], \mathbb{R}^n)$ with norm $|\varphi| = \sup_{-r\leqslant\theta\leqslant 0} |\varphi(\theta)|$ for $\varphi \in C$. If $x \in C([\sigma - r, \sigma + \delta], \mathbb{R}^n)$ for some $\delta > 0$, let $x_t \in C$, $t \in [\sigma, \sigma + \delta]$ be defined by $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$. Suppose $\omega > 0$ fixed, $A: \mathbb{R} \times C \to \mathbb{R}^n$ is continuous, $A(t + \omega)\varphi = A(t)\varphi$ for all $(t, \varphi) \in \mathbb{R} \times C$, $A(t)\varphi$ is linear in φ and there exists a continuous function γ: $[0, \infty) \to \mathbb{R}$, $\gamma(0) = 0$, such that

$$|A(t)\varphi^s| \leqslant \gamma(s)\, |\varphi^s|, \qquad 0 \leqslant s \leqslant r$$

for all $t \in \mathbb{R}$ and all functions $\varphi^s \in C$ such that $\varphi^s(\theta) = 0$ for $\theta \in [-r, -s]$. Let D: $\mathbb{R} \times C \to \mathbb{R}^n$ be defined by $D(t)\varphi = \varphi(0) - A(t)\varphi$. The operator D is said to be *stable* if the zero solution of the functional equation $D(t)\, y_t = 0$ is uniformly asymptotically stable; that is, there are constants K, $\alpha > 0$ such that if $y(\varphi)$ is the solution of $D(t)\, y_t = 0$ with $y_0 = \varphi$, then

$$|y_t(\varphi)| \leqslant Ke^{-\alpha t} |\varphi|, \qquad t \geqslant 0, \quad \varphi \in C. \tag{2.1}$$

Let $\mathscr{P}_\omega = \{x \in C(\mathbb{R}, \mathbb{R}^n): x(t + \omega) = x(t),\ t \in \mathbb{R}\}$, $\mathscr{H}_\omega = \{H \in C(\mathbb{R}, \mathbb{R}^n): H(0) = 0$ and $H(t) = \alpha t + h(t)$ for some $\alpha \in \mathbb{R}^n$, $h \in \mathscr{P}_\omega\}$. For any $h \in \mathscr{P}_\omega$, let $|h| = \sup_{0\leqslant t\leqslant\omega} |h(t)|$ and for any $H \in \mathscr{H}_\omega$, $H(t) = \alpha t + h(t)$, $\alpha \in \mathbb{R}^n$, $h \in \mathscr{P}_\omega$, let $|H| = |\alpha| + |h|$.

Proposition 2.1. *If D is stable, then, for any $c \in \mathbb{R}^n$, there is a unique solution Mc of the equation $D(t)x_t = c$ in $\mathscr{P}_\omega$. Furthermore, M is a continuous linear operator from $\mathbb{R}^n$ to $\mathscr{P}_\omega$.*

Proof. Following the proof of Lemma 3.4 in [10], there are constants $b > 0$, $a > 0$ and an appropriate equivalent norm in C such that the solution $x(\varphi, c)$ of $D(t)x_t = c$, $x_0 = \varphi$, satisfies $|x_t(\varphi, c)| \leqslant |c|b + |\varphi| \exp(-at)$, $t \geqslant 0$, $\varphi \in C$, $c \in \mathbb{R}^n$. If $T\varphi = x_\omega(\varphi, c)$, then T is a contraction mapping. Thus, if $d > 0$ is sufficiently large that $|c|b + d\exp(-a\omega) < d$, then T has a unique fixed point such that $|\varphi| < d$. Consequently, there is a solution of the equation in $\mathscr{P}_\omega$. The fact that D is stable implies the uniqueness, linearity, and continuous dependence on c.

Let us rephrase Proposition 2.1 in a different way. Let L: $\mathscr{P}_\omega \to \mathscr{H}_\omega$ be the continuous linear mapping defined by

$$Lx(t) = D(t)\, x_t - D(0)x_0, \qquad t \in \mathbb{R}.$$

Proposition 2.1 implies that

$$\ker L = \{x \in \mathscr{P}_\omega : \exists c \in \mathbb{R}^n \text{ with } x = Mc\}$$

is an n-dimensional subspace of $\mathscr{P}_\omega$. Let $P: \mathscr{P}_\omega \to \mathscr{P}_\omega$ be a continuous projection onto $\ker L$.

For the statement of the next proposition, let $Q: \mathscr{H}_\omega \to \mathscr{H}_\omega$ be the continuous projection defined by

$$QH(t) = \omega^{-1} H(\omega) t, \qquad t \in \mathbb{R}.$$

PROPOSITION 2.2. *If D is stable, then* $\operatorname{Im} L = \ker Q$ *and there is a continuous linear operator* $K: \operatorname{Im} L \to \ker P$ *such that K is a right inverse of L. Thus, L is a Fredholm operator with index* 0.

Proof. The second proof given in [2] of the Fredholm alternative holds equally well for the equation $D(t)x_t = H(t)$. Thus, from [2] $\dim \ker L = \operatorname{codim} \operatorname{Im} L$. Proposition 2.1 implies $\dim \ker L = n$. For the equation $Lx = H$ to have a solution, it is clearly necessary that $H \in \ker Q$. Since $\operatorname{codim} \ker Q = n$, it it follows that $\operatorname{Im} L = \ker Q$. The existence of the bounded right inverse follows from [2] or one may apply the closed graph theorem to $L \mid (I - P)\mathscr{P}_\omega$.

For the applications, it is necessary to be able to compute $\ker L$. In some simple cases, this is easily accomplished. For example, if $a(t) = a(t + \omega)$, $t \in \mathbb{R}$, is an $n \times n$ matrix with $|a(t)| \leqslant k < 1$ for $t \in \mathbb{R}$, then the unique solution Mc in $\mathscr{P}_\omega$ of

$$x(t) - a(t)\, x(t - r) = c \tag{2.2}$$

is given by

$$(Mc)(t) = \left[I + \sum_{k=0}^{\infty} \prod_{j=0}^{k} a(t - jr)\right] c. \tag{2.3}$$

Another case particularly interesting in the applications is when $D(t)\varphi$ is independent of t. Then $\ker L = \{\text{constant functions in } \mathscr{P}_\omega\}$.

3. EXISTENCE THEOREMS FOR NONLINEAR EQUATIONS

With the above notations, let us consider the neutral functional differential equation

$$\frac{d}{dt} D(t)\, x_t = f(t, x_t) \tag{3.1}$$

where D is stable and $f: \mathbb{R} \times C \to \mathbb{R}^n$ is ω-periodic with respect to t, continuous, and takes bounded sets into bounded sets. If we define $N: \mathscr{P}_\omega \to \mathscr{H}_\omega$ by

$$Nx(t) = \int_0^t f(s, x_s)\, ds, \qquad t \in \mathbb{R},$$

it is clear that finding ω-periodic solutions of (3.1) is equivalent to solving the operator equation $Lx = Nx$ in $\mathscr{P}_\omega$ with L defined in (2.7). To apply coincidence degree theory to this problem still requires that N should be compact, i.e., continuous and taking bounded sets of $\mathscr{P}_\omega$ into relatively compact sets of $\mathscr{P}_\omega$.

PROPOSITION 3.1. *Under the conditions listed above, N is compact.*

Proof. The continuity is obvious. If $S > 0$ and $x \in \mathscr{P}_\omega$ is such that $|x| \leqslant S$, then $|x_t| \leqslant S$ for every $t \in S$ and thus $|f(s, x_s)| \leqslant T$ for some $T > 0$ and every $s \in \mathbb{R}$. It then follows easily that

$$|Nx(t)| \leqslant T(1 + 2\omega), \qquad t \in \mathbb{R}$$

and

$$|Nx(t_1) - Nx(t_2)| \leqslant T|t_1 - t_2|, \qquad t_1, t_2 \in \mathbb{R},$$

and Proposition 3.1 is a consequence of the Arzela–Ascoli theorem.

A direct application of Propositions 2.2, 3.1 above and of Theorem 5.1(i) of [3] gives the following.

THEOREM 3.1. *If there exists an open bounded set $\Omega \subset \mathscr{P}_\omega$ whose boundary $\partial\Omega$ contains no ω-periodic solution of* (3.1) *and if the $\mathscr{L}_+$-coincidence degree $d[(L, N), \Omega]$ is not zero, then Eq.* (3.1) *has at least one ω-periodic solution in Ω.*

This result is quite general but requires the solution of two difficult problems, namely, finding Ω (it is an a priori bound problem) and estimating $d[(L, N), \Omega]$. Theorem 7.2 [3] reduces this last question to the study of Brouwer degree of some well defined finite-dimensional mapping if the a priori estimate condition is slightly strengthened. Let $g: \mathbb{R} \times C \times [0, 1] \to \mathbb{R}^n$, $(t, \varphi, \lambda) \to g(t, \varphi, \lambda)$ be ω-periodic with respect to t, continuous, taking bounded sets into bounded sets and such that

$$g(t, \varphi, 1) \equiv f(t, \varphi), \qquad (t, \varphi) \in \mathbb{R} \times C. \tag{3.2}$$

Let $M: \mathbb{R}^n \to \mathscr{P}_\omega$ be the mapping defined in Proposition 2.1 and define $\mathscr{G}: \mathbb{R}^n \to \mathbb{R}^n$ by

$$\mathscr{G}(a) = \omega^{-1} \int_0^\omega g(t, (Ma)_t, 0)\, dt.$$

If $D(t)\varphi$ is independent of t, one can put $M = I$, the identity in this definition. Theorem 7.2 of [3] implies the following.

THEOREM 3.2. *Suppose there exists an open bounded set $\Omega \subset \mathscr{P}_\omega$ for which the following conditions are satisfied.*

(1) *For each $\lambda \in (0, 1)$, the equation*

$$\frac{d}{dt} D(t)\, x_t = \lambda g(t, x_t, \lambda)$$

has no ω-periodic solution on $\partial\Omega$.

(2) *$\mathscr{G}(a) \neq 0$ for every $a \in \mathbb{R}^n$ such that Ma belongs to $\partial\Omega$.*

(3) *The Brouwer degree $d_B[\mathscr{G}, \tilde{\Omega}, 0]$ is not zero, where $\tilde{\Omega} = \{a \in \mathbb{R}^n$: Ma belongs to $\Omega\}$.*

Then Eq. (3.1) *has at least one ω-periodic solution in Ω.*

Another useful special case of Theorem 3.1 follows at once from Theorem 7.3 [3]. Suppose that the mapping g defined above verifies (3.2) and the supplementary condition

$$g(t, -\varphi, 0) = -g(t, \varphi, 0), \qquad (t, \varphi) \in \mathbb{R} \times C. \tag{3.3}$$

THEOREM 3.3. *Suppose there exists an open bounded set $\Omega \subset \mathscr{P}_\omega$ symmetric with respect to the origin, containing it and such that $\partial\Omega$ contains no ω-periodic solution of each equation*

$$\frac{d}{dt} D(t)\, x_t = g(t, x_t, \lambda), \qquad \lambda \in [0, 1]$$

with g verifying (3.2) *and* (3.3). *Then Eq.* (3.1) *has at least one ω-periodic solution in Ω.*

Let us note that (3.3) will always be satisfied if $g(t, \varphi, 0)$ is linear with respect to φ. Also, Theorems 3.1, 3.2, and 3.3 are respective generalizations of Theorems 2, 3, and 4 [5] which all correspond to the case of retarded functional differential equations, i.e., $D\varphi = \varphi(0)$, and Ω an open ball. Also, an extension to the neutral case of Theorem 1 of [6] is easily obtained by a suitable choice of Ω and the properties of coincidence degree.

4. An Application

Let us consider the neutral equation

$$\frac{d}{dt}\left[x(t) - \sum_{k=1}^{N} A_k x(t-\tau_k)\right] = \operatorname{grad} V[x(t)] + e(t), \tag{4.1}$$

where $V\colon \mathbb{R}^n \to \mathbb{R}$ is of class $\mathscr{C}^1$, $e \in \mathscr{P}_\omega$, $\tau_k \in [-r, 0)$ $(k = 1, 2,\ldots, N)$ and the $n \times n$ constant matrices A_k are such that

$$\sum_{k=1}^{N} | A_k | = 1 - \alpha, \qquad \alpha > 0. \tag{4.2}$$

Let $\bar{e}$ be the mean value of e.

THEOREM 4.1. *If there exists $R > 0$ such that*

$$\bar{e} + \omega^{-1} \int_0^\omega \operatorname{grad} V[x(t)]\, dt \neq 0$$

for every $x \in \mathscr{P}_\omega$ satisfying $\inf_{t\in R} | x(t)| \geqslant R$ and if the Brouwer degree $d_B[\bar{e} + \operatorname{grad} V, B(0, R), 0]$ is not zero, then Eq. (4.1) *has at least one ω-periodic solution.*

Proof. Let $|\cdot|$ and $\langle\cdot,\cdot\rangle$ respectively denote the Euclidean norm and the inner product in $\mathbb{R}^n$. It is well known [10] that condition (4.2) implies that the operator $D\colon \varphi \to \varphi(0) - \sum_{k=1}^{N} A_k\varphi(-\tau_k)$ is stable and the right side of (4.1) clearly takes bounded sets into bounded sets. Let us consider the family of equations

$$\frac{d}{dt}\left[x(t) - \sum_{k=1}^{N} A_k x(t-\tau_k)\right] = \lambda \operatorname{grad} V[x(t)] + \lambda e(t), \qquad \lambda \in (0, 1). \tag{4.3}$$

If x is any ω-periodic solution of (4.3) for some $\lambda \in (0, 1)$ then $x(t)$ must have a continuous first derivative (see [14]) and

$$\omega^{-1} \int_0^\omega \left\langle \dot{x}(t) - \sum_{k=1}^{N} A_k \dot{x}(t-\tau_k), \dot{x}(t)\right\rangle dt$$

$$= \lambda\omega^{-1} \int_0^\omega \langle \operatorname{grad} V[x(t)], \dot{x}(t)\rangle\, dt + \lambda\omega^{-1} \int_0^\omega \langle e(t), \dot{x}(t)\rangle\, dt,$$

which implies, using Schwarz' inequality and (4.2),

$$\left(\omega^{-1} \int_0^\omega | \dot{x}(t)|^2\, dt\right)^{1/2} \leqslant \alpha^{-1}\eta$$

with $\eta^2 = \omega^{-1} \int_0^\omega |e(t)|^2 \, dt$. Then, for every $t, t' \in [0, \omega]$, we have

$$| x(t) - x(t')| \leqslant \omega\alpha^{-1}\eta. \tag{4.4}$$

On the other hand, every ω-periodic solution of (4.3) verifies the equation

$$\bar{e} + \omega^{-1} \int_0^\omega \operatorname{grad} V[x(t)] \, dt = 0,$$

and, hence, there must exist some $\sigma \in [0, \omega]$ for which $| x(\sigma)| < R$. Taking $t' = \sigma$ in (4.4) we obtain

$$| x | < R + \omega\alpha^{-1}\eta = S$$

for every ω-periodic solution of (4.3). The result then follows from Theorem 3.2 by taking for Ω the open ball of center 0 and radius S in $\mathscr{P}_\omega$.

As an application of Theorem 4.1, let us consider the special case of a scalar equation with one delay,

$$\frac{d}{dt} [x(t) + ax(t - r)] = p(x) + e(t),$$

where $| a | < 1$, $e \in \mathscr{P}_\omega$ and $p(x)$ is a given function of x. This equation arises in a transmission line problem with a shunt across the line (see [13, 14]). Then, if p is any continuous function such that $| p(x)| \to \infty$ if $| x | \to \infty$ and $p(x)\, p(-x) < 0$ for all x with $| x |$ sufficiently large, there will exist an ω-periodic solution. Using Liapunov functions, Lopes [14] has obtained the existence of an ω-periodic solution of this special equation for $| a | < \frac{1}{2}$ and $xp(x) \to +\infty$ as $| x | \to \infty$.

5. Neutral Functional Differential Equations with Quasibounded Nonlinearities

We shall consider in this section ω-periodic equations of the form

$$\frac{d}{dt} D(t)\, x_t = b(t, x_t) + f(t, x_t), \tag{5.1}$$

where D satisfies the conditions in Section 2, $b\colon \mathbb{R} \times C \to \mathbb{R}^n$, $(t, \varphi) \to b(t, \varphi)$ is linear with respect to φ and continuous, $f\colon \mathbb{R} \times C \to \mathbb{R}^n$ is continuous, takes bounded sets into bounded sets and is such that

$$\limsup_{|\varphi| \to \infty} (| \varphi |^{-1} \, | f(t, \varphi)|) = \inf_{0 < \rho < \infty} (\sup_{|\varphi| \geqslant \rho} | \varphi |^{-1} \, | f(t, \varphi)|) = 0 \tag{5.2}$$

uniformly in $t \in \mathbb{R}$.

Let us recall that a mapping $F: X \to Y$ between normed spaces is *quasibounded* if the number $\| F \| = \limsup_{|x| \to \infty} |x|^{-1} |Fx|$ is finite, in which case it is called the *quasinorm* of F [12]. We shall use in this section a mapping theorem of Granas for compact quasibounded perturbations of the identity [12] and a special case of its generalization in the frame of coincidence degree theory [4].

PROPOSITION 5.1. *If f satisfies the conditions above, then the mapping $N: \mathscr{P}_\omega \to \mathscr{H}_\omega$ defined by $Nx(t) = \int_0^t f(s, x_s)\, ds$, $t \in \mathbb{R}$, is compact, quasibounded and $\| N \| = 0$.*

Proof. The compactness follows from Proposition 3.1. Now, if $\epsilon > 0$, it follows from (5.1) and the fact that f takes bounded sets into bounded sets that there exist $\gamma(\epsilon) > 0$ such that, for every $(t, \varphi) \in \mathbb{R} \times C$,

$$|f(t, \varphi)| \leqslant \epsilon |\varphi| + \gamma.$$

Hence, for every $x \in \mathscr{P}_\omega$,

$$\begin{aligned} |Nx| &= \left| \omega^{-1} \int_0^\omega f(s, x_s)\, ds \right| \\ &\quad + \sup_{t \in [0, \omega]} \left| \int_0^t \left[f(t', x_{t'}) - \omega^{-1} \int_0^\omega f(s, x_s)\, ds \right] dt' \right| \\ &\leqslant (1 + 2\omega)[\epsilon |x| + \gamma(\epsilon)] \end{aligned}$$

which clearly implies $\| N \| = 0$.

Now we can prove the following.

THEOREM 5.1. *With D, b and f as above, suppose the linear equation*

$$\frac{d}{dt} D(t)\, x_t = b(t, x_t) \tag{5.3}$$

has no nontrivial ω-periodic solution. Then Eq. (5.1) has at least one ω-periodic solution.

Proof. The result is equivalent to solving the equation $Lx - Bx = Nx$ in $\mathscr{P}_\omega$ with $B: \mathscr{P}_\omega \to \mathscr{H}_\omega$ defined by $Bx(t) = \int_0^t b(s, x_s)\, ds$, $t \in \mathbb{R}$, and L, N as above. From Proposition 3.1 we know that B is a compact mapping and L being a continuous Fredholm mapping of index zero, the same is true for $L - B$ [11]. As $L - B$ is one-to-one by our assumption on (5.3) it will necessarily be onto and such that $(L - B)^{-1}: \mathscr{H}_\omega \to \mathscr{P}_\omega$ is continuous. The proof of Theorem 5.1 is then equivalent to the fixed point problem

$x = (L - B)^{-1}Nx$ in $\mathscr{P}_\omega$ with $(L - B)^{-1}N$ clearly compact, quasibounded, and of quasinorm zero. The result then follows from Granas' theorem.

An interesting problem is now to try to drop the assumption about the nonexistence of nontrivial ω-periodic solutions for (5.3). It is clear from the Fredholm alternative that conditions upon f will then be needed. We consider here the simplest case, i.e., $b(t, \varphi) \equiv 0$. Let us define $\mathscr{F}: \mathbb{R}^n \to \mathbb{R}^n$ by $\mathscr{F}(a) = \omega^{-1} \int_0^\omega f(t, (Ma)_t)\, dt$. If $D(t)\varphi$ is independent of t, take $M = I$, the identity, in the definition of $\mathscr{F}$.

THEOREM 5.2. *Let D and f be as above and suppose there exists $\mu > 0$ such that $|(Mc)(t)| \geqslant \mu | c |$ for every $t \in \mathbb{R}$ and every $c \in \mathbb{R}^n$. If there exists $R_1 > 0$ such that $\int_0^\omega f(s, x_s)\, ds \neq 0$ for every $x \in \mathscr{P}_\omega$ verifying $\inf_{t \in \mathbb{R}} | x(t)| \geqslant R_1$ and if $d_B[\mathscr{F}, \Omega_R, 0]$ is not zero, where $\Omega_R = \{a \in \mathbb{R}^n: Ma \in B(0, R)\}$ and $R = \mu^{-1}| M |R_1$, then the equation*

$$\frac{d}{dt} D(t)\, x_t = f(t, x_t) \tag{5.4}$$

has at least one ω-periodic solution.

Proof. We will use Propositions 3.1 and 5.1 above and Theorem 4.1 of [4]. The proof will be complete if we show the existence of $\alpha \geqslant 0$ and $R > 0$ such that every ω-periodic solution x of (5.4) satisfies the inequality

$$| Px | < \alpha|(I - P)x | + R. \tag{5.5}$$

If x is any ω-periodic solution of (5.4), then $\int_0^\omega f(s, x_s)\, ds = 0$, and, hence, there will exist $\sigma \in [0, \omega]$ such that $| x(\sigma)| < R_1$. Therefore, if $c \in \mathbb{R}^n$ is such that $Px = Mc$, we have

$$\mu| c | \leqslant |(Mc)(\sigma)| < R_1 + |(I - P)\, x(\sigma)| \leqslant R_1 + |(I - P)x |,$$

which implies

$$| Px | \leqslant | M | \, | c | < \mu^{-1}| M |R_1 + \mu^{-1}| M | \, |(I - P)x |,$$

and (5.5) holds with $\alpha = \mu^{-1}| M |$ and $R = \mu^{-1}| M |R_1$.

Let us remark that if $D(t)\varphi$ is independent of t, $\ker L$ is the subspace of $\mathscr{P}_\omega$ of constant functions and the positive number μ involved above always exists and can be taken equal to one. Hence, a simple example for Theorem 5.2 is given by the scalar equation

$$\frac{d}{dt} [x(t) - ax(t - r)] = g(x_t) + e(t), \tag{5.6}$$

where $a \in (-1, 1)$, $e \in \mathscr{P}_\omega$ has mean value zero, $g\colon C \to \mathbb{R}^n$ is continuous, quasibounded with quasinorm zero, takes bounded sets into bounded sets and is such that, for some $R > 0$, either $g(\varphi)\,\varphi(\theta) > 0$ or $g(\varphi)\,\varphi(\theta) < 0$, for every $\theta \in [-r, 0]$ and every $\varphi \in C$ such that $\inf_{[-r,0]} |\varphi(\theta)| \geqslant R$. It is the case, for example, for the equation

$$\frac{d}{dt}[x(t) + ax(t-r)] = b\frac{x(t-r)}{|x(t-r)|^{1/2}} + e(t)$$

if $|a| < 1$, $b \neq 0$, $e \in \mathscr{P}_\omega$ has mean value zero and $y/|y|^{1/2}$ is extended by 0 at $y = 0$.

To apply Theorem 5.2 to a scalar equation of the form

$$\frac{d}{dt}[x(t) - a(t)\,x(t-r)] = g(x_t) + e(t)$$

with g and e as above and $a \in \mathscr{P}_\omega$, the crucial point is to prove the existence of $\mu > 0$ such that $|(Mc)(t)| \geqslant \mu|c|$ for every $t \in \mathbb{R}$ and every $c \in \mathbb{R}$. The following propositions give answers to this problem. For the sake of brevity, we shall say that the operator M associated with the scalar equation $x(t) - a(t)\,x(t-r) = c$ has *property* μ if there exists $\mu > 0$ such that $|(Mc)(t)| \geqslant \mu|c|$ for every $t \in \mathbb{R}$ and every $c \in \mathbb{R}$.

PROPOSITION 5.2. *If* $|a(t)| \leqslant k$ *for all* $t \in \mathbb{R}$ *and* $k \in [0, \frac{1}{2})$, *then* M *has property* μ.

Proof. From the relation

$$(Mc)(t) - a(t)(Mc)(t-r) = c$$

one obtains easily $|Mc| \leqslant (1-k)^{-1}|c|$, and, hence,

$$|(Mc)(t)| \geqslant |c| - k|(Mc)(t-r)| \geqslant |c|(1-2k)(1-k)^{-1}$$

for every $t \in \mathbb{R}$ and every $c \in \mathbb{R}$.

The following example will show that Proposition 5.2 is the best possible without supplementary assumptions on the oscillatory character of $a(t)$. Let $\omega = p$, p a positive integer, $r = 1$ and $a(t)$ be a p-periodic continuous function such that $|a(t)| \leqslant k < 1$, $t \in \mathbb{R}$, $a(0) = -k$, $a(m) = k$ $(m = 1, 2,\ldots, p-1)$. Then, if $x(t)$ is the solution of $x(t) - a(t)\,x(t-1) = 1$, property μ clearly will not hold if we exhibit one $t \in [0, p]$ such that $x(t) = 0$. Using formula (2.3) and the form of $a(t)$ we have

$$x(0) = 1 - k(1 + k + k^2 + \cdots + k^{p-1} - k^p(1 + k + \cdots + k^{p-1} - \cdots$$

$$= 1 - k\left(\frac{1 - k^p}{1 - k} - k^p\left(\frac{1 - k^p}{1 - k} - \cdots\right.\right.$$

$$= 1 - k\left(\frac{1 - k^p}{1 - k}\right)(1 - k^p + k^{2p} - \cdots)$$

$$= 1 - k(1 - k^p)(1 - k)^{-1}(1 + k^p)^{-1}$$

$$= (1 + k^p)^{-1}(1 - k)^{-1}(1 - 2k + k^p) = \gamma(k).$$

It is easy to show that $\gamma(k)$ is strictly positive in $[0, 1/2)$ and is strictly negative in a neighborhood of 1. Thus, $x(0) = 0$ for some $k \in [1/2, 1)$ and this zero is arbitrary close to 1/2 if we take p sufficiently large, as follows at once from the form of $\gamma(k)$.

It is, however, possible to improve the condition upon k when $a(t)$ has a constant sign as follows from the following.

PROPOSITION 5.3. *If* $|a(t)| \leqslant k < 1$ *and* $a(t)$ *has constant sign, then* M *has property* μ.

Proof. Let us first consider the case where $0 \leqslant a(t) \leqslant k$ for every $t \in \mathbb{R}$. Then M has property μ because

$$|(Mc)(t)| = |1 + a(t) + a(t)\,a(t - r) + \cdots|\,|c| \geqslant |c|.$$

Now suppose that $-k \leqslant a(t) \leqslant 0$ for every $t \in \mathbb{R}$. It is clear that the unique ω-periodic solution x of $x(t) - a(t)\,x(t - r) = c$ is the limit of the sequence $\{x^m(t)\}$ of ω-periodic functions defined by

$$x^0(t) = c, \qquad x^{m+1}(t) = c + a(t)\,x^m(t - r), \qquad m = 0, 1, 2, \ldots .$$

If $c > 0$, then $x^1(t) = [1 + a(t)]c \geqslant (1 - k)c > 0$, $x^2(t) = c + a(t)x^1(t - r) \geqslant [1 - k(1 - k)]c = (1 - k + k^2)c > 0$, and if

$$x^m(t) \geqslant [1 - k + k^2 + \cdots + (-1)^m k^m]c > 0,$$

then $x^{m+1}(t) = c + a(t)\,x^m(t - r) \geqslant c\{1 - k[1 - k + \cdots + (-1)^m k^m]\} = [1 - k + k^2 + \cdots + (-1)^{m+1}k^{m+1}]c > 0$. Hence by induction and passing to the limit, we have $|(Mc)(t)| \geqslant (1 + k)^{-1}| c |$. Finally, suppose that $c < 0$. Then,

$$c \leqslant x^1(t) = c + a(t)c \leqslant (1 - k)c < 0,$$

and, hence,

$$c \leqslant x^2(t) = c + a(t)\, x^1(t - r) \leqslant (1 - k)c < 0.$$

If we suppose that $c \leqslant x^{m-1}(t) \leqslant (1 - k)c < 0$, then $0 \leqslant a(t)x^{m-1}(t - r) \leqslant -ck$, and, hence,

$$0 > (1 - k)c \geqslant x^m(t) = c + a(t)\, x^{m-1}(t) \geqslant c.$$

By induction and passing to the limit we have $0 > (1 - k)c \geqslant (Mc)(t) \geqslant c$, and, hence, $|(Mc)(t)| \geqslant (1 - k)| c |$, which achieves the proof.

COROLLARY 5.1. *If a is a constant verifying $0 < | a | < 1$ then, for every $b \in \mathscr{P}_\omega$ such that $| b(t)| < \min(| a |, | 1 - a |)$, $t \in \mathbb{R}$, the mapping M associated with $x(t) - [a + b(t)]\, x(t - r) = c$ has property μ.*

REFERENCES

1. J. K. HALE, Functional differential equations, *in* "Analytic Theory of Differential Equations" (P. F. Hsieh and A. W. J. Stoddart, Eds.), pp. 9–22, Springer, Berlin, 1971.
2. J. K. HALE, Oscillations in neutral functional differential equations, *in* "Nonlinear Mechanics," C.I.M.E., Edizioni Cremonese, Roma, 1973.
3. J. MAWHIN, Equivalence theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector spaces, *J. Differential Equations* **12** (1972), 610–636.
4. J. MAWHIN, The solvability of some operator equations with a quasibounded nonlinearity in normed spaces, *J. Math. Anal. Appl.*, to appear.
5. J. MAWHIN, Periodic solutions of nonlinear functional differential equations, *J. Differential Equations* **10** (1971), 240–261.
6. J. CRONIN, Periodic solutions of nonautonomous equations, *Boll. Univ. Mat. Ital.* (4) **6** (1972), 45–54.
7. R. E. FENNELL, Periodic solutions of functional differential equations, *J. Math. Anal. Appl.* **39** (1972), 198–201.
8. J. MAWHIN, Periodic solutions of some vector retarded functional differential equations, *J. Math. Anal. Appl.*, to appear.
9. B. N. SADOVSKII, Application of topological methods in the theory of periodic solutions of nonlinear differential-operator equations of neutral type, (Russian), *Dokl. Akad. Nauk SSSR* **200** (1971); *Soviet Math. Dokl.* **12** (1971), 1543–1547.
10. M. A. CRUZ AND J. K. HALE, Stability of functional differential equations of neutral type, *J. Differential Equations* **7** (1970), 334–355.
11. T. KATO, "Perturbation Theory for Linear Operators," Springer, Berlin, 1966.

12. A. Granas, The theory of compact vector fields and some of its applications to topology of functional spaces (I), *Rozpravy Mat. Přirod Věd. Československé Akad. Věd. Řada* **30** (1962), 1–93.
13. W. Miranker, Periodic solutions of the wave equation with a nonlinear interface condition, *IBM J. Res. Develop* **5** (1961), 2–24.
14. O. Lopes, Asymptotic fixed point theorems and forced oscillations in neutral equations, Ph.D. Thesis, Brown University, Providence, Rhode Island, June 1973.