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Nuclear Physics B 889 (2014) 25–35
NUCLEAR PHYSICS B
www.elsevier.com/locate/nuclphysb

# $q$-Virasoro/W algebra at root of unity and parafermions

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Received 29 August 2014; received in revised form 3 October 2014; accepted 6 October 2014

Available online 13 October 2014

Editor: Herman Verlinde

**Abstract**

We demonstrate that the parafermions appear in the $r$-th root of unity limit of $q$-Virasoro/$W_n$ algebra. The proper value of the central charge of the coset model $\frac{\widehat{\mathfrak{sl}}(n)_r \oplus \widehat{\mathfrak{sl}}(n)_{m-n}}{\widehat{\mathfrak{sl}}(n)_{m-n+r}}$ is given from the parafermion construction of the block in the limit.
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## 1. Introduction

Ever since the AGT relation [1–3] (the correspondence between the correlators of 2d QFT and the 4d instanton sum) was introduced, the both sides of the correspondence have been intensively studied by a number of people. For example, in the 2d side, the $\beta$-deformed matrix model is used in order to control the integral representation of the conformal block [4–10]. There are also some proposals for proving the 2d–4d connection [11–15]. Moreover similar correspondence has been found and examined [16–26]. Among these, we pay our attention, in this paper, to the correspondence between the coset model,

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http://dx.doi.org/10.1016/j.nuclphysb.2014.10.006
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$$\frac{\widehat{\mathfrak{sl}}(n)_r \oplus \widehat{\mathfrak{sl}}(n)_p}{\widehat{\mathfrak{sl}}(n)_{r+p}}, \tag{1.1}$$

and the $\mathcal{N}=2$ $SU(n)$ gauge theory on $\mathbf{R}^4/\mathbf{Z}_r$ [20,23]. Here $\widehat{\mathfrak{sl}}(n)_k$ stands for the affine Lie algebra in the representation of level $k$ and $r$ and $p$ will be specified in this paper.

On the 2d CFT side, a quantum deformation ($q$-deformation) of the Virasoro algebra [27] and the $W_n$ algebra [28,29] is known, while the 4d gauge theories can be lifted to five-dimensional theories with the fifth direction compactified on a circle. There exists a natural generalization to the connection between the 2d theory based on the $q$-deformed Virasoro/W algebra and the five-dimensional $\mathcal{N}=2$ gauge theory [30]. For recent developments, see, for example, [31–37]. In the previous paper [32], we proposed a limiting procedure to get the Virasoro/W block in the 2d side from that in the $q$-deformed version. On the other hand, we saw that the instanton partition function on $\mathbf{R}^4/\mathbf{Z}_r$ is generated from that on $\mathbf{R}^5$ at the same limit. This result means if we assume the 2d–5d connection, it is automatically assured that the Virasoro/W blocks generated by using the limiting procedure agree with the instanton partition function on $\mathbf{R}^4/\mathbf{Z}_r$. Our limiting procedure corresponds to a root of unity limit in $q$. A root of unity limit of the $q$-Virasoro algebra was also considered in [38]. Our limit is slightly different from this and is similar to the one used in order to construct the eigenfunctions of the spin Calogero–Sutherland model from Macdonald polynomials in [39,40].

In the present paper we will elaborate our limiting procedure and show that the $\mathbf{Z}_r$-parafermionic CFT which has the symmetry described by (1.1) appears in the 2d side. We clarify also the relation between the free parameter $p$ and the omega background parameters in the 4d side.

The paper is organized as follows: In the next section, we review the limiting procedure for $q$-Virasoro algebra [32]. In Section 3, we consider the $q$-deformed screening current and charge and show that the $\mathbf{Z}_r$-parafermion currents are derived in a natural way. In Section 4, we consider the generalization to $q$–$W_n$ algebra.

## 2. Root of unity limit of $q$-Virasoro algebra

In this section, we review the root of unity limit [32] of the $q$-deformed Virasoro algebra [27] which has two parameters $q$ and $t=q^\beta$. The defining relation is

$$f(z'/z)\mathcal{T}(z)\mathcal{T}(z') - f(z/z')\mathcal{T}(z')\mathcal{T}(z) = \frac{(1-q)(1-t^{-1})}{(1-p)}\left[\delta(pz/z') - \delta(p^{-1}z/z')\right], \tag{2.1}$$

where $p=q/t$ and

$$f(z) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{(1+p^n)}z^n\right). \tag{2.2}$$

The multiplicative delta function is defined by

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n. \tag{2.3}$$

Using the $q$-deformed Heisenberg algebra $\mathcal{H}_{q,t}$:

$$\begin{aligned}
&[\alpha_n, \alpha_m] = -\frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{(1+p^n)}\delta_{n+m,0}, \quad (n\neq 0),\\
&[\alpha_n, Q] = \delta_{n,0},
\end{aligned} \tag{2.4}$$

the $q$-Virasoro operator $\mathcal{T}(z)$ can be realized as

$$\mathcal{T}(z) = :\exp\left(\sum_{n\neq 0}\alpha_n z^{-n}\right): p^{1/2} q^{\sqrt{\beta}\alpha_0} + :\exp\left(-\sum_{n\neq 0}\alpha_n (pz)^{-n}\right): p^{-1/2} q^{-\sqrt{\beta}\alpha_0}. \tag{2.5}$$

The $q$-deformed chiral bosons are defined in terms of the $q$-deformed Heisenberg algebra as

$$\widetilde{\varphi}^{(\pm)}(z) = \widetilde{\varphi}_0^{(\pm)}(z) + \widetilde{\varphi}_R^{(\pm)}(z), \tag{2.6}$$

where

$$\widetilde{\varphi}_0^{(\pm)}(z) = \beta^{\pm 1/2} Q + \frac{2}{r}\beta^{\pm 1/2}\alpha_0 \log z^r + \sum_{n\neq 0}\frac{(1+p^{-nr})}{(1-\xi_\pm^{nr})}\alpha_{nr} z^{-nr},$$
$$\widetilde{\varphi}_R^{(\pm)}(z) = \sum_{\ell=1}^{r-1}\sum_{n\in\mathbb{Z}}\frac{(1+p^{-nr-\ell})}{1-\xi_\pm^{nr+\ell}}\alpha_{nr+\ell} z^{-nr-\ell}. \tag{2.7}$$

Here $\xi_+ = q$, $\xi_- = t$.

Let us consider the simultaneous $r$-th root of unity limit in $q$ and $t$ which is given by

$$q = \omega e^{-\frac{1}{\sqrt{\beta}}h}, \qquad t = \omega e^{-\sqrt{\beta}h}, \qquad p = e^{Q_E h}, \qquad h \to 0, \tag{2.8}$$

where $\omega = e^{\frac{2\pi i}{r}}$ and $Q_E = \sqrt{\beta} - \frac{1}{\sqrt{\beta}}$. Since $t = q^\beta$, this limit is possible if the parameter $\beta$ takes the rational number such as

$$\beta = \frac{rm_- + 1}{rm_+ + 1}, \tag{2.9}$$

where $m_\pm$ are non-negative integers. In the limit, we have two types of bosons $\phi(w)$ and $\varphi(w)$ [32] respectively given by

$$\lim_{h\to 0}\widetilde{\varphi}_0^{(\pm)}(z) = \sqrt{\frac{2}{r}}\beta^{\pm 1/2}\phi(w),$$
$$\lim_{h\to 0}\widetilde{\varphi}_R^{(\pm)}(z) = \sqrt{\frac{2}{r}}\varphi(w), \tag{2.10}$$

where $w = z^r$ and

$$\phi(w) = Q_0 + a_0 \log w - \sum_{n\neq 0}\frac{a_n}{n} w^{-n}, \tag{2.11}$$

$$\varphi(w) = \sum_{\ell=1}^{r-1}\varphi^{(\ell)}(w), \qquad \varphi^{(\ell)}(w) = \sum_{n\in\mathbb{Z}}\frac{\tilde{a}_{n+\ell/r}}{n+\ell/r} w^{-n-\ell/r}. \tag{2.12}$$

The commutation relations are

$$[a_m, a_n] = m\delta_{m+n,0}, \qquad [a_n, Q_0] = \delta_{n,0},$$
$$\left[\widetilde{a}_{n+\ell/r}, \widetilde{a}_{-m-\ell'/r}\right] = (n+\ell/r)\delta_{m,m'}\delta_{\ell,\ell'}. \tag{2.13}$$

The boson $\phi(w)$ and the twisted boson $\varphi(w)$ play an important role for the appearance of the $\mathbf{Z}_r$-parafermions.

## 3. $Z_r$-parafermionic CFT

The $q$-deformed screening current and the charge are defined respectively by

$$S^{(\pm)}(z) = :\mathrm{e}^{\tilde{\varphi}^{(\pm)}(z)}:, \qquad Q^{(\pm)}_{[a,b]} = \int_a^b \mathrm{d}_{\xi_\pm} z S^{(\pm)}(z), \tag{3.1}$$

where the Jackson integral is defined by

$$\int_0^a \mathrm{d}_q z f(z) = a(1-q) \sum_{k=0}^{\infty} f\big(aq^k\big) q^k. \tag{3.2}$$

From now on we choose $Q^{(+)}_{[a,b]}$. Multiplying the regularization factor, we obtain the screening charge in the root of unity limit, up to normalization,

$$Q^{(+)}_{[a^r,b^r]} \equiv \lim_{h\to 0} \frac{(1-q^r)}{(1-q)} Q^{(+)}_{[a,b]} = \int_{a^r}^{b^r} \mathrm{d}w \psi_1(w) : \mathrm{e}^{\sqrt{\beta}\phi(w)}, \tag{3.3}$$

where we have defined [41]

$$\psi_1(w) = \frac{A_r}{w^{(r-1)/r}} \sum_{k=0}^{r-1} \omega^k :\exp\left\{\sqrt{\frac{2}{r}} \phi^{(k)}(w)\right\}:. \tag{3.4}$$

Here $A_r$ is the normalization factor and we have introduced

$$\phi^{(k)}(w) \equiv \varphi\big(e^{2\pi \mathrm{i} k} w\big). \tag{3.5}$$

The correlation function is given by

$$\begin{aligned} \big\langle \phi^{(k)}(w)\phi^{(k')}\big(w'\big)\big\rangle &= \log \frac{(1-\omega^{k'-k}(w'/w)^{1/r})^r}{1-w'/w} \\ &= \log \frac{(1-w'/w)^{r-1}}{\prod_{j=1}^{r-1}(1-\omega^{k'-k+j}(w'/w)^{1/r})^r}. \end{aligned} \tag{3.6}$$

Note that

$$\phi^{(k+1)}(w) = \phi^{(k)}\big(e^{2\pi \mathrm{i}} w\big), \qquad \phi^{(r+k)}(w) = \phi^{(k)}(w), \qquad \sum_{k=0}^{r-1} \phi^{(k)}(w) = 0. \tag{3.7}$$

For example, we consider the $r = 2$ case. In the limit, we obtain

$$\lim_{q\to -1} S(z) = :\mathrm{e}^{\sqrt{\beta}\phi(w)} \mathrm{e}^{\varphi(w)}:, \tag{3.8}$$

and after the appropriate normalization, we obtain the following screening charge for the superconformal block [42,43]:

$$Q_{[a^2,b^2]} = \int_{a^2}^{b^2} \mathrm{d}w \psi(w) :\mathrm{e}^{\sqrt{\beta}\phi(w)}:, \tag{3.9}$$

where

$$\psi(w) \equiv \frac{\mathrm{i}}{2\sqrt{2w}}\left(:\mathrm{e}^{\varphi(w)}: - :\mathrm{e}^{-\varphi(w)}:\right), \qquad \langle \psi(w_1)\psi(w_2)\rangle = \frac{1}{w_1 - w_2}, \tag{3.10}$$

is the NS fermion.

From now on we will show that the $\boldsymbol{Z}_r$-parafermions appear in the general $r$-th root of unity limit. In particular, $\psi_1(w)$ will be shown to work as the first parafermion current.

The $\boldsymbol{Z}_r$-parafermion algebra consists of $(r-1)$ currents $\psi_\ell(w)$ $(\ell = 1, \cdots, r-1)$ satisfying the following defining relations [44]:

$$\psi_\ell(w)\psi_{\ell'}(w') = \frac{c_{\ell,\ell'}}{(w-w')^{2\ell\ell'/r}}\left\{\psi_{\ell+\ell'}(w') + \mathcal{O}(w-w')\right\}, \quad \ell + \ell' < r, \tag{3.11}$$

$$\psi_\ell^\dagger(w)\psi_{\ell'}(w') = c_{\ell,r-\ell'}(w-w')^{-2\ell(r-\ell')/r}\left\{\psi_{\ell-\ell'}(w') + \mathcal{O}(w-w')\right\}, \quad \ell' < \ell \tag{3.12}$$

$$\psi_\ell^\dagger(w)\psi_\ell(w') = (w-w')^{-2\Delta_\ell}\left\{1 + \frac{2\Delta_\ell}{c_p}(w-w')^2 T_{\mathrm{PF}}(w) + \mathcal{O}\left((w-w')^3\right)\right\}, \tag{3.13}$$

where $\psi_\ell^\dagger(w) = \psi_{r-\ell}(w)$ and

$$\Delta_\ell = \frac{\ell(r-\ell)}{r}, \qquad c_p = \frac{2(r-1)}{r+2}, \tag{3.14}$$

are the conformal dimension of $\psi_\ell(w)$ and the central charge of the parafermionic stress tensor $T_{\mathrm{PF}}$. The explicit form of $T_{\mathrm{PF}}(w)$ is given in [45]. The coefficients $c_{\ell\ell'}$ are given by

$$c_{\ell\ell'} = \sqrt{\frac{(\ell+\ell')!(r-\ell)!(r-\ell')!}{\ell!\ell'!(r-\ell-\ell')!r!}}. \tag{3.15}$$

The OPE of (3.4) is

$$\psi_1(w)\psi_1(w') \equiv \frac{c_{1,1}}{(w-w')^{2/r}}\left\{\psi_2(w) + \mathcal{O}(w-w')\right\}. \tag{3.16}$$

Here we have defined the second parafermion,

$$\psi_2(w) = \frac{A_r^2}{c_{1,1}w^{2(r-2)/r}}\sum_{k,k'=0}^{r-1}\omega^{k+k'}\left(1-\omega^{k'-k}\right)^2 :\mathrm{e}^{\sqrt{\frac{2}{r}}(\phi^{(k)}(w)+\phi^{(k')}(w))}:. \tag{3.17}$$

Similarly, the $(\ell+1)$-th parafermion is obtained from $\ell$-th parafermion by

$$\psi_{\ell+1}(w) \equiv \lim_{w'\to w}\frac{(w-w')^{2\ell/r}}{c_{1,\ell}}\psi_1(w')\psi_\ell(w). \tag{3.18}$$

In particular,

$$\psi_1^\dagger(w) \equiv \psi_{r-1}(w) = \frac{B_r}{w^{(r-1)/r}}\sum_{\ell=0}^{r-1}\omega^{-\ell}\exp\left\{-\sqrt{\frac{2}{r}}\phi^{(\ell)}(w)\right\}, \tag{3.19}$$

where $B_r$ is a constant which can be determined by the relation

$$\langle\psi_1^\dagger(w)\psi_1(w')\rangle = \frac{1}{(w-w')^{2(r-1)/r}}. \tag{3.20}$$

After all, we have the chiral boson $\phi(w)$ coupled to $Q_E$ and the $\mathbf{Z}_r$-parafermion $\psi_\ell(w)$. Therefore, the stress tensor of the whole system is

$$T(w) = T_{\mathrm{B}}(w) + T_{\mathrm{PF}}(w), \tag{3.21}$$

where $T_{\mathrm{B}}(w)$ stands for the usual stress tensor for the chiral boson field. The central charge is

$$c^{(r)} = 1 - \frac{6Q_E^2}{r} + \frac{2(r-1)}{r+2} = \frac{3r}{r+2} - \frac{6Q_E^2}{r}. \tag{3.22}$$

Because $\beta$ is restricted to the rational number (2.9), (3.22) is written as

$$c^{(r,m,s)} = \frac{3r}{r+2} - \frac{6rs^2}{m(m+rs)}, \tag{3.23}$$

where we have set $m = rm_+ + 1$ and $s = m_- - m_+$. Especially, when $s = 1$,

$$c^{(r,m,1)} = \frac{3r}{r+2} - \frac{6r}{m(m+r)}, \tag{3.24}$$

is the central charge of the unitary series of the $\mathbf{Z}_r$-parafermionic CFT [46].

The form of the screening charge in the case of general $r$ is the same as that of Eq. (3.9).

## 4. Root of unity limit of $q$–$W_n$ algebra

In this section, we consider the generalization to the $q$–$W_n$ algebra [29]. We denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{sl}(n)$ Lie algebra. The $q$–$W_n$ algebra is expressed in terms of the following $\mathfrak{h}$-valued $q$-deformed boson,

$$\langle e_a, \widetilde{\varphi}^{(\pm)}(z)\rangle \equiv \widetilde{\varphi}_a^{(\pm)}(z) = \widetilde{\varphi}_{0,a}^{(\pm)}(z) + \widetilde{\varphi}_{R,a}^{(\pm)}(z), \tag{4.1}$$

where

$$\widetilde{\varphi}_{0,a}^{(\pm)}(z) = \beta^{\pm\frac{1}{2}} Q_a + \beta^{\pm\frac{1}{2}}\alpha_{0,a}\log z + \sum_{n\neq 0}\frac{1}{\xi_\pm^{nr/2} - \xi_\pm^{-nr/2}}\alpha_{nr,a} z^{-nr}, \tag{4.2}$$

$$\widetilde{\varphi}_{R,a}^{(\pm)}(z) = \sum_{\ell=1}^{r-1}\widetilde{\varphi}_{\ell,a}^{(\pm)}(z) = \sum_{\ell=1}^{r-1}\sum_{n\in\mathbf{Z}}\frac{1}{\xi_\pm^{(nr+\ell)/2} - \xi_\pm^{-(nr+\ell)/2}}\alpha_{nr+\ell,a} z^{-(nr+\ell)}, \tag{4.3}$$

and $e_a$ $(a = 1, \cdots, n-1)$ are the simple roots and $\langle , \rangle : \mathfrak{h}^* \otimes \mathfrak{h} \to \mathbf{C}$ is the canonical pairing. The commutation relations are given by

$$\begin{aligned}
&[Q_a, \alpha_{0,b}] = C_{ab},\\
&[\alpha_{n,a}, \alpha_{m,b}] = \frac{1}{n}\big(q^{n/2} - q^{-n/2}\big)\big(t^{n/2} - t^{-n/2}\big)C_{ab}(p)\delta_{n+m,0},\\
&[Q_a, Q_b] = 0, \qquad [\alpha_{0,a}, \alpha_{0,b}] = 0,
\end{aligned} \tag{4.4}$$

where $C_{ab}$ is the Cartan matrix of $A$ type and

$$C_{ab}(p) = [2]_p\delta_{a,b} - p^{1/2}\delta_{a,b-1} - p^{-1/2}\delta_{a-1,b}. \tag{4.5}$$

The $q$-number is defined by

$$[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}. \tag{4.6}$$

Similar to the $q$-Virasoro case, we consider the limit,

$$q=\omega^k \mathrm{e}^{-\frac{h}{\sqrt{\beta}}}, \qquad t=\omega^k \mathrm{e}^{-\sqrt{\beta}h}, \qquad p=q/t=\mathrm{e}^{Q_E h},$$
$$\omega=\mathrm{e}^{\frac{2\pi \mathrm{i}}{r}}, \qquad h\to +0, \tag{4.7}$$

where $\omega=\mathrm{e}^{\frac{2\pi \mathrm{i}}{r}}$ and $k$ is a natural number mutually prime to $r$. The condition to be able to take this limit is that $\beta$ is a rational number,

$$\beta=\frac{rm_- + k}{rm_+ + k}, \tag{4.8}$$

where $m_\pm$ are non-negative integers. Taking this limit,

$$\lim_{h\to 0}\widetilde{\varphi}_0^a(z)=\frac{1}{\sqrt{r}}\beta^{1/2}\phi^a(w), \tag{4.9}$$

$$\lim_{h\to 0}\widetilde{\varphi}_R^a(z)=\frac{1}{\sqrt{r}}\varphi^a(w), \tag{4.10}$$

we obtain

$$\phi^a(w)=Q_0^a + a_0^a \log w - \sum_{n\neq 0}\frac{1}{n}a_n^a w^{-n}, \tag{4.11}$$

$$\varphi^a(w)=\sum_{\ell=1}^{r-1}\varphi_\ell(w), \qquad \varphi_\ell(w)=\sum_{\ell=1}^{r-1}\sum_{n\in\mathbf{Z}}\frac{1}{n+\ell/r}\widetilde{a}^a_{n+\ell/r}w^{-(n+\ell/r)}. \tag{4.12}$$

Here we have normalized as

$$Q^a=\frac{1}{\sqrt{r}}Q_0^a, \qquad \alpha_0^a=\sqrt{r}a_0^a, \tag{4.13}$$

$$\alpha_{nr}^a=-(-1)^{nk}\sqrt{r}ha_n^a, \tag{4.14}$$

$$\alpha^a_{nr+\ell}=\frac{e^{\mathrm{i}\pi k(nr+\ell)/2}-e^{-\mathrm{i}\pi k(nr+\ell)/2}}{\sqrt{r}(n+\ell/r)}\widetilde{a}^a_{n+\ell/r}. \tag{4.15}$$

The commutation relations are

$$\left[Q^a,\alpha_0^b\right]=C_{ab}, \qquad \left[Q^a,Q^b\right]=0, \qquad \left[\alpha_0^a,\alpha_0^b\right]=0, \tag{4.16}$$

$$\left[a_n^a,a_m^b\right]=nC_{ab}\delta_{n+m,0}, \tag{4.17}$$

$$\left[\widetilde{a}^a_{n+\ell/r},\widetilde{a}^b_{-m-\ell'/r}\right]=\left(n+\frac{\ell}{r}\right)C_{ab}\delta_{n,m}\delta_{\ell,\ell'}. \tag{4.18}$$

The correlation functions are

$$\left\langle \phi^a(w)\phi^b(w')\right\rangle=C_{ab}\log(w-w'), \tag{4.19}$$

$$\left\langle \varphi_\ell^a(w)\varphi_{\ell'}^b(w')\right\rangle=\delta_{\ell+\ell',r}C_{ab}\sum_{k=0}^{r-1}\omega^{-k\ell}\log\left[1-\omega^k\left(\frac{w'}{w}\right)^{\frac{1}{r}}\right], \tag{4.20}$$

$$\left\langle \varphi^a(w)\varphi^b(w')\right\rangle=C_{ab}\log\left[\frac{(1-(w'/w)^{1/r})^r}{1-(w'/w)}\right]. \tag{4.21}$$

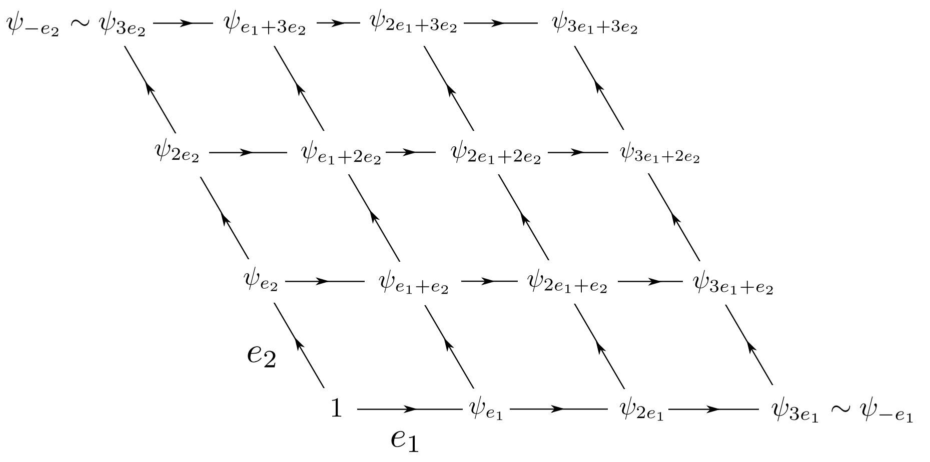


Fig. 1. The parafermions in the case of $\mathfrak{sl}(3)$ and $r = 4$.

For each $e_a$, we define

$$\psi_{e_a}(w) = \frac{A_r}{w^{(r-1)/r}} \sum_{\ell=0}^{r-1} \omega^\ell :\exp\left[\sqrt{\frac{1}{r}}\phi_a^{(\ell)}(w)\right]:, \tag{4.22}$$

where $A_r$ is a normalization factor and

$$\phi_a^{(\ell)}(w) \equiv \varphi_a\left(e^{2\pi i \ell} w\right). \tag{4.23}$$

Let $\alpha = \sum_{a=1}^{n-1} n_a e_a \in Q$, where $n_a$ are non-negative integers and $Q$ denotes the root lattice. We obtain the corresponding parafermion, up to its normalization,

$$\psi_\alpha \sim \prod \psi_{e_a}^{n_a}. \tag{4.24}$$

The independent parafermion can be given only for the case $\alpha \in Q/rQ$. Not of all $\psi_\alpha$ are independent;

$$1 \sim \underbrace{\psi_{e_a} \cdots\cdots \psi_{e_a}}_{r}. \tag{4.25}$$

For example, in the case of $\mathfrak{sl}(3)$ algebra and $r = 4$, the corresponding parafermions are drawn in Fig. 1. We define the parafermion associated with negative of a simple root by

$$\psi_{-e_a} \sim \underbrace{\psi_{e_a}\psi_{e_a} \cdots \psi_{e_a}}_{r-1}. \tag{4.26}$$

The normalization can be determined by the correlation functions [47],

$$\left\langle \psi_{-\alpha}(w)\psi_\alpha\left(w'\right)\right\rangle = \left(w - w'\right)^{-2+\frac{\alpha^2}{r}}, \tag{4.27}$$

where $\alpha^2 = (\alpha, \alpha)$. In particular,

$$\left\langle \psi_{-e_a}(w)\psi_{e_a}\left(w'\right)\right\rangle = \left(w - w'\right)^{-2\frac{r-1}{r}}. \tag{4.28}$$

In the case of the $\mathfrak{sl}(2)$ algebra, we obtain the first $\mathbf{Z}_r$-parafermion,

$$\psi_1(w) = \psi_{e_1}(w). \tag{4.29}$$

Similar to the case of $n = 2$ (3.22), the central charge is given by

$$\begin{aligned} c_n^{(r)} &= \frac{n(n-1)(r-1)}{r+n} + (n-1)\left(1 - n(n+1)\frac{Q_E^2}{r}\right) \\ &= \frac{r(n^2-1)}{r+n} - n\left(n^2-1\right)\frac{Q_E^2}{r}. \end{aligned} \tag{4.30}$$

When we set $m = rm_+ + k$, $m_- = m_+ + s$ in (4.8), this central charge becomes

$$\begin{aligned} c_n^{(r,m,s)} &= \frac{r(n^2-1)}{r+n} - \frac{rs^2n(n^2-1)}{m(m+rs)} \\ &= \frac{(n^2-1)r(\frac{m}{s}-n)(\frac{m}{s}+n+r)}{(r+n)\frac{m}{s}(\frac{m}{s}+r)}, \end{aligned} \tag{4.31}$$

which is the same as that of the coset model,

$$\frac{\widehat{\mathfrak{sl}}(n)_r \oplus \widehat{\mathfrak{sl}}(n)_{\frac{m}{s}-n}}{\widehat{\mathfrak{sl}}(n)_{\frac{m}{s}-n+r}}. \tag{4.32}$$

Compared with (1.1) we find

$$p = \frac{m}{s} - n. \tag{4.33}$$

In the case of $s = 0$ corresponding to $Q_E = 0$, we have the central charge of the usual Sugawara stress tensor for $\widehat{\mathfrak{sl}}(n)_r$,

$$c_n^{(r,m,0)} = \frac{r(n^2-1)}{r+n} = c_{\widehat{\mathfrak{sl}}(n)_r}. \tag{4.34}$$

It is well-known that the affine Lie algebra $\widehat{\mathfrak{sl}}(n)_r$ is represented by parafermions and an auxiliary boson [47]. In the case of $s = 1$, because (4.31) becomes

$$c_n^{(r,m,1)} = \frac{(n^2-1)r(m-n)(m+n+r)}{(r+n)m(m+r)}, \tag{4.35}$$

the model gives us the unitary series of the coset,

$$\frac{\widehat{\mathfrak{sl}}(n)_r \oplus \widehat{\mathfrak{sl}}(n)_{m-n}}{\widehat{\mathfrak{sl}}(n)_{m-n+r}}. \tag{4.36}$$

We can see how the level $p$ is related with the omega-background parameters $\epsilon_1$ and $\epsilon_2$ in the 4-d side. Since $\beta = -\epsilon_1/\epsilon_2$, (4.8) yields the condition to the ratio of these parameters. Therefore, when we introduce the free parameter $\epsilon$, $\epsilon_{1,2}$ can be written respectively as

$$\epsilon_1 = \epsilon(p+n+r), \qquad \epsilon_2 = -\epsilon(p+n). \tag{4.37}$$

This result suggests that the Nekrasov–Shatashvili limit $\epsilon_1 \to 0$ (resp. $\epsilon_2 \to 0$) of the $\mathcal{N} = 2$ gauge theory on the $\mathbf{R}^4/\mathbf{Z}_r$ corresponds to the critical level limit $p + r \to -n$ (resp. $p \to -n$) of the coset model.

## Acknowledgements

We thank D. Serban for valuable discussions. The authors' research is supported in part by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture, Japan (23540316).

## References

[1] L.F. Alday, D. Gaiotto, Y. Tachikawa, Liouville correlation functions from four-dimensional gauge theories, Lett. Math. Phys. 9 (2010) 167–197, arXiv:0906.3219 [hep-th].
[2] N. Wyllard, $A_{N-1}$ conformal Toda field theory correlation functions from conformal $\mathcal{N}=2$ $SU(N)$ quiver gauge theories, J. High Energy Phys. 0911 (2009) 002, arXiv:0907.2189 [hep-th].
[3] A. Mironov, A. Morozov, On AGT relation in the case of U(3), Nucl. Phys. B 825 (2010) 1–37, arXiv:0908.2569 [hep-th].
[4] R. Dijkgraaf, C. Vafa, Toda theories, matrix models, topological strings, and $N=2$ gauge systems, arXiv:0909.2453 [hep-th].
[5] H. Itoyama, K. Maruyoshi, T. Oota, The quiver matrix model and 2d–4d conformal connection, Prog. Theor. Phys. 123 (2010) 957–987, arXiv:0911.4244 [hep-th].
[6] A. Mironov, A. Morozov, Sh. Shakirov, Matrix model conjecture for exact BS periods and Nekrasov functions, J. High Energy Phys. 1002 (2010) 030, arXiv:0911.5721 [hep-th].
[7] A. Mironov, A. Morozov, Sh. Shakirov, Conformal blocks as Dotsenko–Fateev integral discriminants, Int. J. Mod. Phys. A 25 (2010) 3173–3207, arXiv:1001.0563 [hep-th].
[8] H. Itoyama, T. Oota, Method of generating $q$-expansion coefficients for conformal block and $\mathcal{N}=2$ Nekrasov function by $\beta$-deformed matrix model, Nucl. Phys. B 838 (2010) 298–330, arXiv:1003.2929 [hep-th].
[9] A. Mironov, A. Morozov, And. Morozov, Matrix model version of AGT conjecture and generalized Selberg integrals, Nucl. Phys. B 843 (2011) 534–557, arXiv:1003.5752 [hep-th].
[10] H. Itoyama, T. Oota, N. Yonezawa, Massive scaling limit of the $\beta$-deformed matrix model of Selberg type, Phys. Rev. D 82 (2010) 085031, arXiv:1008.1861 [hep-th].
[11] A. Mironov, A. Morozov, Sh. Shakirov, A direct proof of AGT conjecture at $\beta=1$, J. High Energy Phys. 1102 (2011) 067, arXiv:1012.3137 [hep-th].
[12] S. Kanno, Y. Matsuo, H. Zhang, Extended conformal symmetry and recursion formulae for Nekrasov partition function, J. High Energy Phys. 1308 (2013) 028, arXiv:1306.1523 [hep-th].
[13] A. Morozov, A. Smirnov, Towards the proof of AGT relations with the help of the generalized Jack polynomials, Lett. Math. Phys. 104 (2014) 585–612, arXiv:1307.2576 [hep-th].
[14] S. Mironov, And. Morozov, Y. Zenkevich, Generalized Jack polynomials and the AGT relations for the $SU(3)$ group, JETP Lett. 99 (2014) 109–113, arXiv:1312.5732 [hep-th].
[15] Y. Matsuo, C. Rim, H. Zhang, Construction of Gaiotto states with fundamental multiplets through degenerate DAHA, arXiv:1405.3141 [hep-th].
[16] V. Belavin, B. Feigin, Super Liouville conformal blocks from $\mathcal{N}=2$ $SU(2)$ quiver gauge theories, J. High Energy Phys. 1107 (2011) 079, arXiv:1105.5800 [hep-th].
[17] T. Nishioka, Y. Tachikawa, Central charges of para-Liouville and Toda theories from M-5-branes, Phys. Rev. D 84 (2011) 046009, arXiv:1106.1172 [hep-th].
[18] A. Belavin, V. Belavin, M. Bershtein, Instantons and 2d superconformal field theory, J. High Energy Phys. 1109 (2011) 117, arXiv:1106.4001 [hep-th].
[19] G. Bonelli, K. Maruyoshi, A. Tanzini, Instantons on ALE spaces and super Liouville conformal field theories, J. High Energy Phys. 1108 (2011) 056, arXiv:1106.2505 [hep-th];
G. Bonelli, K. Maruyoshi, A. Tanzini, Gauge theories on ALE space and super Liouville correlation functions, Lett. Math. Phys. 101 (2012) 103–124, arXiv:1107.4609 [hep-th].
[20] N. Wyllard, Coset conformal blocks and $\mathcal{N}=2$ gauge theories, arXiv:1109.4264 [hep-th].
[21] B. Estienne, V. Pasquier, R. Santachiara, D. Serban, Conformal blocks in Virasoro and W theories: duality and the Calogero–Sutherland model, Nucl. Phys. B 860 (2012) 377–420, arXiv:1110.1101 [hep-th].
[22] Y. Ito, Ramond sector of super Liouville theory from instantons on an ALE space, Nucl. Phys. B 861 (2012) 387–402, arXiv:1110.2176 [hep-th].
[23] M.N. Alfimov, G.M. Tarnopolsky, Parafermionic Liouville field theory and instantons on ALE spaces, J. High Energy Phys. 1202 (2012) 036, arXiv:1110.5628 [hep-th].

[24] A.A. Belavin, M.A. Bershtein, B.L. Feigin, A.V. Litvinov, G.M. Tarnopolsky, Instanton moduli spaces and bases in coset conformal field theory, Commun. Math. Phys. 319 (2013) 269–301, arXiv:1111.2803 [hep-th].

[25] A.A. Belavin, M.A. Bershtein, G.M. Tarnopolsky, Bases in coset conformal field theory from AGT correspondence and Macdonald polynomials at the roots of unity, arXiv:1211.2788 [hep-th].

[26] M.N. Alfimov, A.A. Belavin, G.M. Tarnopolsky, Coset conformal field theory and instanton counting on $\mathbb{C}^2/\mathbb{Z}_p$, J. High Energy Phys. 1308 (2013) 134, arXiv:1306.3938 [hep-th].

[27] J. Shiraishi, H. Kubo, H. Awata, S. Odake, A quantum deformation of the Virasoro algebra and the Macdonald symmetric functions, Lett. Math. Phys. 38 (1996) 33–51, arXiv:q-alg/9507034.

[28] B. Feigin, E. Frenkel, Quantum $\mathcal{W}$-algebras and elliptic algebras, Commun. Math. Phys. 178 (1996) 653–678, arXiv:q-alg/9508009.

[29] H. Awata, H. Kubo, S. Odake, J. Shiraishi, Quantum $\mathcal{W}_N$ algebras and MacDonald polynomials, Commun. Math. Phys. 179 (1996) 401–416, arXiv:q-alg/9508011.

[30] H. Awata, Y. Yamada, Five-dimensional AGT relation and the deformed $\beta$-ensemble, Prog. Theor. Phys. 124 (2010) 227–262, arXiv:1004.5122 [hep-th].

[31] F. Nieri, S. Pasquetti, F. Passerini, 3d & 5d gauge theory partition functions as $q$-deformed CFT correlators, arXiv:1303.2626 [hep-th].

[32] H. Itoyama, T. Oota, R. Yoshioka, 2d–4d connection between $q$-Virasoro/W block at root of unity limit and instanton partition function on ALE space, Nucl. Phys. B 877 (2013) 506–537, arXiv:1308.2068 [hep-th];
H. Itoyama, T. Oota, R. Yoshioka, $q$-Virasoro algebra at root of unity limit and 2d–4d connection, J. Phys. Conf. Ser. 474 (2013) 012022.

[33] M.-C. Tan, An M-theoretic derivation of a 5d and 6d AGT correspondence, and relativistic and elliptized integrable systems, J. High Energy Phys. 1312 (2013) 031, arXiv:1309.4775 [hep-th].

[34] D. Orlando, A stringy perspective on the quantum integrable model/gauge correspondence, arXiv:1310.0031 [hep-th].

[35] L. Bao, V. Mitev, E. Pomoni, M. Taki, F. Yagi, Non-Lagrangian theories from brane junctions, J. High Energy Phys. 1401 (2014) 175, arXiv:1310.3841 [hep-th].

[36] F. Nieri, S. Pasquetti, F. Passerini, A. Torrielli, 5D partition functions, $q$-Virasoro systems and integrable spin-chains, arXiv:1312.1294 [hep-th].

[37] H. Itoyama, A. Mironov, A. Morozov, Matching branches of non-perturbative conformal block at its singularity divisor, arXiv:1406.4750 [hep-th].

[38] P. Bouwknegt, K. Pilch, The deformed Virasoro algebra at roots of unity, Commun. Math. Phys. 196 (1998) 249–288, arXiv:q-alg/9710026.

[39] K. Takemura, D. Uglov, The orthogonal eigenbasis and norms of eigenvectors in the spin Calogero–Sutherland model, J. Phys. A 30 (1997) 3685–3718.

[40] D. Uglov, Yangian Gelfand-Zetlin bases, $\mathfrak{gl}_N$-Jack polynomials and computation of dynamical correlation functions in the spin Calogero–Sutherland model, Commun. Math. Phys. 191 (1998) 663–696.

[41] G. Cristofano, G. Maiella, V. Marotta, A twisted conformal field theory description of the quantum Hall effect, Mod. Phys. Lett. A 15 (2000) 547–555, arXiv:cond-mat/9912287.

[42] Y. Kitazawa, N. Ishibashi, A. Kato, K. Kobayashi, Y. Matsuo, S. Odake, Operator product expansion coefficients in $N=1$ superconformal theory and slightly relevant perturbation, Nucl. Phys. B 306 (1988) 425–444.

[43] L. Alvarez-Gaumé, Ph. Zaugg, Structure constants in the $N=1$ superoperator algebra, Ann. Phys. 215 (1992) 171–230, arXiv:hep-th/9109050.

[44] A.B. Zamolodchikov, V.A. Fateev, Nonlocal (parafermion) currents in two-dimensional conformal quantum field theory and self-dual critical points in $Z_N$-symmetric statistical systems, Zh. Eksp. Teor. Fiz. 89 (1985) 380–399, Sov. Phys. JETP 62 (1985) 215–225;
A.B. Zamolodchikov, V.A. Fateev, Representations of the algebra of "parafermion currents" of spin 4/3 in two-dimensional conformal field theory. Minimal models and the tricritical potts $Z_3$ model, Teor. Mat. Fiz. 71 (1987) 163–178, Theor. Math. Phys. 71 (1987) 451–462.

[45] V. Marotta, Stress-tensor for parafermions from winding subalgebras of affine algebras, Mod. Phys. Lett. A 13 (1998) 853–860, arXiv:hep-th/9712031.

[46] A.B. Zamolodchikov, Exact solutions of conformal field theory in two dimensions and critical phenomena, Rev. Math. Phys. 1 (1989) 197–234.

[47] D. Gepner, New conformal field theories associated with Lie algebras and their partition function, Nucl. Phys. B 290 (1987) 10–24.