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A characterization of the components of the graphs $D(k, q)$ ☆

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Abstract

We study the graphs $D(k, q)$ of [4] with particular emphasis on their connected components when q is odd. In [6] the authors proved that these components (most often) provide the best-known asymptotic lower bound for the greatest number of edges in graphs of their order and girth. It was further shown in [6] that $D(k, q)$ has at least q^{t-1} components, where $t = \lfloor (k+2)/4 \rfloor$. In this paper we prove that the value q^{t-1} is precise and that the numerical invariant introduced in [6] completely characterizes the components of $D(k, q)$. Some general results regarding the relationship between $D(l, q)$ and $D(k, q)$ $(l < k)$ are also obtained.

Résumé

On étudie les graphes $D(k, q)$ de [4], en prêtant une attention partciluière à leurs composantes connexes dans le cas où q est impair. Dans l'article [6], les auteurs ont prouvé que ces composantes produisent (dans la plupart des cas) la meilleure borne inférieure asymptotique que l'on connaisse pour le nombre maximum d'arêtes dans un graphe dont le nombre de sommets et la longueur du cycle le plus court sont spécifiés. Toujours dans [6] il est démontré que $D(k, q)$ a au moins q^{t-1} composantes, où $t = \lfloor (k+2)/4 \rfloor$. Dans l'article présent nous montrons que la valeur q^{t-1} est atteinte et qu'un certain paramètre introduit dans [6] permet de caractériser complètement les composantes de $D(k, q)$. On obtient aussi des résultats généraux portant sur la relation entre $D(l, q)$ et $D(k, q)$ pour $l < k$.

1. Introduction

In this paper we investigate the graphs $D(k, q)$ introduced in [4], with particular emphasis on the connected components of such graphs when q is odd. These graphs

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first arose in the context of extremal graph theory because of their large girth (i.e., length of a shortest cycle) for graphs of their order (number of vertices) and size (number of edges). Their definition appears in Section 2. Recently, it was discovered that these graphs possess many interesting properties both related and unrelated to extremal graph theory, see [4,3]. Each graph $D(k,q)$ is q-regular, bipartite, of order $2q^k$, girth at least $k+5$ (for k odd), and its automorphism group is transitive on each of its bipartition sets, as well as on its set of edges. Recently, the authors showed in [6] that $D(k,q)$ is disconnected for $k \geqslant 6$, and the number $N_{k,q}$ of its connected components is at least $q^{\lfloor (k+2)/4 \rfloor - 1}$. This finding was especially important from the extremal graph theoretic points of view, the sense in which we shall briefly describe below. A more thorough discussion can be found in [6].

Since all connected components are isomorphic (by edge transitivity), we denote by $CD(k,q)$ any component of $D(k,q)$. For odd $k \geqslant 1$, graphs $CD(k,q)$ provide the best-known asymptotic lower bound for the greatest number of edges in graphs of order v and girth $g \geqslant 5$, $g \neq 11, 12$, namely $\Omega\left(v^{1+(k-\lfloor (k+2)/4 \rfloor+1)^{-1}}\right)$. For $g \geqslant 24$, this represents a slight improvement to the bounds independently established by Margulis [9] and Lubotzky et al. [8]; for $5 \leqslant g \leqslant 23$, $g \neq 11, 12$, it improves or ties existing bounds.

Graphs $CD(k,q)$ also form a family of graphs of large girth in the sense of Biggs [1]: Let $\{G_i\}_{i \geqslant 1}$ be a family of graphs such that each G_i is an r-regular graph of girth g_i and increasing order v_i. We say that $\{G_i\}$ is a family of graphs with *large girth* if

$$g_i \geqslant \gamma \log_{r-1}(v_i)$$

for some positive constant γ. It is well known (e.g., see [2]) that $\gamma \leqslant 2$, but no family has been found for which $\gamma = 2$. The results from the previous paragraph imply that for the graphs $CD(k,q)$, we have $\gamma \geqslant \frac{4}{3} \log_q(q-1)$. Currently, the largest known value of γ is $\frac{4}{3}$, see [8,9].

The goal of this paper is to better understand the structure of $D(k,q)$ and that of its components $CD(k,q)$, thereby strengthening the results of [6]. In Section 2 we define the family of graphs $D(k,q)$ and discuss the component vector introduced in [6]. In Section 3 we discuss the notion of projecting larger graphs from the family onto smaller ones, as well as the inverse operation of lifting graphs. In Section 4 certain automorphisms of $D(k,q)$ are investigated. These automorphisms will turn out to be critical in proving that the component vector mentioned above completely characterizes the components of the graph. This and related results are proved in Section 5.

2. The family $D(k,q)$

In this section we give the definition of the graphs $D(k,q)$. (For additional information, see [4]. For motivation behind their construction, see [5].)

Let q be a prime power, and let P and L be two copies of the countably infinite-dimensional vector space V over $GF(q)$. Elements of P will be called *points* and those of L *lines*. In order to distinguish points from lines we introduce the use of parentheses and brackets: If $v \in V$, then $(v) \in P$ and $[v] \in L$. It will also be advantageous to adopt the notation for coordinates of points and lines introduced in [4]:

$$(p) = (p_1, p_{11}, p_{12}, p_{21}, p_{22}, p'_{22}, p_{23}, p_{32}, \ldots, p_{ii}, p'_{ii}, p_{i,i+1}, p_{i+1,i}, \ldots),$$
$$[l] = [l_1, l_{11}, l_{12}, l_{21}, l_{22}, l'_{22}, l_{23}, l_{32}, \ldots, l_{ii}, l'_{ii}, l_{i,i+1}, l_{i+1,i}, \ldots].$$

We now define an incidence structure (P,L,I) as follows. We say point (p) is *incident* to line $[l]$, and we write $(p)I[l]$, if the following relations on their coordinates hold:

$$l_{11} - p_{11} = l_1 p_1,$$
$$l_{12} - p_{12} = l_{11} p_1,$$
$$l_{21} - p_{21} = l_1 p_{11},$$
$$l_{ii} - p_{ii} = l_1 p_{i-1,i},$$
$$l'_{ii} - p'_{ii} = l_{i,i-1} p_1,$$
$$l_{i,i+1} - p_{i,i+1} = l_{ii} p_1,$$
$$l_{i+1,i} - p_{i+1,i} = l_1 p'_{ii}.$$

(The last four relations are defined for $i \geqslant 2$.) These incidence relations for (P,L,I) become adjacency relations for a related bipartite graph, namely the *incidence graph* of (P,L,I), which has vertex set $P \cup L$ and edge set consisting of all pairs $\{(p),[l]\}$ for which $(p)I[l]$.

For each integer $k \geqslant 2$ we obtain an incidence structure (P_k, L_k, I_k) as follows. First, P_k and L_k are obtained from P and L, respectively, by projecting each vector onto its k initial coordinates. Incidence I_k is then defined by imposing the first $k-1$ incidence relations and ignoring all others. For fixed q, the incidence graph corresponding to the structure (P_k, L_k, I_k) is denoted by $D(k,q)$. Obviously, $D(k,q)$ is bipartite of order $2q^k$, and one easily shows it is q-regular. In [4] it was established that girth of $D(k,q)$ is at least $k+5$ (k odd), and that its automorphism group is transitive on each of its bipartition sets and on its set of edges.

Let $k \geqslant 6$, $t = \lfloor (k+2)/4 \rfloor$, and $u = (u_1, u_{11}, \ldots, u'_{tt}, \ldots)$ be a vertex of $D(k,q)$ (it does not matter whether u is a point or line). For every r, $2 \leqslant r \leqslant t$, let

$$a_r = a_r(u) = \sum_{i=0}^{r} (u_{ii} u'_{r-i,r-i} - u_{i,i+1} u_{r-i,r-i-1}),$$

and $\boldsymbol{a} = \boldsymbol{a}(u) = (a_2, a_3, \ldots, a_t)$. (Here we define $p_{0,-1} = l_{0,-1} = p_{10} = l_{01} = 0$, $p_{00} = l_{00} = -1$, $p'_{00} = l'_{00} = 1$, $p_{01} = p_1$, $l_{10} = l_1$, $l'_{11} = l_{11}$, $p'_{11} = p_{11}$.)

In [7] the following result is proved, see also [6].

Proposition 2.1. *Let u and v be vertices from the same component of $D(k,q)$. Then $\boldsymbol{a}(u) = \boldsymbol{a}(v)$. Moreover, for any $t-1$ field elements $x_i \in GF(q)$, $2 \leqslant i \leqslant t = \lfloor (k+2)/4 \rfloor$,* *there exists a vertex v of $D(k,q)$ for which*

$$a(v) = (x_2, x_3, \dots, x_t).$$

Our goal is to establish the converse of Proposition 2.1, which we are able to do for q odd. This proves that the vector a of Proposition 2.1 serves to characterize the components of $D(k,q)$, q odd (see Section 5). The case when q is even requires further investigation and is not presented here.

3. Projections and lifts

Denote the vertex set and edge set of graph $D(k,q)$ by $V(k,q)$ and $E(k,q)$, respectively. In what follows we shall use the "non-brace" notation for edges, i.e., vu in place of $\{v,u\}$.

For $k > l$, define the *canonical projection*

$$\pi_l^k\colon V(k,q) \to V(l,q)$$

as the map which sends each vertex $v \in V(k,q)$ to the vertex $v' \in V(l,q)$ of the same type (point or line) whose coordinates coincide with the initial l coordinates of v. In this case, we also refer to v as a *lift* of v' from $D(l,q)$ to $D(k,q)$. For simple graphs G and H, a *graph homomorphism* of G to H is a mapping $\phi\colon V(G) \to V(H)$ such that adjacent vertices of G are mapped to adjacent vertices of H. Note that according to this definition, endpoints of an edge of G must have distinct images.

Proposition 3.1. *π_l^k is a (q^{k-l})-to-1 surjective graph homomorphism.*

Proof. Clearly, π_l^k is a (q^{k-l})-to-1 surjective map as the lift of any vertex to $D(k,q)$ is uniquely determined by its $k-l$ final coordinates. To show it is a graph homomorphism we must prove that π_l^k preserves adjacency, i.e., that $\pi_l^k(v)\pi_l^k(u) \in E(l,q)$ whenever $vu \in E(k,q)$. But this follows from the fact that any point-line pair of vertices which satisfies the first $k-1$ incidence relations must *a fortiori* satisfy the first $l-1$ relations. □

In [4] it was observed that for any $v \in V(k,q)$ and $x \in GF(q)$, there exists a unique $u \in V(k,q)$ having x as its first coordinate such that $vu \in E(k,q)$. (This follows immediately from the system defining adjacency in $D(k,q)$.) Using this it is trivial to prove

Lemma 3.1. *Let $v \in V(k,q)$, $u' \in V(l,q)$ with $\pi_l^k(v)u' \in E(l,q)$. Then there exists unique $u \in V(k,q)$ with $\pi_l^k(u) = u'$ and $vu \in E(k,q)$.*

Proof. Omitted. □

As a consequence of Proposition 3.1, π_l^k induces a map on edges,

$$\tilde{\pi}_l^k\colon E(k,q) \to E(l,q),$$

defined by $\tilde{\pi}_l^k\colon vu \mapsto \pi_l^k(v)\pi_l^k(u)$.

Proposition 3.2. *$\tilde{\pi}_l^k$ is a (q^{k-l})-to-1 surjective map. Moreover, the q^{k-l} edges of $D(k,q)$ which are preimages of a fixed edge of $D(l,q)$ are pairwise vertex-disjoint.*

Proof. Fix $v'u' \in E(l,q)$. By Proposition 3.1 there are q^{k-l} lifts v of v'. By Lemma 3.1 each lift v gives rise to a unique preimage vu of $v'u'$ in $D(k,q)$, and all preimages arise in this manner. Now suppose there exist edges $e, f \in E(k,q)$, each a preimage of $v'u' \in E(l,q)$, which share a common vertex $w \in V(k,q)$. Without loss of generality, $\pi_l^k(w) = v'$. By Lemma 3.1 there exists unique $u \in V(k,q)$ with $\pi_l^k(u) = u'$ and $wu \in E(k,q)$. But then $e = wu = f$. □

For a subgraph H of $D(k,q)$, we define $\pi_l^k(H)$ to be the subgraph of $D(l,q)$ with vertex set $\pi_l^k(V(H))$ and edge set $\tilde{\pi}_l^k(E(H))$. The following proposition allows us to extend the notion of lifts to trees.

Proposition 3.3. *Let T' be a tree in $D(l,q)$ and fix $v' \in V(T')$. Then for each lift v of v' to $D(k,q)$ there exists a unique tree T in $D(k,q)$ with $v \in V(T)$ such that $\pi_l^k(T) = T'$. Moreover, the q^{k-l} trees in $D(k,q)$ which are preimages of T' (and so contain a lift of v') are pairwise vertex-disjoint.*

Proof. Induct on the number of edges of T'. Proposition 3.1 treats the case of no edges; Proposition 3.2 treats the case of a single edge. □

Any tree T which projects onto T' will be called a *lift* of T'. A moment's reflection will reveal that the set of lifts of T' does not depend on the vertex $v \in V(T')$ chosen; thus Proposition 3.3 can be restated as 'Each tree in $D(l,q)$ lifts to q^{k-l} trees in $D(k,q)$ which are pairwise vertex-disjoint.' Note also that $T \cong T'$ for each lift T of T'.

Proposition 3.4. *Let C be a component of $D(k,q)$. Then $\pi_l^k(C)$ is a component of $D(l,q)$.*

Proof. By Proposition 3.1, π_l^k preserves adjacency, so also connectedness. Thus $\pi_l^k(C)$ is connected. Let C' be the component of $D(l,q)$ which contains $\pi_l^k(C)$. We want to show that $C' = \pi_l^k(C)$. Let $u \in V(C)$, $u' = \pi_l^k(u)$, and v' be an arbitrary vertex of C' distinct from u'. Let now P' be a u'–v' path in C'. By Proposition 3.3, P' lifts to a u–w path in $D(k,q)$, where w is a preimage of v'. As w is clearly a vertex in C, we have $v' = \pi_l^k(w) \in \pi_l^k(C)$, i.e., $C' \subseteq \pi_l^k(C)$. Thus, $C' = \pi_l^k(C)$ and the proposition is proved. □

For any $A \subset V(k,q)$, let $\pi_l^k|_A$ denote the restriction of π_l^k to A.

Proposition 3.5. *Let C be a fixed component of $D(k,q)$. Then $\pi_l^k|_C$ is a t-to-1 graph homomorphism for some t, $1 \leqslant t \leqslant q^{k-l}$.*

Proof. That $\pi_l^k|_C$ is a homomorphism follows from the fact that π_l^k is. Let now v' and u' be two vertices of $\pi_l^k(C)$ and denote by $t_{v'}$ and $t_{u'}$, respectively, the number of lifts of v' and u' to C. We show $t_{v'} = t_{u'}$, whence $\pi_l^k|_C$ is a t-to-1 map for $t = t_{v'}$. By Proposition 3.4, $\pi_l^k(C)$ is a component of graph $D(l,q)$; fix a path P' from v' to u'. By Proposition 3.3, each lift v of v' determines a unique path P in $D(k,q)$ which contains v and is a lift of P'. Then there are precisely $t_{v'}$ such paths P. But Proposition 3.3 further asserts that the paths P are pairwise vertex-disjoint. Thus, the remaining endpoints of the paths P (i.e., those which are not lifts of v') must be *distinct* lifts of u', whence $t_{u'} \geqslant t_{v'}$. By symmetry, $t_{v'} \geqslant t_{u'}$, so $t_{v'} = t_{u'}$. The inequality $1 \leqslant t \leqslant q^{k-l}$ is obvious. □

We should mention at this point that the notion of lifts cannot be extended beyond forests; indeed, cycles do not generally lift to cycles. One simple argument for this follows from the inequality $g(D(k,q)) \geqslant k+5$, k odd, where $g(D(k,q))$ denotes the girth of $D(k,q)$ (see [4]). However, we prefer to illustrate precisely where any attempt to extend the notion of lifts to cycles breaks down. So suppose K' is a cycle in $D(l,q)$ and let $v', u' \in V(K')$. We can consider K' as the union of two paths, P' and Q', each between v' and u', which have distinct internal vertices. Any lift K of K' coincides with the mutual lifting of P' and Q' to certain paths P and Q. Furthermore, if K is to be a cycle we must impose that P and Q are 'doubly joined,' i.e., that each of v' and u' has a common lift in both P and Q. While this can certainly be done for either v' or u', there is no general procedure which achieves this simultaneously for both vertices. Thus, K' may well lift to a (non-closed) path.

We have observed that $g(D(k,q))$ tends to infinity as k does. That this is also true of the diameter of any component of $D(k,q)$ follows from the inequality $\operatorname{diam}(CD(k,q)) \geqslant g(D(k-1,q))$. In actuality, both $g(D(k,q))$ and $\operatorname{diam}(CD(k,q))$ are nondecreasing functions of k. This can be established using nothing stronger than the notions of projection and lifts. Proofs can be found in [10].

4. Automorphisms

Many of the proofs we present in Section 5 depend heavily on the existence of certain automorphisms of the graph $D(k,q)$. The purpose of this brief section is to provide the reader with a detailed description of these automorphisms. In Table 1 we reproduce a list that initially appeared in [4]. An entry of the table illustrates the action of the map which heads its column on the coordinate of a line or a point which heads its row. If the action of a map on a specific coordinate is not listed, it means that the

Table 1
Certain automorphisms of $D(k,q)$

$i \geq 0$	$t_1(x)$	$t_2(x)$	$t_{1,1}(x)$	$t_{m,m+1}(x)$ $m \geq 1$	$t_{m+1,m}(x)$ $m \geq 1$	$t_{m,m}(x)$ $m \geq 2$	$t'_{m,m}(x)$ $m \geq 2$
l_{ii}		$+l_{i,i-1}x$	$-l_{i-1,i-1}x$	$+l_{r,r-1}x$, $r = i-m \geq 1$		$-l_{rr}x$, $r = i-m \geq 0$	
$l_{i,i+1}$		$+(l_{ii}+l'_{ii})x$ $+l_{i,i-1}x^2$	$-l_{i-1,i}x$	$+l'_{rr}x$, $r = i-m \geq 0$		$-l_{r,r+1}x$, $r = i-m \geq 0$	
$l_{i+1,i}$	$+l_{ii}x$		$+l_{i,i-1}x$		$-l_{rr}x$, $r = i-m \geq 0$		$+l_{r+1,r}x$, $r = i-m \geq 0$
l'_{ii}	$+l_{i-1,i}x$	$+l_{i,i-1}x$	$+l'_{i-1,i-1}x$		$-l_{r-1,r}x$, $r = i-m \geq 1$		$+l'_{rr}x$, $r = i-m \geq 0$
p_{ii}	$+p_{i-1,i}x$	$+p_{i,i-1}x$	$-p_{i-1,i-1}x$	$+p_{r,r-1}x$, $r = i-m \geq 1$		$-p_{rr}x$, $r = i-m \geq 0$	
$p_{i,i+1}$		$+p'_{ii}x$	$-p_{i+1,i}x$	$+p'_{rr}x$, $r = i-m \geq 0$		$-p_{r,r+1}x$, $r = i-m \geq 0$	
$p_{i+1,i}$	$+(p_{ii}+p'_{ii})x$ $+p_{i-1,i}x^2$		$+p_{i,i-1}x$		$-p_{rr}x$, $r = i-m \geq 0$		$+p_{r+1,r}x$, $r = i-m \geq 0$
p'_{ii}	$+p_{i-1,i}x$		$+p'_{i-1,i-1}x$		$-p_{r-1,r}x$, $r = i-m \geq 1$		$+p'_{rr}x$, $r = i-m \geq 0$

$l_{00} = p_{00} = -1,\ l_{01} = p_{10} = 0,\ l_{10} = l_1,\ p_{01} = p_1,\ l'_{11} = l_{11},\ p'_{11} = p_{11},\ p'_{00} = l'_{00} = 1,\ p_{-1,0} = l_{0,-1} = p_{0,-1} = l_{-1,0} = 0.$

coordinate is fixed by the map. For example, the map $t_2(x)$ changes every coordinate $l_{i,i+1}$, $i \geqslant 1$, of a line $[l]$ according to the rule $l_{i,i+1} \to l_{i,i+1} + (l_{ii} + l'_{ii})x + l_{i,i-1}x^2$, and leaves every coordinate $p_{i+1,i}$, $i \geqslant 1$, of a point (p) fixed; the map $t_{1,1}(x)$ changes every coordinate p_{ii}, $i \geqslant 1$, of a point (p) according to the rule $p_{ii} \to p_{ii} - p_{i-1,i-1}x$; the map $t_{5,6}(x)$ does not change the coordinate of any line $[l]$ and point (p) which precedes the coordinate l_{56} or p_{56}, respectively.

Table 1 is by no means intended to be complete; in fact, new 'multiplicative' automorphisms $m(x,y)$ are defined below. We later prove that each $m(x,y)$ stabilizes the component $C(0)$ of $D(k,q)$ to which the point $(0) := (0,\dots,0)$ belongs.

4.1. Multiplicative automorphisms

Let point (p) and line $[l]$ be given, respectively, by

$$(p) = (p_1, p_{11}, p_{12}, p_{21}, \dots, p_{ii}, p'_{ii}, p_{i,i+1}, p_{i+1,i}, \dots),$$

$$[l] = [l_1, l_{11}, l_{12}, l_{21}, \dots, l_{ii}, l'_{ii}, l_{i,i+1}, l_{i+1,i}, \dots].$$

For $x, y \in GF(q)$, $x \neq 0$, $y \neq 0$, we define $m(x,y)$: $V(k,q) \to V(k,q)$ as the map which sends (p) and $[l]$, respectively, to

$$(xp_1, xyp_{11}, x^2yp_{12}, xy^2p_{21}, \dots, x^iy^ip_{ii}, x^iy^ip'_{ii}, x^{i+1}y^ip_{i,i+1}, x^iy^{i+1}p_{i+1,i}, \dots),$$

$$[yl_1, xyl_{11}, x^2yl_{12}, xy^2l_{21}, \dots, x^iy^il_{ii}, x^iy^il'_{ii}, x^{i+1}y^il_{i,i+1}, x^iy^{i+1}l_{i+1,i}, \dots].$$

Proposition 4.1. *$m(x,y)$ is an automorphism of $D(k,q)$ for each choice of $x, y \in GF(q)$, $x \neq 0$, $y \neq 0$.*

Proof. From the incidence relations (see Section 2), it is immediate that each $m(x,y)$ preserves adjacency. It is also easy to see that $m(1,1)$ is the identity automorphism and that $m(x,y)m(x^{-1},y^{-1}) = m(1,1)$. The result follows. □

Remark. We refer to the automorphisms $m(x,y)$ as multiplicative since each $m(x,y)$ is the unique automorphism of $D(k,q)$ determined by multiplying the initial coordinate of every point by x and every line by y.

Let $C(v)$ denote the component of $D(k,q)$ which contains the vertex $v \in V(k,q)$.

Proposition 4.2. *Let τ be an automorphism of $D(k,q)$. Then τ stabilizes $C(v)$ if and only if $v^\tau \in C(v)$.*

Proof. First observe that $u \in C(v)$ if and only if $u^\tau \in C(v^\tau)$. (Indeed, if P is a path from u to v then P^τ is a path from u^τ to v^τ, and conversely.) But clearly, $C(v^\tau) = C(v)^\tau$. Thus, τ stabilizes $C(v)$ if and only if $C(v^\tau) = C(v)$, i.e., if and only if $v^\tau \in C(v)$. □

Corollary 4.1. *$m(x,y)$ stabilizes $C(0)$.*

Proof. Immediate from Proposition 4.2 as $(0)^{m(x,y)} = (0)$. □

5. Main results

In this section we consider only projections and lifts between $D(k,q)$ and $D(k-1,q)$. We shall denote π^k_{k-1} simply by π throughout. Also C (respectively, C') shall denote the component of $D(k,q)$ (respectively, $D(k-1,q)$) which contains $(0) \in V(k,q)$ (respectively, $(0) \in V(k-1,q)$), and $\pi|_C$ will be the restriction of π to C. Finally, let $N_{k,q}$ denote the number of distinct components of $D(k,q)$. Note that $2q^k = |V(k,q)| = N_{k,q}|C|$ since all components of $D(k,q)$, being isomorphic, have the same cardinality.

Proposition 5.1. *Let $k \equiv 2 \pmod 4$, $k \geqslant 6$. Then $\pi|_C$ is a bijection, so an isomorphism.*

Proof. By Proposition 3.5, $\pi|_C$ is t-to-1 for some t, $1 \leqslant t \leqslant q$. By Proposition 3.4, $\pi(C) = C'$. We claim that $(0) \in V(C') \subseteq V(k-1,q)$ has a unique lift to $V(C)$, namely to $(0) \in V(C)$. Indeed, if $(p) = (0,\ldots,0,x) \in V(C)$ is a lift of $(0) \in V(C')$, then by Proposition 2.1 $\boldsymbol{a}(p) = \boldsymbol{a}(0) = \mathbf{0}$, which forces $x = 0$. Thus $t = 1$, and $\pi|_C$ is a bijection. □

Proposition 5.2. *Let $k \equiv 0,3 \pmod 4$, and suppose $\pi|_C$ is a t-to-1 mapping for some $t \neq 1$. Then $t = q$.*

Proof. As $t \neq 1$, $(0) \in V(C')$ has at least two distinct lifts to $V(C)$. Clearly, $(0) \in V(C)$ is one such lift; let $(p) = (0,\ldots,0,x) \in V(C)$, $x \neq 0$, be another.

Case 1: $k \equiv 0 \pmod 4$. For $a \in GF(q)$, $a \neq 0$, observe that the image of (p) under $m(xa^{-1},x^{-1}a)$ is $(0,\ldots,0,a)$. It follows from Corollary 4.1 that $(0,\ldots,0,a) \in V(C)$ for all $a \in GF(q)$. Thus, $(0) \in V(C')$ has q lifts to $V(C)$, whence $t = q$ by Proposition 3.5.

Case 2: $k \equiv 3 \pmod 4$. Replace $m(xa^{-1},x^{-1}a)$ by $m(ax^{-1},a^{-1}x)$, and proceed as in case 1. □

Proposition 5.3. *$D(k,q)$ is connected for $2 \leqslant k \leqslant 5$.*

Proof. For $2 \leqslant k \leqslant 4$ it is not difficult to write down an explicit path from (0) to any point or line. For $k = 5$, imitate case 3 of the proof of Theorem 5.1 below. □

Theorem 5.1. *Let q be odd, $k \geqslant 6$. If $v \in V(k,q)$ satisfies $\boldsymbol{a}(v) = \mathbf{0}$, then $v \in V(C)$.*

Proof. The proof proceeds by induction on k. First, let $v \in V(6,q)$ satisfy $\boldsymbol{a}(v) = \mathbf{0}$, and set $v' = \pi(v) \in V(5,q)$. As $D(5,q)$ is connected (Proposition 5.3) it is clear that v' lies in $C' = D(5,q)$. By surjectivity of $\pi|_C$, there exists $w \in V(C)$ with $\pi(w) = v' = \pi(v)$. But this implies $w = v$ since the sixth coordinate of any vertex u is uniquely determined by its initial five coordinates and the value $\boldsymbol{a}(u)$. Thus, $v \in V(C)$ as claimed.

The inductive step is treated in four separate cases, depending on the congruence class of k modulo 4. In each case we assume that $v \in V(k,q)$ with $\boldsymbol{a}(v) = \mathbf{0}$ and we set $v' = \pi(v)$.

Case 1: $k \equiv 3 \pmod 4$, $k \geqslant 7$. Write $k = 4j-1$, $j \geqslant 2$. The form of points in $D(k,q)$ now becomes

$$(p) = (p_1, p_{11}, p_{12}, p_{21}, \ldots, p_{j,j-1}, p_{jj}, p'_{jj}, p_{j,j+1}).$$

Let (p') be the point of $D(k-1,q)$ with $p_{jj} = p'_{jj} = x$ and zeros elsewhere, i.e.,

$$(p') = (0,\ldots,0,x,x).$$

As $\boldsymbol{a}(p') = \mathbf{0}$, we have $(p') \in V(C')$ by induction. By surjectivity of $\pi|_C$ there exists $(p) \in V(C)$ with $\pi(p) = p'$, say

$$(p) = (0,\ldots,0,x,x,y).$$

By Corollary 4.1, we also have

$$(0,\ldots,0,x,x,-y) = (p)^{m(-1,-1)} \in V(C).$$

Thus if $y \neq 0$, then (p) and $(p)^{m(-1,-1)}$ are two distinct lifts of (p') to C (note that $y \neq -y$ as q is odd), whence $\pi|_C$ is a q-to-1 map by Proposition 5.2. But this implies all lifts of v' to $D(k,q)$ lie in C, i.e., $v \in V(C)$ as desired. So assume $y = 0$, whence

$$(p) = (0,\ldots,0,x,x,0) \in V(C)$$

for all $x \in GF(q)$. As (0) is mapped to (p) under the composite automorphism $t_{j,j}(x)t'_{j,j}(x)$, we conclude from Proposition 4.2 that $t_{j,j}(x)t'_{j,j}(x)$ is an element of the stabilizer Stab(C) of C, for all $x \in GF(q)$. Also $t_2(x) \in$ Stab(C) for all $x \in GF(q)$ as $t_2(x)$ fixes the zero line $[0]$, which is obviously in $V(C)$. Let now

$$(m) = (1, 0, \ldots, 0) \in V(k, q).$$

From the incidence relations one immediately sees that (m) is adjacent to $[0]$, whence $(m) \in V(C)$. As $t_{j,j}(1)t'_{j,j}(1)t_2(-1) \in$ Stab(C) we have $(0, \ldots, 0, 1, 1, -2) \in V(C)$, as it is the image of (m) under $t_{j,j}(1)t'_{j,j}(1)t_2(-1)$. We have now produced two distinct lifts of $(0, \ldots, 0, 1, 1)$ to C, namely $(0, \ldots, 0, 1, 1, 0)$ and $(0, \ldots, 0, 1, 1, -2)$, whence $\pi|_C$ is q-to-1 by Proposition 5.2. As above, this implies $v \in V(C)$ as desired.

Case 2: $k \equiv 0 \pmod 4$, $k \geqslant 8$. Write $k = 4j$, $j \geqslant 2$. Here the form of points is

$$(p) = (p_1, p_{11}, p_{12}, p_{21}, \ldots, p_{jj}, p'_{jj}, p_{j,j+1}, p_{j+1,j}).$$

Let (p') be the point of $D(k-1, q)$ which has $p_{j,j-1} = x$ and zeros elsewhere, i.e.,

$$(p') = (0, \ldots, 0, x, 0, 0, 0).$$

As $\boldsymbol{a}(p') = \mathbf{0}$, we have $(p') \in V(C')$ by induction. Let

$$(p) = (0, \ldots, 0, x, 0, 0, 0, y)$$

be a lift of (p') to C. One easily checks that

$$(p)^{m(-1,1)} = (0, \ldots, 0, x, 0, 0, 0, -y) \text{ for } j \text{ odd,}$$

$$(p)^{m(1,-1)} = (0, \ldots, 0, x, 0, 0, 0, -y) \text{ for } j \text{ even.}$$

In either case, Corollary 4.1 asserts that

$$(0, \ldots, 0, x, 0, 0, 0, -y) \in V(C).$$

As in case 1 we are done unless $y = 0$, so assume

$$(p) = (0, \ldots, 0, x, 0, 0, 0, 0) \in V(C).$$

This implies that $t_{j,j-1}(x) \in$ Stab(C) as $t_{j,j-1}(x)$ maps (0) to (p).

The above argument applies equally well when (p') is replaced by $(m') = (0, \ldots, 0, x) \in V(C')$. The conclusion here is that $t_{j,j+1}(x) \in$ Stab(C) for all $x \in GF(q)$.

Finally, let (r') be the point in $D(k-1, q)$ which has $p_{11} = x$, $p'_{ii} = x^i$ for $2 \leqslant i \leqslant j$, and zeros elsewhere, i.e.,

$$(r') = (0, x, 0, 0, 0, x^2, 0, 0, 0, x^3, 0, 0, 0, \ldots, 0, 0, 0, x^j, 0).$$

The reader should verify that $\boldsymbol{a}(r') = \mathbf{0}$, which is the condition that inductively gives $(r') \in V(C')$. Letting (r) be an arbitrary lift of (r') to C, and then applying to (r) the automorphism $m(-1, -1)$, one concludes that

$$(r) = (0, x, 0, 0, 0, x^2, 0, 0, 0, x^3, 0, 0, 0, \ldots, 0, 0, 0, x^j, 0, 0).$$

It is now straightforward to verify that $(0)^{\tau(x)} = (r)$, where

$$\tau(x) = t_{1,1}(x)t'_{2,2}(x^2)t'_{4,4}(x^4)\cdots t'_{s,s}(x^s),$$

$$s = \begin{cases} j, & j \text{ even}, \\ j-1, & j \text{ odd}. \end{cases}$$

We conclude that $\tau(x) \in \mathrm{Stab}(C)$.

At this point we have shown that each of $t_{j,j-1}(x)$, $t_{j,j+1}(x)$, and $\tau(x)$ stabilizes C. Thus, the composite automorphism

$$\sigma = \tau(1)t_{j,j-1}(-1)t_{j,j+1}(1)$$

stabilizes C. As $(v) = (0,\ldots,0,1,0,0,0,0)$ is in $V(C)$ (indeed, set $x = 1$ in (p) to obtain (v)), we have also that

$$(v)^\sigma = (0,1,0,0,0,1,0,0,0,1,0,0,0,\ldots,0,0,0,1,0,2)$$

is in $V(C)$. But

$$(u) = (0,1,0,0,0,1,0,0,0,1,0,0,0,\ldots,0,0,0,1,0,0)$$

is in $V(C)$ as well (indeed, (u) is obtained by setting $x = 1$ in (r)). Thus, $(v)^\sigma$ and (u) are distinct lifts of

$$(0,1,0,0,0,1,0,0,0,1,0,0,0,\ldots,0,0,0,1,0) \in V(C'),$$

whence $\pi|_C$ is a q-to-1 map by Proposition 5.2. As in case 1, the result follows.

Case 3: $k \equiv 1 \pmod 4$, $k \geqslant 9$. Write $k = 4j-3$, $j \geqslant 2$. The form of points in $D(k,q)$ is then given by

$$(p) = (p_1, p_{11}, p_{12}, p_{21}, \ldots, p'_{j-1,j-1}, p_{j-1,j}, p_{j,j-1}, p_{jj}).$$

Let (p') be the point of $D(k-1,q)$ with $p_{j-1,j} = x$ and zeros elsewhere, i.e.,

$$(p') = (0,\ldots,0,x,0).$$

As $\boldsymbol{a}(p') = \mathbf{0}$, $(p') \in V(C')$ by induction. Let (p) be a lift of (p') to C, i.e.,

$$(p) = (0,\ldots,0,x,0,y)$$

for some $y \in GF(q)$. As the reader can easily verify, (p) is fixed by the composite automorphism $t_1(1)t_{j,j}(-x)$. Thus, by Proposition 4.2, $t_1(1)t_{j,j}(-x) \in \mathrm{Stab}(C)$. But $t_1(1) \in \mathrm{Stab}(C)$ as well, since $t_1(1)$ fixes the zero point (0). Thus, $t_{j,j}(-x) \in \mathrm{Stab}(C)$ for all $x \in GF(q)$, whence, by Proposition 4.2, $(0)^{t_{j,j}(-x)} \in V(C)$ for all $x \in GF(q)$. As

$$(0)^{t_{j,j}(-x)} = (0,\ldots,0,-x),$$

we see that the zero point of $D(k-1,q)$ has q distinct lifts to C. Thus, $\pi|_C$ is a q-to-1 map and we are done as in the two previous cases.

Case 4: $k \equiv 2 \pmod 4$, $k \geqslant 10$. Clearly $\boldsymbol{a}(v') = \boldsymbol{0}$, whence $v' \in V(C')$ by induction. Let w be a lift of v' to C. As $\boldsymbol{a}(w) = \boldsymbol{0} = \boldsymbol{a}(v)$ and $\pi(w) = v' = \pi(v)$, we have $w = v$ as in the base case $k = 6$. Thus, $v \in V(C)$ and the theorem is proved. □

Corollary 5.1.

$$|C| = \begin{cases} q|C'|, & k \not\equiv 2 \pmod 4, \\ |C'|, & k \equiv 2 \pmod 4, \end{cases}$$

$$N_{k,q} = \begin{cases} N_{k-1,q}, & k \not\equiv 2 \pmod 4, \\ qN_{k-1,q}, & k \equiv 2 \pmod 4. \end{cases}$$

Proof. Both results follow at once from the observation that $\pi|_C$ is q-to-1 for $k \not\equiv 2 \pmod 4$, and an isomorphism for $k \equiv 2 \pmod 4$ (see the proof of Theorem 5.1). □

We are now ready to prove the main result of the paper, namely that the converse of Proposition 2.1 holds in the case q odd. As previously discussed, this implies that the vector $\boldsymbol{a}$ completely characterizes the components of $D(k,q)$, q odd.

Corollary 5.2. *For $x \in V(k,q)$, $k \geqslant 6$ and q odd, let $C(x)$ denote the component of $D(k,q)$ which contains x. Then, for all $u, v \in V(k,q)$,*

$$\boldsymbol{a}(u) = \boldsymbol{a}(v) \iff C(u) = C(v).$$

Proof. Let $t = \lfloor (k+2)/4 \rfloor$. From Corollary 5.1 and Proposition 5.3, one easily establishes that $N_{k,q} = q^{t-1}$. Now consider the mapping

$$f : \mathscr{C} \to Im(\boldsymbol{a})$$

defined by $f(C(v)) = \boldsymbol{a}(v)$, where $\mathscr{C}$ is the set of components of $D(k,q)$. Proposition 2.1 asserts that f is both well-defined and surjective. Since $|\mathscr{C}| = N_{k,q} = q^{t-1} = |Im(\boldsymbol{a})|$, we see that f is bijective. But this means that $C(u) = C(v)$ whenever $\boldsymbol{a}(u) = \boldsymbol{a}(v)$, whence the converse of Proposition 2.1 is established. □

As a final result, we determine precisely when $D(l,q)$ is isomorphic to a subgraph of $D(k,q)$ for $l < k$.

Corollary 5.3. *Let $k \geqslant l+1 \geqslant 3$. Then $D(l,q)$ is isomorphic to a subgraph of $D(k,q)$ if and only if $k = l+1 \equiv 2 \pmod 4$.*

Proof. Assume $k = l+1 \equiv 2 \pmod 4$. By Proposition 5.1, $C \cong C'$. As all components of $D(k,q)$ are isomorphic to C, it follows that $D(k,q) \cong N_{k,q} \cdot C$. Similarly, $D(k-1,q) \cong N_{k-1,q} \cdot C' \cong N_{k-1,q} \cdot C$. As $N_{k,q} = qN_{k-1,q}$ (Corollary 5.1), we conclude that

$$D(k,q) \cong N_{k,q} \cdot C \cong (qN_{k-1,q}) \cdot C \cong (qN_{k-1,q}) \cdot C' \cong q \cdot (N_{k-1,q} \cdot C') \cong q \cdot D(k-1,q),$$

and the result follows.

We now suppose, by way of contradiction, that ϕ is an isomorphism from $D(l,q)$ into $D(k,q)$ with either $k > l+1$ or $k = l+1 \not\equiv 2 \pmod 4$. Letting C_l denote a fixed component of $D(l,q)$, we see that $\phi(C_l)$ is connected, whence it is a subgraph of some component C_k of $D(k,q)$. But as $\phi(C_l)$ and C_k are each q-regular, we have $\phi(C_l) = C_k$, i.e., $C_l \cong C_k$. By Corollary 5.1, $|C_k| = q|C_{k-1}| \geqslant q|C_l|$ if $k \not\equiv 2 \pmod 4$, and $|C_k| = |C_{k-1}| \geqslant q|C_l|$ if $k \equiv 2 \pmod 4$ and $k > l+1$. In either case we obtain a contradiction, and the corollary is proved. □

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