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Spurious oscillation in a uniform Euler discretisation of linear stochastic differential equations with vanishing delay

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Abstract

We investigate the oscillatory behaviour of a random Euler-type difference equation, intended to serve as a discrete model of a linear Itô stochastic differential equation with vanishing delay. The oscillatory behaviour of the continuous process satisfying this differential equation was partially described in Appleby and Kelly [Asymptotic and oscillatory properties of linear stochastic delay differential equations with vanishing delay, Funct. Differential Equation 11(3–4) (2004) 235–265.] The construction of a discrete model that successfully mimics some of the properties of the continuous process would simplify the analysis, allowing the partial description to be completed. However, care must be taken; a uniform Euler discretisation yields spurious oscillatory behaviour. We present a complete analysis of the uniform scheme.

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1. Introduction

The purpose of this paper is to characterise the oscillatory behaviour of the solutions of an Euler-type delay-difference equation of the form

$$Y_{n+1} = Y_n - \Delta P_n Y_{n-r_{\mathrm{M}_\Delta}(n)}, \quad n \geqslant 0, \tag{1}$$

evolving on a uniform mesh M_Δ. Roughly speaking, r is a discrete delay function, depending on M_Δ, which vanishes in finite time, and P_n is a random variable with a time-inhomogeneously scaled lognormal distribution.

The specific structure of (1) is motivated by a desire to use a discrete model to map out the oscillatory behaviour of solutions of an Itô-type stochastic differential equation with asymptotically vanishing delay. Consider

$$\mathrm{d}X(t) = (aX(t) + bX(t-\tau(t)))\,\mathrm{d}t + \sigma X(t)\,\mathrm{d}B(t), \quad t > 0, \tag{2a}$$

$$X(t) = \psi(t), \quad -\bar{\tau} \leqslant t \leqslant 0. \tag{2b}$$

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Ideally it would be possible to define the mesh in such a way that the a.s. oscillatory behaviour of solutions of the class of equations defined by (1) corresponds to that of the solutions of (2). In fact this is possible, and details can be found in [3]. Our purpose here is to consider the class of uniform meshes; to show that they can fail in this task, and to discuss why they fail. In rough terms, our main result shows that when the uniform mesh size is too long, spurious oscillations can result.

To put the results for the difference equations in context, we first give some background on (2) and quote some results on the oscillatory behaviour of its solutions. In (2), $B=\{B(t);\mathscr{F}^B(t); 0\leqslant t<\infty\}$ is a standard one-dimensional Brownian motion defined on the probability triple $(\Omega,\mathscr{F},\mathbb{P})$. Here $(\mathscr{F}^B(t))_{t\geqslant 0}$ is the natural filtration of B: $\mathscr{F}^B(t)=\sigma(\{B(s)\ :\ 0\leqslant s\leqslant t\})$. The drift parameters satisfy $a\in\mathbb{R}$ and $b<0$, and the diffusion parameter $\sigma\neq 0$. The evolutionary variable $t\geqslant -\bar{\tau}$ marks the passage of time, and $\psi\in C([-\bar{\tau},0];\mathbb{R}^+)$ is a continuous positive initial data function. Finally, $\tau\in C([0,\infty);\mathbb{R}^+)$ is a continuous positive delay function satisfying

$$\lim_{t\to\infty}\tau(t)=0,\quad \tau(t)>0,\ \ t\mapsto t-\tau(t) \text{ is nondecreasing,} \tag{3}$$

and $-\bar{\tau}=\inf_{t\geqslant 0} t-\tau(t)$. Note that Condition (3) requires the delay function τ to be positive everywhere on its domain, and allows it to have intervals of increase. Asymptotically however, τ must vanish.

Before summarising the main results on oscillatory behaviour of solutions of (2), we observe that the connection between Eqs. (1) and (2) is not immediately evident, as the equations do not seem to have similar structure. However, in [2], it proved useful to study the solutions of another equation having coincident zeros and differentiable sample paths, of the form

$$y'(t)=-p(t)y(t-\tau(t)),\quad t>0. \tag{4}$$

The nonobvious form of (1) arises from our decision to discretise the *auxiliary* process satisfying (4), rather than discretising (2) directly. Our model need only reproduce oscillatory behaviour, and the techniques employed in [2] are easily adapted here to a discrete setting.

There is a partial description of the oscillatory behaviour of solutions of (2) given by Theorem 13 in [2]. We summarise these results here, as they allow us to examine the usefulness of (1) as a discrete model of (2). In order to ensure that solutions of (2) are almost surely (a.s.) oscillatory for all continuous positive initial data functions ψ, it is sufficient that τ be *eventually decreasing* (in the sense that there exists a time $t^*<\infty$ such that τ is a strictly decreasing function for all $t\geqslant t^*$), satisfy (3), and

$$\lim_{t\to\infty}\frac{\log(1/\tau(t))}{\sqrt{\tau(t)\log t}}=0. \tag{5}$$

It should be noted that these oscillations are the result of an interaction between the delayed feedback in the drift and the noise perturbation. They do not survive the deletion of the noise term: for the equation

$$x'(t)=ax(t)+bx(t-\tau(t)),\quad t>0,$$

$$x(t)=\psi(t),\quad -\bar{\tau}\leqslant t\leqslant 0,$$

where $b<0$ and τ is a continuous function which satisfies (3), it is always possible to pick ψ so that x is nonoscillatory. Oscillations are also precluded by the absence of delayed feedback in the drift coefficient. If we take the limiting case and set $\tau(t)\equiv 0$, then (2a) is solved by a geometric Brownian motion, which is nonoscillatory for any nonzero initial value $\psi(0)$.

Theorem 13 of [2] also seems to indicate that the critical rate of decay of the delay function—the rate at which the behaviour of solutions of (2) switches from oscillatory to nonoscillatory—is in some sense close to $(\log t)^{-1}$. It is shown that if τ is decreasing, satisfies (3) and

$$\lim_{t\to\infty}\tau(t)\log t=0, \tag{6}$$

then all nontrivial solutions of (2a), with an appropriate choice of positive initial data, are positive with probability arbitrarily close to one. In this paper, we test whether the oscillatory behaviour of solutions of the discretised equation (1) is consistent with that of solutions of the continuous equation (2), by seeing whether the discrete scheme can reproduce nonoscillatory solutions when τ obeys (6), and oscillatory solutions when τ obeys (5).

While the classification of oscillatory behaviour given by Theorem 13 of [2] is useful as a guide, it is incomplete. The positivity result applies to a process similar to that described by (2a), but with strong restrictions on the initial data. Additionally, the positivity result cannot be applied with probability one to any such process. Nor does the theorem give a precise decay rate of τ where solutions of (2) switch from oscillatory to nonoscillatory behaviour. The continuous analysis has limitations that are not necessarily shared by a discrete analysis. Our analysis here seeks to establish a classification of oscillatory behaviour of a class of stochastic difference equation, which cannot necessarily be obtained for a related class continuous equations.

Our work here relates to other research which cautions on the use of Euler methods when discretising even the simplest of ordinary differential equations. A nice illustration of this can be found in the introduction to the paper by Mohamad and Gopalsamy [10]. There, examples are given of ordinary differential equations, including the logistic equation and the simple linear equation

$$y'(t) = -y(t), \quad t > 0,$$

that have an Euler discretisation displaying spurious behaviour that arises from the discretisation process. This uncharacteristic behaviour is misleading, and its occurrence must carefully be avoided.

As stated, the purpose of this paper is to show how such misleading behaviour can arise. Discretisation over a uniform mesh removes the effect of the delay after a finite interval. In the absence of noise, this approach would be perfectly valid: see, for example, Karoui and Vaillancourt [7], who apply it to general deterministic nonlinear vanishing delay equations. In the presence of a noise perturbation, the evidence suggests that the delay can qualitatively affect the behaviour of the process regardless of how small it is: see Appleby and Buckwar [1] for further details. If the asymptotic effect of the delay is not present after discretisation, the interaction between the delay and the noise that determines the oscillatory behaviour is ignored.

2. Zero-set oscillation and the discretisation of (2)

In [2], we worked with a definition of oscillation based on the finiteness of the zero-set of a continuous process. In this section we revisit that definition, and show how an auxiliary process with identical oscillatory behaviour and differentiable sample paths can be constructed. We also consider a natural definition of oscillation in the discrete realm, and carry out a preliminary discretisation of the paths of the auxiliary process. In Section 3, we model the resulting discrete process with a random difference equation displaying the same oscillatory behaviour as that constructed in Section 2.4, but which allows us to account for the lack of independence displayed by consecutive terms of the solution of the difference equation. This lack of independence arises from the delay.

2.1. Oscillation of continuous processes

We say that a nontrivial continuous function $y : [t_0, \infty) \to \mathbb{R}$ is *oscillatory* if the set $Z_y = \{t \geqslant t_0 : y(t) = 0\}$ satisfies $\sup Z_y = \infty$. A function which is not oscillatory is called *nonoscillatory*. In [2], a continuous stochastic process was said to be a.s. oscillatory if these notions were extended in the following intuitive manner:

A stochastic process $\{X(t,\omega)\}_{t \geqslant t_0}$ defined on a probability triple $(\Omega, \mathscr{F}, \mathbb{P})$ with continuous sample paths is said to be a.s. *oscillatory* if there exists $\Omega^* \subseteq \Omega$ with $\mathbb{P}[\Omega^*] = 1$ such that for all $\omega \in \Omega^*$, the path $X(\cdot, \omega)$ is oscillatory.

A stochastic process is a.s. *nonoscillatory* if there exists $\Omega^* \subseteq \Omega$ with $\mathbb{P}[\Omega^*] = 1$ such that for all $\omega \in \Omega^*$, the path $X(\cdot, \omega)$ is nonoscillatory.

2.2. The decomposition of solutions of (2)

The structure of our discrete process relies on the following decomposition. Using the same standard Brownian motion B that drives the noise perturbation in (2a), define a geometric Brownian motion $\{\varphi(t)\}_{t \geqslant -\overline{\tau}}$ solving

$$\mathrm{d}\varphi(t) = a\varphi(t)\,\mathrm{d}t + \sigma\varphi(t)\,\mathrm{d}B(t), \quad t > 0,$$

$$\varphi(t) = 1, \quad t \in [-\overline{\tau}, 0].$$

Define $y(t)=X(t)/\varphi(t)$ for $t\geqslant-\overline{\tau}$, where X solves (2). By stochastic integration by parts, y satisfies

$$y(t)=y(0)+\int_0^t by(s-\tau(s))\varphi(s-\tau(s))\varphi(s)^{-1}\,\mathrm{d}s,\quad t\geqslant 0,$$

which can be written as

$$y'(t)=b\varphi(t)^{-1}\varphi(t-\tau(t))y(t-\tau(t)),\quad t>0, \tag{7a}$$

$$y(t)=\psi(t),\quad t\in[-\overline{\tau},0]. \tag{7b}$$

Clearly, $y\in C^1((0,\infty);\mathbb{R})$, and moreover, we have

$$y'(t)=-p(t)y(t-\tau(t)),\quad t>0, \tag{8}$$

where p is a random process. Since $t\mapsto t-\tau(t)$ is nondecreasing, there exists $t^*=\inf\{t>0:t-\tau(t)=0\}$, so that $t-\tau(t)\geqslant 0$ for all $t>t^*$. Then, letting $\lambda=a-\frac{1}{2}\sigma^2$, the path $p(\cdot,\omega)$ can be defined by

$$p(t,\omega)=\begin{cases}-b\mathrm{e}^{-\lambda\tau(t)}\mathrm{e}^{-\sigma(B(t,\omega)-B(t-\tau(t),\omega))}, & t>t^*,\\ -b\mathrm{e}^{-\lambda t-\sigma B(t,\omega)}, & t\leqslant t^*.\end{cases} \tag{9}$$

Since we will discretise the individual *paths* of (8), we represent each possible outcome of the random process p in (9) as a separate parameter ω, indicating that this definition holds for each individual ω.

The solution of (2) can thus be written as the product of the geometric Brownian motion φ and the solution of a random delay differential equation which admits a continuously differentiable solution. The zeros of the process y correspond a.s. to the zeros of the process X, and we can apply a deterministic Euler method to each path of (8).

2.3. Oscillation of discrete processes

For completeness, let us first define several notions relating to sequences of events $\{A_n\}_{n\geqslant 0}$.

If infinitely many of the events A_n occur, then we say that the event 'A_n *infinitely often* (i.o.)' has occurred, where

$$\text{'}A_n\ \text{i.o.'}=\bigcap_{n=0}^{\infty}\bigcup_{j=n}^{\infty}A_j.$$

If all the events A_n occur from a certain rank on, then we say that the event 'A_n *eventually* (ev.)' has occurred, where

$$\text{'}A_n\ \text{ev.'}=\bigcup_{n=0}^{\infty}\bigcap_{j=n}^{\infty}A_j.$$

We can now define the notions of a.s. oscillation and nonoscillation for discrete processes.

Let $\{Y_n\}_{n\geqslant 0}$ be a real valued stochastic process. We say that $\{Y_n\}_{n\geqslant 0}$ is a.s. *oscillatory* if $\mathbb{P}[Y_nY_{n+1}<0\text{ i.o.}]=1$.

The process is a.s. *nonoscillatory* if $\mathbb{P}[Y_nY_{n+1}>0\text{ ev.}]=1$.

The definition of a.s. oscillation for discrete processes uses products of pairs of successive solution values in order to detect sign changes. This construction has precedent in the literature: see Koplatadze [9] for an example. However, since we show in Lemma 5 that the solutions of the discrete process are nonzero-valued at every time step, a.s., it would be equally valid to use quotients of pairs of successive solution values instead of products. In fact this approach is taken in [8], and does not significantly change the analysis in the proof of our main result, Theorem 6.

2.4. Discretising the auxiliary process

A discrete process can be characterised by the sequence $\{Y_n\}$ representing the solution of a difference equation, with appropriate initial data, and the mesh on which $\{Y_n\}$ evolves. We apply an Euler scheme to each path of the auxiliary process (8) with initial data (7b), yielding the difference equation

$$Y_{n+1}=Y_n-\varDelta_nP_n(\omega)Y_{n-r_{\mathrm{M}}(n)},\quad n\geqslant 0, \tag{10}$$

with solutions evolving on some mesh M, with positive initial data on a finite discrete subset of $[-\overline{\tau}, 0]$ that includes the endpoints. The form of the delay function r_{M} depends on the structure of the mesh. Each term of the sequence of random variable instances $\{P_n(\omega)\}_{n=0}^{\infty}$ is defined to be (9) on the corresponding path, sampled at the nth mesh point. We will prove results that apply over almost all paths of (10), and therefore we can generally suppress the ω-dependence and write $\{P_n\}_n$. The lack of differentiability in almost every path of the process p, defined in (9), ensures that the convergence results in [5] do not apply.

Finally, recall that Condition (3), although requiring that τ go to zero asymptotically, allows it to have intervals of increase. We will disallow this possibility, and require that τ be a continuous function which satisfies

$$\lim_{t\to\infty} \tau(t) = 0, \quad \tau(t) > 0, \quad t \mapsto \tau(t) \text{ is strictly decreasing on } [0, \infty). \tag{11}$$

An explanation for this monotonicity restriction is given in Remark 1, in Section 3.1. Hereinafter τ will obey (11), even if this is not explicitly stated.

Let $\mathrm{M}_\varDelta$ be a uniform mesh of mesh size $\varDelta$. The difference equation (10) with ω suppressed becomes

$$Y_{n+1} = Y_n - \Delta P_n Y_{n-r(n)}, \quad n \geqslant 0, \tag{12}$$

where

$$r(n) = \sup\{k > 0 : k\varDelta \leqslant \tau(n\varDelta)\}. \tag{13}$$

By (11), we can define a constant $N_0 < \infty$ to be

$$N_0 = \inf\{n \in \mathbb{N} : n - r(n) > 0\}.$$

Thus, the initial data for (12) is an ordered set

$$\boldsymbol{\psi} = (\psi_0, \psi_1, \ldots, \psi_{N_0-1}, Y_0), \tag{14}$$

where $\psi_i = Y_{i-r(i)}$ for all $i < N_0$, and $\psi_{N_0-1} = Y_0$ if

$$(N_0 - 1)\varDelta - r((N_0 - 1)\varDelta) = 0,$$

a condition that can be guaranteed for any τ satisfying (11) by choosing $\varDelta$ appropriately. However, in general it will not hold. Nonetheless, (14) is well defined for any given τ, regardless of the size of $\varDelta$. Note that it is enough to associate an initial data value with each mesh point up to and including the mesh point at $N_0\varDelta$, without specifying the location of the initial data values on $\mathbb{R}$.

To ensure that $Y_1 \neq 0$, we require that

$$\psi_0 \neq \frac{Y_0}{\varDelta|b|}. \tag{15}$$

Eq. (15) can be satisfied by choosing $\varDelta$ appropriately. We also require that

$$\psi_j \in \mathbb{R}^+ \quad \text{for all } j < N_0, \tag{16}$$

a natural condition, given that the initial data function (2b) is itself positive on $[-\overline{\tau}, 0]$.

The random variable P_n satisfies

$$P_n = \begin{cases} |b|\mathrm{e}^{-\lambda\tau(n\varDelta)}\mathrm{e}^{|\sigma|(B(n\varDelta)-B(n\varDelta-\tau(n\varDelta)))}, & N_0 \leqslant n, \\ |b|\mathrm{e}^{-\lambda n\varDelta}\mathrm{e}^{|\sigma|B(n\varDelta)}, & 0 < n < N_0, \\ |b|, & n = 0. \end{cases} \tag{17}$$

By (11), we can define a constant $N_1 > N_0$ by

$$N_1 = \inf\{n > N_0 : n - r(n) - 1 - r(n - r(n) - 1) > 0\}.$$

By (11) and (13), there exists some $N^* < \infty$ large enough that, for all $n > N^*$, the delay function τ satisfies $\tau(n\Delta) < \Delta$. Consequently, $r(n) = 0$ for all $n > N^*$. So, for all $n > N^*$, each term of $\{Y_n\}$ satisfies

$$Y_{n+1} = Y_n(1 - \Delta P_n).$$

The problems with the mesh M_Δ begin at N^*, when the length of the delay drops below the mesh size. From this point onwards $r_{\mathrm{M}_\Delta}(n) \equiv 0$, and the delay no longer has an appropriate effect on the evolution of the solution.

3. Setting up the discrete-time model

In order that the final analysis be as straightforward as possible, we will define in this section a discrete model of (12) that will allow us to deal effectively with the lack of independence between terms of $\{Y_n\}$ and terms of $\{P_n\}$, arising from the form of the discrete delay function r_{M_Δ}.

3.1. Constructing a discrete-time filtration

To prove a result describing the behaviour of solutions of (12), we will replace each P_n with a random variable $\widetilde{P}_n$ of identical distribution. Thus we will be studying the solution $\widetilde{Y}$ of a discrete model of (8), rather than the solution of the direct discretisation Y. To do this, we must define a new filtration.

Consider a sequence of standard normal random variables $\{\xi_k\}_{k=0}^\infty$, mutually independent and, because we are constructing a model, not necessarily related to the specific Brownian motion B. The distribution of each element in the sequence $\{\xi_k\}_{k=0}^\infty$ will, scaled appropriately, coincide with the distribution of a particular increment of the Brownian motion B. The filtration generated by this sequence is $\{\mathscr{G}_k^\xi\}_{k=0}^\infty$, where

$$\mathscr{G}_k^\xi = \sigma(\{\xi_i\}_{i=0}^j; 0 \leqslant j \leqslant k).$$

We associate with each ξ_k a number $\delta_k^2 > 0$. We will first define the sequence $\{\delta_k^2\}_{k=0}^\infty$ explicitly and concisely, before giving a full explanation for this definition in Section 4.1.

(1) Define a sequence $\{a_k\}_{k=0}^\infty$, where $a_k = k\Delta$ for every k.
(2) Define a sequence $\{b_k\}_{k=0}^\infty$, where $b_k = (k + N_0)\Delta - \tau((k + N_0)\Delta)$ for every k.
(3) From $\{a_k\}_{k=0}^\infty$ and $\{b\}_{k=0}^\infty$ we can construct a new sequence $\{c_k\}_{k=0}^\infty$ as follows: for every $n < \infty$, let $c_{2n} = a_n$, and
  - if there exists $0 \leqslant j < \infty$ such that $b_j \in (a_n, a_{n+1})$, then $c_{2n+1} = b_j$;
  - otherwise $c_{2n+1} = (a_n + a_{n+1})/2$.
(4) Now, for every $0 \leqslant k < \infty$, let $\delta_k^2 = c_{k+1} - c_k$.

Remark 1. The terms of the sequence $\{b_k\}_{k=0}^\infty$ act as dividers, splitting each mesh interval into two. Requiring that τ satisfy (11) prevents there being more than one element of $\{b_k\}_{k=0}^\infty$ between any two mesh points. If τ were merely required to satisfy the less restrictive (3), such a possibility would not be prevented; hence the restriction. If there is no term of $\{b_k\}_{k=0}^\infty$ on a given mesh interval, then we construct an artificial divider at the halfway point of that interval in order to guarantee that exactly two terms of $\{\delta_k^2\}_{k=0}^\infty$ can be associated with it. Further discussion of the motivation behind this construction can be found in Section 4.1.

Finally, we introduce a sequence of independent, $\mathscr{G}_k^\xi$-measurable random variables $\{\zeta_k\}_{k=0}^\infty$, defined so that, for each k, $\zeta_k = \mathrm{e}^{|\sigma|\delta_k\xi_k}$.

A schematic visualisation of this construction is given for an arbitrary vanishing delay function τ in Figs. 1 and 2.

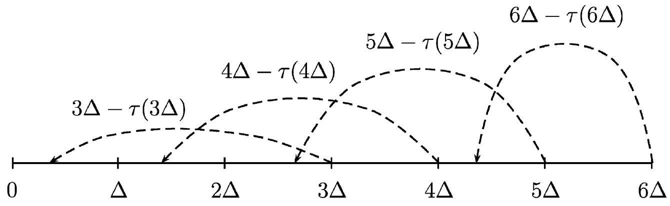

Fig. 1. A uniform mesh of size Δ overlaid with the feedback positions of an arbitrary continuous vanishing delay function τ at each mesh point. $N_0 = 3$, $\widetilde{N} = 6$.

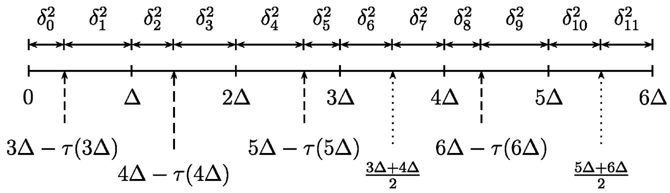

Fig. 2. The construction of the sequence $\{\delta_k^2\}_{k=0}^{\infty}$.

3.2. Final construction of the discrete model

We introduce the functions $h, i, j : \mathbb{N}_0 \to \mathbb{N}_0$, where

$$h(n) = 2n - 1,$$

$$i(n) = \begin{cases} 2(n - r(n)), & \dfrac{\tau(n\Delta)}{\Delta} \in \mathbb{N}_0, \\ 2(n - r(n)) - 1 & \text{otherwise,} \end{cases}$$

$$j(n) = \begin{cases} 2(n - r(n) - 1 - r(n - r(n) - 1)), & \dfrac{\tau((n - r(n) - 1)\Delta)}{\Delta} \in \mathbb{N}_0, \\ 2(n - r(n) - 1 - r(n - r(n) - 1)) - 1 & \text{otherwise.} \end{cases}$$

For every $n < N^*$, there is a $\mathscr{G}_{2n}^{\xi}$-measurable random variable $\widetilde{P}_n$ with an identical distribution to that of P_n, defined to be

$$\widetilde{P}_n = \begin{cases} |b| e^{-\lambda \tau(n\Delta)} \zeta_{h(n)}, & n \geqslant N^*, \\ |b| e^{-\lambda \tau(n\Delta)} \zeta_{i(n)} \cdots \zeta_{h(n)}, & N_0 \leqslant n < N^*, \\ |b| e^{-\lambda n \Delta} \zeta_0 \cdots \zeta_{h(n)}, & 0 < n < N_0, \\ |b|, & n = 0, \end{cases}$$

We consider the oscillatory behaviour of the process $\{\widetilde{Y}_n\}_{n \geqslant 0}$ obeying

$$\widetilde{Y}_{n+1} = \widetilde{Y}_n - \Delta \widetilde{P}_n \widetilde{Y}_{n-r(n)}, \tag{18}$$

with initial data (14) satisfying (15) and (16). By (18) and the definition of $\widetilde{P}_n$, each $\widetilde{Y}_n$ is $\mathscr{G}_{2(n-1)}^{\xi}$-measurable.

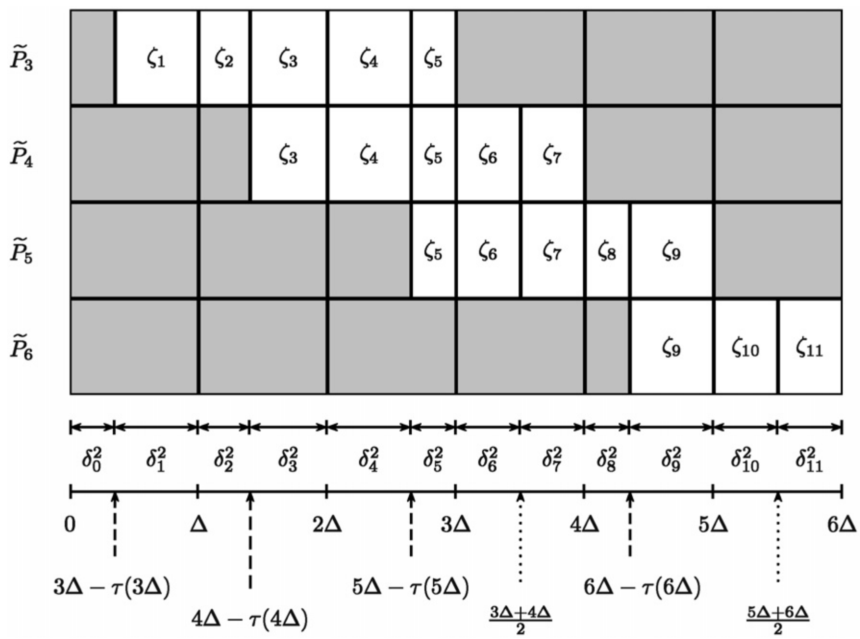

Fig. 3. The decomposition of nonindependent distributions of $\widetilde{P}_3$, $\widetilde{P}_4$, $\widetilde{P}_5$, and $\widetilde{P}_6$, into functions of independent lognormal random variables.

4. Spurious behaviour arising from a uniform discretisation

First, we use the model designed in Section 3 to show that the solutions of (18) are a.s. nonzero if the initial data is positive. If $Y_m = 0$ for some $m > N^*$ then the process $\{Y_n\}_{n \geqslant 0}$ would remain zero-valued for all $n \geqslant m$. This degenerate collapse to equilibrium could not possibly arise in the solutions of the differential equation (8) with positive initial data, and consequently we must show that it cannot arise in the solutions of (18).

Second, we prove a theorem (Theorem 6) that characterises the oscillatory behaviour of the solutions of (18) with respect to different vanishing rates of τ. It will be seen from this theorem that the difference equation fails to reproduce the oscillatory behaviour of solutions of (2), but does suggest a reason for this failure, and points to an alternative model design that yields better results.

4.1. Solution-values of discrete process are a.s. nonzero

The filtration $\{\mathscr{G}_k^{\xi}\}_{k=0}^{\infty}$ is the basis for the model described by (18), and at first glance appears to be unnecessary. However, it is essential in order to show that terms of the sequence $\{Y_n\}_{n \geqslant 0}$ will never be zero-valued.

The main barrier to analysis here is the lack of independence between each Y_n, P_n, and $Y_{n-r(n)}$ on the right-hand side of (12), for $n < N^*$. P_n depends on an increment of Brownian motion longer than $\varDelta$. Therefore, each successive P_n is not independent of its predecessors. Neither is P_n independent of Y_n. These dependencies must be explicitly handled. The Brownian increment must be subdivided into a sequence of independent standard normal random variables.

Although the mesh itself imposes a natural partition on the Brownian increment, we must go further, splitting each subdivision of the Brownian increment of length $\varDelta$ into two smaller subdivisions. This is necessary because, in general, $n\varDelta - \tau(n\varDelta)$ is not a multiple of $\varDelta$, and thus the mesh $\mathrm{M}_{\varDelta}$ is not sufficient to define a sequence of independent normal random variables that will allow us to analyse the dependencies of the components of (12). Consider also that (11) places no upper limit on the rate at which τ can converge to zero, and therefore the delay function may 'jump' across mesh intervals. This motivates the construction of the sequence (δ_k^2). A need to define $\{\mathscr{G}_k^{\xi}\}_{k=0}^{\infty}$ precisely, in spite of these considerations, determines the structure of the filtration, and therefore requires that we study the discrete model (18), rather than the initial discretisation (12).

We can explicitly show the dependencies between $\widetilde{Y}_n$, $\widetilde{P}_n$, and $\widetilde{Y}_{n-r(n)}$ with the example illustrated in Fig. 3. This is a schematic representation of the dependence of each $\widetilde{P}_k$, for $N_0 \leqslant k \leqslant \widetilde{N}$, on lognormal random variables, for some arbitrary delay function τ. For instance, let $n=5$. The term $\widetilde{Y}_5$ can be written in terms of the sequence of lognormal random variables $\{\zeta_i\}_{i\geqslant 0}$ by iterating (18) back to the zeroth term. We know from (18) and from Fig. 3 that

$$\widetilde{Y}_5 = \widetilde{Y}_4 - \Delta \widetilde{P}_4 \widetilde{Y}_2.$$

Clearly $\widetilde{Y}_4$ depends on $\{\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$, $\widetilde{P}_4$ depends on $\{\zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7\}$, and $\widetilde{Y}_2$ depends on $\{\zeta_0, \zeta_1\}$. Although $\widetilde{Y}_4$, $\widetilde{P}_4$, and $\widetilde{Y}_2$ are not fully independent, we can see that $\widetilde{Y}_4$ depends on $\{\zeta_2\}$, whereas $\widetilde{P}_4$ and $\widetilde{Y}_2$ do not, and $\widetilde{P}_4$ depends on $\{\zeta_6, \zeta_7\}$, whereas $\widetilde{Y}_4$ and $\widetilde{Y}_2$ do not. If we can prove that this partial independence is a characteristic of all terms of the sequence defined by (18), then we can finally prove that all terms are a.s. nonzero.

The final proof is given in Lemma 5. First however, Lemmas 2–4 use this partial independence argument to show that all terms $\widetilde{Y}_n$ up to, but not including, $\widetilde{Y}_{N^*}$, are a.s. nonzero.

Lemma 2. *If* $\{\widetilde{Y}_n\}_{n\geqslant 0}$ *is the process defined by* (18), *and with an initial data set* (14) *obeying* (15) *and* (16), *then for all* $0<n\leqslant N_0$,

$$\mathbb{P}[\widetilde{Y}_n = 0] = 0.$$

Lemma 2 shows that $\widetilde{Y}_n$ is a.s. nonzero on the interval $0<n\leqslant N_0$. Its proof is given, and explained, in Section 6.

Lemma 3. *If* $\{\widetilde{Y}_n\}_{n\geqslant 0}$ *is the process defined by* (18), *and with an initial data set* (14) *obeying* (15) *and* (16), *then for all* $N_0<n<N_1$,

$$\mathbb{P}[\widetilde{Y}_n = 0] = 0.$$

Lemma 4. *If* $\{\widetilde{Y}_n\}_{n\geqslant 0}$ *is the process defined by* (18), *and with an initial data set* (14) *obeying* (15) *and* (16), *then for all* $N_1\leqslant n<N^*$,

$$\mathbb{P}[\widetilde{Y}_n = 0] = 0.$$

Lemmas 3 and 4 show that $\widetilde{Y}_n$ is a.s. nonzero for the remaining terms, from $\widetilde{Y}_{N_0+1}$ to $\widetilde{Y}_{N^*-1}$. The proofs of Lemmas 3 and 4 are similar to that of Lemma 2, and are not presented.

One can intuitively see why, if $\mathbb{P}[\widetilde{Y}_{N^*-1}\neq 0]=1$, then it should be possible to show that $\mathbb{P}[\widetilde{Y}_n\neq 0 \text{ for all } n\geqslant N^*]=1$. Once the delay drops below the mesh size, all of the dependency issues discussed above disappear. Lemmas 2–4 get us to that stage, and we can now show that the solutions of (18) never display spurious degenerate behaviour.

Lemma 5. *If* $\{\widetilde{Y}_n\}_{n\geqslant 0}$ *is the sequence of random variables defined by* (18), *and with an initial data set* (14) *obeying* (15) *and* (16), *then*

$$\mathbb{P}[\widetilde{Y}_n \neq 0 \textit{ for all } n] = 1.$$

Proof. When $n\geqslant N^*$, $\tau(n\Delta)<\Delta$, so $r_{\mathrm{M}_\Delta}(n)=0$. Additionally, by Lemma 4, $\widetilde{Y}_{N^*-1}\neq 0$ a.s. So, with $\widetilde{P}_n=|b|\mathrm{e}^{-\lambda\tau(n\Delta)}\zeta_{h(n)}$, (12) can be rewritten as

$$\widetilde{Y}_{n+1} = \widetilde{Y}_n(1-\Delta\widetilde{P}_n), \quad n\geqslant N^*, \quad \widetilde{Y}_{N^*-1}\neq 0.$$

Since $\{\widetilde{P}_n\}$ are independent, $\widetilde{P}_n$ is $\mathscr{G}^{\xi}_{h(n)}$-measurable and $\mathscr{G}^{\xi}_{h(n-1)}$-independent. $\widetilde{Y}_{n+1}$ is $\mathscr{G}^{\xi}_{h(n)}$-measurable and so $\widetilde{Y}_n$ is $\mathscr{G}^{\xi}_{h(n-1)}$-measurable. Thus $\widetilde{P}_n$ and $\widetilde{Y}_n$ are independent, $n\geqslant N^*$. Define the event $A_n=\{\omega : Y_n(\omega)=0\}$, $n\geqslant N^*$. By the definition of $\{\widetilde{P}_n\}$, $\mathbb{P}[A_n|\overline{A_{n-1}}]=\mathbb{P}[\widetilde{P}_n=1/\Delta]=0$, for each $n>N^*$. Since $Y_{N^*-1}\neq 0$, we can infer, by induction, that $\mathbb{P}[A_n]=0$ for all $n\geqslant N^*$. Finally, we extend the definitions of A_n and $\overline{A}_n$ to define the event $B_n=\{\omega : \widetilde{Y}_n(\omega)=0\}$, for $n\geqslant 0$. Then $\mathbb{P}[\widetilde{Y}_n\neq 0 \text{ for all } n]=\mathbb{P}[\bigcap_n \overline{B_n}]\geqslant 1-\sum_n \mathbb{P}[B_n]=1$. □

4.2. Main result

The main result of the paper shows that, unless the vanishing rate of τ is slow, the oscillatory behaviour of $\widetilde{Y}$ depends on the mesh size Δ.

Theorem 6. *Let* M_Δ *be a uniform mesh, and* $\widetilde{Y}$ *be the process defined by* (18) *with initial data* ψ *given by* (14) *obeying* (15) *and* (16). *Suppose that*

$$\lim_{t\to\infty} \tau(t)\log t =: c \in [0,\infty].$$

(i) *Let* $c=0$.
   (a) *If* $\Delta > 1/|b|$, *then* $\widetilde{Y}$ *is a.s. oscillatory.*
   (b) *If* $\Delta < 1/|b|$, *then* $\widetilde{Y}$ *is a.s. nonoscillatory.*

(ii) *Let* $c \in (0,\infty)$.
   (a) *If* $\Delta > 1/|b|\mathrm{e}^{|\sigma|\sqrt{2c}}$, *then* $\widetilde{Y}$ *is a.s. oscillatory.*
   (b) *If* $\Delta < 1/|b|\mathrm{e}^{|\sigma|\sqrt{2c}}$, *then* $\widetilde{Y}$ *is a.s. nonoscillatory.*

(iii) *If* $c=\infty$, *then* $\widetilde{Y}$ *is a.s. oscillatory.*

Before giving a proof, we make a direct comparison between the known oscillatory behaviour of the solutions of (2) for various decay rates of τ, as discussed in Section 1, and the statement of Theorem 6.

If τ obeys Condition (6), then Case (i) holds. However, Case (i) Part (a) is contradicted by Theorem 13 of [2], which does not allow a.s. oscillation under (6). Case (i) Part (b) is consistent with Theorem 13 of [2], but differs in that it deals with nonoscillatory solutions rather than positive solutions, no restrictions on the initial data are required. Furthermore, the result applies with probability one.

Theorem 13 of [2] does not cover Case (ii), and the mesh dependence makes it impossible to draw any conclusions about oscillatory behaviour.

If τ obeys Condition (5), then Case (iii) holds. This case is consistent with the behaviour categorised in [2].

The inconsistency displayed in Case (i) Part (a) is evidence of spurious behaviour arising from the procedure of discretising over a uniform mesh. Note that the value of the mesh size Δ at which the process moves from a.s. oscillation to a.s. nonoscillation is deterministic. Consequently, one can choose Δ a priori in order to ensure a particular type of behaviour.

Proof. By the definitions of the delay functions τ and r, there exists $N^* < \infty$ large enough that, for all $n \geqslant N^*$, $r(n)=0$. Therefore, for all $n\geqslant N^*$, (12) can be rewritten as $\widetilde{Y}_n\widetilde{Y}_{n+1} = \widetilde{Y}_n^2(1-\Delta\widetilde{P}_n)$. By Lemma 5, $\widetilde{Y}_n \neq 0$ for all n a.s., and therefore, in order to prove that $\widetilde{Y}$ is a.s. oscillatory, it is enough to show that $\mathbb{P}[\widetilde{P}_n > (1/\Delta) \text{ i.o.}]=1$. Similarly, in order to prove that $\widetilde{Y}$ is a.s. nonoscillatory, it is enough to show that $\mathbb{P}[\widetilde{P}_n < (1/\Delta) \text{ ev.}]=1$.

Define $\vartheta_n = \xi_{h(n)}$ for all $n\geqslant N^*$, and consider that, since $\tau(t)\to 0$,

$$\limsup_{n\to\infty} \widetilde{P}_n = |b|\exp\left\{|\sigma|\limsup_{n\to\infty}\sqrt{2\tau(n\Delta)\log n}\frac{\vartheta_n}{\sqrt{2\log n}}\right\}.$$

Since $\{\vartheta_n\}_{n\geqslant N^*}$ is a sequence of independent, standard normal random variables, it is well known that $\limsup_{n\to\infty} \vartheta_n/\sqrt{2\log n}=1$, a.s., and therefore we need only consider the asymptotic behaviour of $\sqrt{2\tau(n\Delta)\log n}$.

Case (i): Since $\lim_{n\to\infty}\tau(n\Delta)\log n\Delta = 0$, and

$$\tau(n\Delta)\log n\Delta = \tau(n\Delta)\log n + \tau(n\Delta)\log\Delta,$$

it follows that $\limsup_{n\to\infty}\tau(n\Delta)\log n = 0$. Thus $\limsup_{n\to\infty}\widetilde{P}_n = |b|$, a.s. Therefore, if $\Delta > 1/|b|$, $\widetilde{Y}$ is a.s. oscillatory, and if $\Delta < 1/|b|$, $\widetilde{Y}$ is a.s. nonoscillatory.

Case (ii): Since $\lim_{n\to\infty}\tau(n\Delta)\log n\Delta = c > 0$, we have

$$\lim_{n\to\infty}\tau(n\Delta)\log n = c.$$

Thus $\limsup_{n\to\infty}\widetilde{P}_n = |b|\mathrm{e}^{|\sigma|\sqrt{2c}}$, a.s. Therefore, if $\Delta > 1/(|b|\mathrm{e}^{|\sigma|\sqrt{2c}})$, $\widetilde{Y}$ is a.s. oscillatory, and if $\Delta < 1/(|b|\mathrm{e}^{|\sigma|\sqrt{2c}})$, $\widetilde{Y}$ is a.s. nonoscillatory.

Case (iii): Let $\lim_{n\to\infty}\tau(n\Delta)\log n\Delta = \infty$. Letting $c\to\infty$ in Case (ii) implies that $\widetilde{Y}$ is a.s. oscillatory. □

5. Conclusions

The oscillatory behaviour of the solutions of (2) was partially described in [2]. In that paper, an auxiliary process with differentiable sample paths and coincident zeros was constructed, and the properties of its paths were analysed using results from the oscillation theory of linear first order inhomogeneous delay differential equations. This behaviour differs from both limiting equations (each achieved by deleting either the delay term or the noise perturbation) of (2), and therefore appears to derive from the influence of the delay and the noise in tandem.

Attempting to extend the scope of this analysis, by applying a uniform Euler discretisation to the paths of the auxiliary process, fails. A uniform mesh forces the discrete delay function to vanish after a finite time. Since the presence of a noise perturbation allows the delay to affect the qualitative behaviour of the process regardless of how small it has become, the asymptotic effect of the delay must be present after discretisation. By removing the delay after a finite interval we are effectively ignoring the delay-noise interaction that we are trying to reproduce. A nonuniform mesh that allows the effect of the delay to persist asymptotically, as used in [3], would give a more accurate representation of the oscillatory behaviour.

Further work—A direct discretisation of (2): The method applied here, and in [3], is rather indirect. It might be informative to discretise equation (2) directly, using a stochastic Euler scheme on a nonuniform mesh, and to study the oscillatory behaviour of the resulting discrete process. Such a direct method could be applied to a wider class of equations. Details of Euler schemes as applied directly to stochastic delay differential equations can be found in [4].

Further work—"Coloured" noise: Since the presence of Gaussian noise in (2) engenders oscillatory behaviour that is not characteristic of the corresponding deterministic equation, it would be interesting to replace the Gaussian perturbation with a temporally correlated stochastic process. For example, in contrast to Brownian motion, an Ornstein–Uhlenbeck perturbation would have correlated increments, while remaining susceptible to Itô methods.

6. Proof of Lemma 2

We begin this technical section with a result showing that an aggregate random variable which depends on a sequence of mutually independent random variables in a specific way, and which is described by (19), will take on a value of zero with probability zero. It is an application of Jacobi's transformation formula [6] to the density functions of random variables. The proof is standard, and thus omitted.

Lemma 7. *Suppose that the independent random variables $\eta_0, \eta_1, \ldots, \eta_N$ have joint density $f_{(\eta_0,\ldots,\eta_N)}$, and*

$$X = h(\eta_0, \eta_1, \ldots, \eta_N) := h_1(\eta_0, \ldots, \eta_{N-1}) + h_2(\eta_0, \ldots, \eta_{N-1})\eta_N. \tag{19}$$

Define

$$S_0 := \{(y_0, y_2, \ldots, y_N) : h_2(y_0, \ldots, y_{N-1}) = 0,\, y_N \in \mathbb{R}^+\}. \tag{20}$$

If S_0 has measure 0 in $\mathbb{R}^{N+1}$, then X has density f_X defined by

$$f_X(x) = \int_{y_0}\int_{y_1}\cdots\int_{y_{N-1}} f_{(\eta_0,\eta_1,\ldots,\eta_N)}\left(y_0, y_1, \ldots, y_{N-1}, \frac{x - h_1(y_0, y_1, \ldots y_{N-1})}{h_2(y_0, y_1, \ldots, y_{N-1})}\right) \times \frac{1}{|h_2(y_0, y_1, \ldots, y_{N-1})|}\, dy_{N-1}dy_{N-2}\cdots dy_0,$$

and therefore $\mathbb{P}[X = 0] = 0$.

Thus, in order to prove Lemma 2, we must show that each random variable $\widetilde{Y}_n$ can be decomposed into the form given by (19). The role of each η_i will be played by the product of pairs of lognormal random variables $\zeta_{2i}\zeta_{2i+1}$, where the sequence $\{\zeta_k\}_{k=0}^{\infty}$ is as defined in Section 3.1.

Patterns of dependence on a sequence of random variables: Given a positive sequence $\{x_k\}_{k=0}^{h(N_0)}$ we can define the following interdependent sequences of functions.

(1) For each $2 \leqslant k \leqslant N_0$, define the function $q_k : \mathbb{R}^{h(k-1)+1} \to \mathbb{R}$ so that

$$q_k(x_0, \ldots, x_{h(k-1)}) = -\Delta|b|\mathrm{e}^{-\lambda k\Delta} x_0 \cdots x_{h(k-1)}\psi_k. \tag{21}$$

If $N_0 - r(N_0) = 1$, then further define $q_{N_0+1} : \mathbb{R}^{h(N_0)+1} \to \mathbb{R}$ so that

$$q_{N_0}(x_0, \ldots, x_{h(N_0-1)}) = -\Delta|b|\mathrm{e}^{-\lambda N_0\Delta} x_0 \cdots x_{h(N_0-1)}\tilde{y}_1^{\psi}, \tag{22}$$

where $\tilde{y}_1^{\psi}$ satisfies (24). By (21) and (22) for each $2 \leqslant k \leqslant N_0 + 1$, the surface

$$S^k := \{(x_0, \ldots, x_{h(k)}) : q_k(x_0, \ldots, x_{h(k-1)}) = 0,\ x_{h(k)-1}x_{h(k)} \in \mathbb{R}^+\} \tag{23}$$

has measure 0 in $\mathbb{R}^{h(k)+1}$.

(2) Now define the sequence of functions $\{\tilde{y}_k^{\psi}\}_{k=2}^{N_0-1}$ recursively so that

$$\tilde{y}_1^{\psi} = \widetilde{Y}_0 - \Delta|b|\psi_0, \tag{24}$$

and, for $2 \leqslant k < N_0$,

$$\begin{aligned}&\tilde{y}_{k+1}^{\psi}(x_0, \ldots, x_{h(k)-2}, x_{h(k)-1}x_{h(k)})\\&\quad = \tilde{y}_k^{\psi}(x_0, \ldots, x_{h(k-1)-2}, x_{h(k-1)-1}x_{h(k-1)}) + q_k(x_0, \ldots, x_{h(k-1)})x_{h(k)-1}x_{h(k)}.\end{aligned} \tag{25}$$

Remark 8. Note that definition of $\tilde{y}^{\psi}$ in (25) has the product of two terms as its last argument. This deliberately reflects the fact that, when representing terms of the sequence $\{\widetilde{Y}_n\}_{n\geqslant 0}$ in terms of the functions $\tilde{y}^{\psi}$, there is always a pair of independent lognormal terms appearing as a product. Together they play the role of η_N in (19).

We use the recursively defined functions $\{\tilde{y}_n^{\psi}\}_{n\geqslant 0}$ to show that each $\widetilde{Y}_n$ depends on a sequence of independent lognormal random variables specifically as described in (19). This will allow us to apply Lemma 7 directly. Lemma 2 demonstrates this on the interval $0 < n \leqslant N_0$. A more complex definition of each $\tilde{y}_n^{\psi}$ is required in order to show that the same result holds for $N_0 < n \leqslant N_1$ and $N_1 < n \leqslant N^*$, and therefore to prove Lemmas 3 and 4, but the spirit is the same, and these function definitions and proofs are omitted from this paper for reasons of clarity. All details can be found in Chapter 6 of [8].

Finally, we give the proof of Lemma 2.

Proof (*Lemma 2*). For every $0 < n \leqslant N_0$, $\widetilde{P}_n = |b|\mathrm{e}^{-\lambda n\Delta}\zeta_0 \cdots \zeta_{h(n)}$. Note first that $\widetilde{Y}_1 = \widetilde{Y}_0 - \Delta|b|\widetilde{Y}_{0-r(0)} = \widetilde{Y}_0 - \Delta|b|\psi_0$. By (15) and (16), $\widetilde{Y}_1 \neq 0$. We proceed by induction. Assume that, for $2 \leqslant k < N_0$,

$$\widetilde{Y}_k = \tilde{y}_k^{\psi}(\zeta_0, \ldots, \zeta_{h(k-1)-2}, \zeta_{h(k-1)-1}\zeta_{h(k-1)}).$$

Since $k < N_0$, $\widetilde{Y}_{k-r(k)} = \psi_k$. Since $\widetilde{Y}_{k+1} = \widetilde{Y}_k - \Delta\widetilde{P}_k\widetilde{Y}_{k-r(k)}$, we have

$$\widetilde{Y}_{k+1} = \tilde{y}_k^{\psi}(\zeta_0, \ldots, \zeta_{h(k-1)-2}, \zeta_{h(k-1)-1}\zeta_{h(k-1)}) - \Delta|b|\mathrm{e}^{-\lambda k\Delta}\zeta_0 \cdots \zeta_{h(k)}\psi_k.$$

Since $k < N_0$, $\widetilde{Y}_{k-r(k)} \in \boldsymbol{\psi}$. Therefore, by (21) and (25),

$$\begin{aligned}\widetilde{Y}_{k+1} &= \tilde{y}_k^{\psi}(\zeta_0, \ldots, \zeta_{h(k-1)-2}, \zeta_{h(k-1)-1}\zeta_{h(k-1)}) + q_k(\zeta_0, \ldots, \zeta_{h(k-1)})\zeta_{h(k)-1}\zeta_{h(k)}\\&= \tilde{y}_{k+1}^{\psi}(\zeta_0, \ldots, \zeta_{h(k)-2}, \zeta_{h(k)-1}\zeta_{h(k)}).\end{aligned}$$

We now consider the base case when $k = 2$. By (24),

$$\widetilde{Y}_2 = \widetilde{Y}_1 - \Delta\widetilde{P}_1\widetilde{Y}_{1-r(1)} = \tilde{y}_1^{\psi} - \Delta|b|\mathrm{e}^{n\Delta}\zeta_0\zeta_1\psi_1 = \tilde{y}_2^{\psi}(\zeta_0, \zeta_1).$$

Therefore, by induction, for all $0 < k \leqslant N_0$,

$$\widetilde{Y}_k = \tilde{y}_k^{\psi}(\zeta_0, \ldots, \zeta_{h(k-1)-2}, \zeta_{h(k-1)-1}\zeta_{h(k-1)}).$$

Since the surface S^k defined in (23) has measure 0 in $\mathbb{R}^{h(k)+1}$, Lemma 7 applies, completing the proof. □

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