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Biorthogonality of the Lagrange interpolants

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Abstract

We show that the Lagrange interpolation polynomials are biorthogonal with respect to a set of rational functions whose poles coincide with interpolation points.

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The Newton–Lagrange interpolation is a well-known problem in elementary calculus. Recall basic facts concerning this problem [2,5].

Let $A_k,\ k=0,1,2,\dots$ and $a_k,\ k=0,1,2,\dots$ be two arbitrary sequences of complex numbers (we assume that all a_k are distinct $a_k \neq a_j$ if $k \neq j$. By interpolation polynomial we mean a n-degree polynomial $P_n(z)$ whose values at points $a_0, a_1, \dots, a_n$ coincide with $A_0, A_1, \dots, A_n$, i.e.

$$P_n(a_k) = A_k, \quad k = 0, 1, 2, \dots, n. \tag{1}$$

Usually, the parameters A_k are interpreted as values of some function $F(z)$ at fixed points a_k, i.e.,

$$A_k = F(a_k). \tag{2}$$

In this case polynomials $P_n(z)$ interpolate the function $F(z)$ at points a_k.

Explicit expression for interpolation polynomial $P_n(z)$ can be presented in two forms. In the Newtonian form we have [2,5]

$$P_n(z) = \sum_{k=0}^{n} [a_0, a_1, \dots, a_k]\omega_k(x), \tag{3}$$

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where

$$\omega_0 = 1,\ \omega_k(x) = (x-a_0)(x-a_1)\dots(x-a_{k-1})$$

and $[a_0, a_1, \dots, a_k]$ denotes the kth Newtonian divided difference which is defined as

$$[a_0] = A_0,\ [a_0, a_1] = \frac{A_1 - A_0}{a_1 - a_0}, \dots, [a_0, a_1, \dots, a_k] = \sum_{s=0}^{k} \frac{A_s}{\omega'_{k+1}(a_s)},$$

where

$$\omega'_{k+1}(a_s) = (a_s - a_1)(a_s - a_2)\dots(a_s - a_{s-1})(a_s - a_{s+1})\dots(a_s - a_k) = \prod_{i=0, i\neq s}^{k} (a_s - a_i).$$

If representation (2) holds where $F(z)$ is a meromorphic function then the Hermite formula is useful [2]

$$[a_0, a_1, \dots, a_k] = (2\pi i)^{-1} \int_{\Gamma} \frac{F(\zeta)\,\mathrm{d}\zeta}{\omega_{k+1}(\zeta)}, \tag{4}$$

where the contour Γ in complex plane is chosen such that points $a_0, a_1, \dots, a_k$ lie inside the contour whereas all singularities of the function $F(z)$ lie outside the contour.

In the Lagrangian form we have [2,5]

$$P_n(z) = \omega_{n+1}(z) \sum_{k=0}^{n} \frac{A_k}{(z-a_k)\omega'_{n+1}(a_k)}. \tag{5}$$

Introduce the monic interpolation polynomials $\hat{P}_n(z) = P_n(z)/\alpha_n$, where

$$\alpha_n = [a_0, a_1, \dots, a_n].$$

In what follows, we will assume that $\alpha_n \neq 0$ for all $n = 1, 2, \dots$. This condition guarantees that polynomials $P_n(z)$ are indeed of the nth degree. It is easily seen that $P_n(z) = z^n + O(z^{n-1})$. For polynomials $\hat{P}_n(z)$ one has the recurrence relation [3]

$$\hat{P}_{n+1}(z) = \left(z - a_n + \frac{\alpha_n}{\alpha_{n+1}}\right)\hat{P}_n(z) - \frac{\alpha_{n-1}}{\alpha_n}(z-a_n)\hat{P}_{n-1}(z) \tag{6}$$

with the initial conditions

$$\hat{P}_{-1} = 0,\ \hat{P}_0(z) = 1. \tag{7}$$

It is clear that the set of monic interpolation polynomials $\hat{P}_0, \hat{P}_1(z), \dots, \hat{P}_n(z), \dots$ does not belong to a set of orthogonal polynomials (OP), because OP satisfy 3-term recurrence relations of the form [6]

$$P_{n+1}(z) + b_n P_n(z) + u_n P_{n-1}(z) = z P_n(z) \tag{8}$$

which does not have the form (6).

Nevertheless, recurrence relation (6) belongs to the so-called R_I-type recurrence relations (in terminology of [4]). It was shown in [4] that polynomials satisfying R_I-type relations possess some orthogonality property. In our case this orthogonality property is well known [5]:

Lemma 1. *Polynomials $\hat{P}_n(z)$ satisfy formal orthogonality relation*

$$I_{nj} = (2\pi\mathrm{i})^{-1} \int_\Gamma \frac{\zeta^j \hat{P}_n(\zeta)\,\mathrm{d}\zeta}{\omega_{n+1}(\zeta)F(\zeta)} = \frac{\delta_{nj}}{\alpha_n}, \quad j = 0, 1, \ldots, n, \tag{9}$$

where the contour Γ encompasses points $a_0, a_1, \ldots, a_N$ with $N \geqslant n$ and all singularities of the function $1/F(z)$ lie outside the contour.

For the proof it is sufficient to note, that under conditions upon choice of the contour Γ, the intergral can be presented as a sum of residues

$$I_{nj} = \sum_{s=0}^{n} \frac{a_s^j \hat{P}_n(a_s)}{A_s \omega'_{n+1}(a_s)} = \sum_{s=0}^{n} \frac{a_s^j}{\alpha_n \omega'_{n+1}(a_s)}, \tag{10}$$

where we used interpolation property (1). Hence, in integral (9) one can replace $P_n(\zeta)/F(\zeta) = 1/\alpha_n$ and we have

$$I_{nj} = (2\pi\mathrm{i})^{-1} \int_\Gamma \frac{\zeta^j \mathrm{d}\zeta}{\alpha_n \omega_{n+1}(\zeta)} = 0, \quad j = 0, 1, \ldots, n-1,$$

because the value of the integral from pure rational function does not depend on choice of the contour Γ (provided that all poles of the function lie inside the contour) and we can choose Γ as a circle of a great radius. For $j = n$ we have analogously

$$I_{nn} = (2\pi\mathrm{i})^{-1} \int_\Gamma \frac{\zeta^n \mathrm{d}\zeta}{\alpha_n \omega_{n+1}(\zeta)} = (2\pi\mathrm{i})^{-1} \int_\Gamma \frac{\mathrm{d}\zeta}{\alpha_n \zeta} = 1/\alpha_n.$$

As was shown in [7,8], polynomials of R_I type possess not only orthogonality of the form (9) but also nice *biorthogonality* property with respect to some set of rational functions.

In order to derive this biorthogonality property in our case, we construct auxiliary polynomials

$$T_n(z) = \hat{P}_{n+1}(z) - (z - a_{n+1})\hat{P}_n(z). \tag{11}$$

Clearly, degree of these polynomials $\leqslant n$. More exactly,

$$T_n(z) = v_n z^n + \mathrm{O}(z^{n-1}),$$

where

$$v_n = a_{n+1} - a_n + \frac{\alpha_n}{\alpha_{n+1}} - \frac{\alpha_{n-1}}{\alpha_n}. \tag{12}$$

In what follows we will assume that $v_n \neq 0$. This means that degree of $T_n(z)$ is n and it is possible to introduce monic polynomials

$$\hat{T}_n = T_n(z)/v_n = z^n + \mathrm{O}(z^{n-1}). \tag{13}$$

We have

Theorem 1. *Let $\hat{P}_n(z)$ be Lagrange interpolation polynomials for the function $F(z)$ and $\hat{T}_n(z)$ defined by* (11) *and* (13). *Define a set of rational functions*

$$V_n(z) = \frac{\hat{T}_n(z)}{\omega_{n+2}(x)}. \tag{14}$$

Then Lagrange interpolation polynomials $\hat{P}_n(z)$ and functions $V_n(z)$ form a biorthogonal system in the following sense:

$$\int_\Gamma \frac{\hat{P}_n(\zeta)V_m(\zeta)\,\mathrm{d}\zeta}{F(\zeta)} = \alpha_n^{-1}\delta_{nm}, \tag{15}$$

where the contour Γ should be chosen such that interpolation points $a_0, a_1, \ldots, a_N$ lie inside the contour ($N \geqslant \max(n, m+1)$), and the function $1/F(z)$ is regular inside and on the contour.

Proof. Assume first that $m < n-1$. Then we have, obviously

$$\int_\Gamma \frac{\hat{P}_n(\zeta)V_m(\zeta)\,\mathrm{d}\zeta}{F(\zeta)} = \int_\Gamma \frac{\hat{P}_n(\zeta)\hat{T}_m(\zeta)(z-a_{m+2})(z-a_{m+3})\ldots(z-a_n)\,\mathrm{d}\zeta}{\omega_{n+1}(z)F(\zeta)}$$

$$= \int_\Gamma \frac{\hat{P}_n(\zeta)q_{n-1}(z)\,\mathrm{d}\zeta}{F(\zeta)} = 0, \tag{16}$$

where $q_{n-1}(z)$ is a polynomial of degree $\leqslant n-1$ and in the last equality in (16) we used orthogonality property (9).

If $m = n-1$ then

$$\int_\Gamma \frac{\hat{P}_n(\zeta)V_{n-1}(\zeta)\,\mathrm{d}\zeta}{F(\zeta)} = \int_\Gamma \frac{\hat{P}_n(\zeta)\hat{T}_{n-1}(\zeta)\,\mathrm{d}\zeta}{\omega_{n+1}(z)F(\zeta)} = 0$$

again by (9).

If $m > n$ then we can write down

$$\int_\Gamma \frac{\hat{P}_n(\zeta)V_m(\zeta)\,\mathrm{d}\zeta}{F(\zeta)} = \int_\Gamma \frac{\hat{P}_n(\zeta)(\hat{P}_{m+1}(\zeta)-(\zeta-a_{m+1})\hat{P}_m(\zeta))\,\mathrm{d}\zeta}{\omega_{m+2}(\zeta)F(\zeta)}$$

$$= \int_\Gamma \frac{\hat{P}_n(\zeta)\hat{P}_{m+1}(\zeta)\,\mathrm{d}\zeta}{\omega_{m+2}(\zeta)F(\zeta)} - \int_\Gamma \frac{\hat{P}_n(\zeta)\hat{P}_m(\zeta)\,\mathrm{d}\zeta}{\omega_{m+1}(\zeta)F(\zeta)} = 0$$

because both terms in the last relation vanish due to (9) for $n < m$.

Finally, consider the case $m = n$. We have

$$\int_\Gamma \frac{\hat{P}_n(\zeta)V_n(\zeta)\,\mathrm{d}\zeta}{F(\zeta)} = \int_\Gamma \frac{\hat{P}_n(\zeta)\hat{T}_n(\zeta)\,\mathrm{d}\zeta}{\omega_{n+1}(\zeta)F(\zeta)}$$

$$= \int_\Gamma \frac{\hat{P}_n(\zeta)\hat{P}_{n+1}(\zeta)\,\mathrm{d}\zeta}{\omega_{n+2}(\zeta)F(\zeta)} - \int_\Gamma \frac{\hat{P}_n^2(\zeta)\,\mathrm{d}\zeta}{\omega_{n+1}(\zeta)F(\zeta)}$$

$$= \int_\Gamma \frac{\hat{P}_n^2(\zeta)\,\mathrm{d}\zeta}{\alpha_n\omega_n(z)} = 1/\alpha_n.$$

The theorem is proven. □

The biorthogonality property can be rewritten in another form

$$\sum_{s=0}^{N} \lim_{z=a_s}((z-a_s)P_n(z)V_m(z))/A_s = \delta_{nm}/\alpha_n, \tag{17}$$

where N is any positive integer such that $N \geqslant \max(n, m+1)$.

This result allows one to find coefficients ξ_k in expansion of a given polynomial $Q(z)$ of degree n in terms of interpolation polynomials $P_n(z)$

$$Q(z)=\sum_{k=0}^{n}\xi_k\hat{P}_k(z).$$

Indeed, from (17) we have

$$\xi_k=(2\pi\mathrm{i})^{-1}\alpha_k\int_{\Gamma}Q(\zeta)V_k(\zeta)\,\mathrm{d}\zeta. \tag{18}$$

Consider an example. For the exponential function $F(z)=\exp(hz)$ (with an arbitrary nonzero real parameter h) choose uniform grid of the interpolation points $a_k=k,=0,1,\ldots.$ We then have (cf. [3])

$$P_n(z)=\sum_{k=0}^{n}\frac{(-z)_k}{k!}(1-\mathrm{e}^h)^k, \tag{19}$$

where $(b)_k=b(b+1)\ldots(b+k-1)$ is the Pochhammer symbol. From (19) it is found

$$\alpha_n=\frac{(\mathrm{e}^h-1)^n}{n!}. \tag{20}$$

Construct auxiliary polynomials $T_n(z)=\hat{P}_{n+1}(z)-(z-a_{n+1})\hat{P}_n(z)$. For the leading coefficient v_n of the polynomials $T_n(z)=v_nz^n+\mathrm{O}(z^{n-1})$ we have from (12) and (20)

$$v_n=\frac{\mathrm{e}^h}{\mathrm{e}^h-1}\neq 0.$$

So polynomials $T_n(z)$ are indeed of degree n and for monic polynomials $\hat{T}_n(z)=T_n(z)/v_n$ it is not difficult to obtain a rather attractive closed formula

$$T_n(z)=\frac{(n+1)!}{(\mathrm{e}^h-1)^n}\,{}_2F_1\left(\begin{matrix}-n,-z\\ -1-n\end{matrix};\quad 1-\mathrm{e}^h\right). \tag{21}$$

Thus for rational corresponding rational functions $V_n(z)$ we have from (14)

$$V_n(z)=\frac{(n+1)!}{(1-\mathrm{e}^h)^n(-z)_{n+2}}\,{}_2F_1\left(\begin{matrix}-n,-z\\ -1-n\end{matrix};\quad 1-\mathrm{e}^h\right). \tag{22}$$

Using standard transformation formulas for the Gauss hypergeometric function [1], we can present the functions $V_n(z)$ in a slightly different form

$$V_n(z)=\frac{1}{(1-\mathrm{e}^h)^nz(z-1)}\,{}_2F_1\left(\begin{matrix}-n,-z\\ 2-z\end{matrix};\quad \mathrm{e}^h\right). \tag{23}$$

Thus rational functions $V_n(z)$ form a biorthogonal set with respect to the Lagrangian interpolation polynomials (19):

$$\int_{\Gamma}P_n(\zeta)V_m(\zeta)\exp(-h\zeta)\,\mathrm{d}\zeta=\delta_{nm}, \tag{24}$$

where Γ is an arbitrary contour containing the points $0,1,\ldots,\max(n,m+1)$ inside.

Note finally, that recurrence relation (6) *completely characterizes* the Lagrange interpolation polynomials $P_n(z)$. More exactly, we have the

Theorem 2. *Assume that a set of monic nth degree polynomials* $\hat{P}_n(z)$ *satisfies recurrence relation* (6) *with initial conditions* (7), *where parameters* $\alpha_n, a_n,\ n=0,1,\ldots$ *are arbitrary with the restrictions that all* a_i *are distinct*: $a_i \neq a_j$, *for* $i \neq j$ *and all* α_n *are nonzero* $\alpha_n \neq 0,\ n=0,1,\ldots$. *Then polynomials* $P_n(z)=\alpha_n\hat{P}_n(z)$ *satisfy interpolation condition* $P_n(a_k)=A_k,\ k=0,1,\ldots,n$ *for all* $n=0,1,\ldots$, *where*

$$A_n = P_n(a_n) = \sum_{s=0}^{n} \alpha_s \omega_s(a_n), \quad n=0,1,2,\ldots. \tag{25}$$

Proof. From recurrence relation (6) and initial conditions (7) it can be easily found

$$\hat{P}_{n+1}(z) - \frac{\alpha_n}{\alpha_{n+1}}\hat{P}_n(z) = \omega_{n+1}(z), \quad n=0,1,\ldots. \tag{26}$$

Hence for $P_n(z)=\alpha_n\hat{P}_n(z)$ we have the conditions

$$P_{n+1}(a_k) = P_n(a_k), \quad k=0,1,\ldots,n, \quad n=0,1,2,\ldots$$

From these relations, by induction, we obtain

$$P_n(a_k) = P_k(a_k) = A_k, \quad k=0,1,\ldots,n.$$

Thus interpolation conditions are fulfilled. Expression (25) for A_k follows then from Newton formula (3).

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