Journal of Pure and Applied Algebra 213 (2009) 1865–1889

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Geometry of obstructed equisingular families of algebraic hypersurfaces

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A R T I C L E I N F O

Article history:
Received 31 March 2008
Received in revised form 19 January 2009
Available online 19 March 2009
Communicated by A.V. Geramita

MSC:
14J10
14J17
14J70
32G11

A B S T R A C T

We study geometric properties of certain obstructed equisingular families of projective hypersurfaces with quasihomogeneous singularity with emphasis on smoothness, reducibility, being reduced, and having expected dimension.

In the case of minimal obstructedness, we give a detailed description of such families corresponding to quasihomogeneous singularities.

Next we study the behavior of these properties with respect to stable equivalence of singularities.

We show that under certain conditions, stabilization of singularities ensures the existence of a reduced component of expected dimension. For minimally obstructed families the whole family becomes irreducible.

As an application we show that if the equisingular family of a projective hypersurface H has a reduced component of expected dimension then the deformation of H induced by the equisingular family $|H|$ is complete with respect to one-parameter deformations.

1. Introduction

The study of equisingular families of algebraic curves and hypersurfaces with given invariants and given set of singularities is an old, but still attractive and widely open problem. Already at the beginning of the 20th century, the foundation was made in the works of Plücker, Severi, Segre and Zariski. Later the theory of equisingular families has been in focus of the numerous studies by algebraic geometers and has found important applications in singularity theory, topology of complex algebraic curves and surfaces, and in real algebraic geometry.

This paper is devoted to the study of the so-called *obstructed* families of projective hypersurfaces, of a given degree, having one isolated singularity of prescribed type.

Let Σ be a smooth projective variety over the complex field $\mathbb{C}$. Let D be an ample divisor on Σ. Denote by $V = V_{|D|}(S_1, \ldots, S_r)$ the set of hypersurfaces in the linear system $|D|$ having r singular points of analytic types $S_1, \ldots, S_r$ (as their only singularities). One knows that $V_{|D|}(S_1, \ldots, S_r)$ can be identified with a (locally closed) subscheme ("*equisingular stratum*") in the Hilbert scheme of hypersurfaces on Σ. The main questions concerning this space are

- *Existence problem:* Is $V_{|D|}(S_1, \ldots, S_r)$ non-empty, that is, does there exist a hypersurface $F \in |D|$ with the given collection of singularities?
- *Smoothness problem:* If $V_{|D|}(S_1, \ldots, S_r)$ is non-empty, is it smooth?
- *Dimension problem:* If $V_{|D|}(S_1, \ldots, S_r)$ is non-empty, does it have the "expected" dimension (expressible via local invariants of the singularities)?

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0022-4049/$ – see front matter
doi:10.1016/j.jpaa.2009.02.012

- *Irreducibility problem:* Is $V_{|D|}(S_1, \ldots, S_r)$ irreducible?
- *Versality problem:* Is the deformation of the multisingularity of a hypersurface $H \in V_{|D|}(S_1, \ldots, S_r)$ induced by the linear system $|D|$ versal (see Section 2.2.1)?

If $V_{|D|}(S_1, \ldots, S_r)$ is non-empty, smooth and has expected dimension, it is said to be "T-smooth". It is known that in this case the deformation induced by $|D|$ is versal.

The case of plane nodal curves has been settled completely. In 1920, Severi gave answers to the first three questions: there exists an irreducible plane curve of degree d having n nodes as their only singularities if and only if

$$0 \le n \le \frac{(d-1)(d-2)}{2}.$$

Furthermore, if $V_d^{irr}(n \cdot A_1)$ is non-empty, then it is smooth of the expected dimension $\frac{d(d+3)}{2} - n$. In 1985, Harris proved that $V_d^{irr}(n \cdot A_1)$ is irreducible.

Already in the case of curves with nodes and cusps there is no complete answer. Segre [1,2] gave an example of an equisingular stratum of such curves (having $6m^2$ cusps as their only singularities) which has a component of non-expected dimension. Zariski [3] gave the first example of a reducible equisingular stratum (of sextic curves with 6 cusps). Finally, Wahl [4] gave the first example of a non-smooth equisingular stratum (of curves of degree 104 having 3636 nodes and 900 cusps). Some other examples can be found in [5–8].

So far, the main effort in the study of equisingular families has been concentrated on obtaining criteria for the T-smoothness. In turn, the obstructed (i.e., non-T-smooth) equisingular families and non-versal deformations have not been studied systematically.

The deformation theory leads to the following result: Suppose that $h^1(\mathcal{O}_\Sigma(D)) = 0$. Then the variety $V_{|D|}(S_1, \ldots, S_r)$ is T-smooth at $H \in V_{|D|}(S_1, \ldots, S_r)$ if and only if

$$h^1(\mathcal{J}_{Z^{ea}(H)/\Sigma}(H)) = 0,$$

where $Z^{ea}(H)$ is a certain zero-dimensional scheme and $\mathcal{J}_{Z^{ea}(H)/\Sigma}(H)$ is its defining ideal (see precise definition in Section 2.1.6). In our research we focus on the minimal obstructedness case, i.e. $h^1(\mathcal{J}_{Z^{ea}(H)/\Sigma}(H)) = 1$.

One of the interesting recent examples is due to du Plessis and Wall [9]:

Example 1.1. (a) For any $d \ge 5$ the curve $C \subset \mathbb{P}^2$ given by the equation $(x_1^d + x_2^5 x_0^{d-5} + x_2^d = 0)$ has a unique singular point $z = (0 : 0 : 1)$ with Tjurina number $\tau(C, z) = 4d - 4$, and satisfies

$$h^1(\mathcal{J}_{Z^{ea}(C)/\mathbb{P}^2}(d)) > 0$$

(b) Denote by S the analytic type of the plane curve singularity (C, z) in (a). If $d \ge 10$ then the family $V_d(S)$ is singular at C.

In [10] this example has been generalized and studied in detail. The following result has been obtained:

Example 1.2. Let C be a projective plane curve, given in local coordinates $x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}$ by the equation $x^k + y^l = 0$. Suppose for convenience $k \ge l$. Let $d = k + l - 5$ and let $V_{d,C}(S)$ be the germ at C of the equianalytic family of plane curves $V_d(S)$, where S is the analytic type of the plane curve singularity (C, z). For any $k, l \ge 5$ such that $d > 5$, $V_{d,C}(S)$ is non-T-smooth and $h^1(\mathcal{J}_{Z^{ea}(C)/\mathbb{P}^2}(d)) = 1$.

Furthermore

(i) If $d = 6$ (i.e. $l = 5$, $k = 6$), the germ $V_{6,C}(S)$ is non-reduced. It is a double $[V_{6,C}(S)]_{red}$, and $[V_{6,C}(S)]_{red}$ is smooth of expected codimension.
(ii) If $d = 7$ (i.e. $l = 5$, $k = 7$ or $k = l = 6$), the germ $V_{7,C}(S)$ is reducible and decomposes into two smooth components of expected codimension that intersect non-transversally with multiplicity one. The intersection locus is smooth. Moreover, the sectional singularity is of type A_1.
(iii) If $d \ge 8$, the germ $V_{d,C}(S)$ is a reduced irreducible non-smooth variety of expected codimension which has a smooth singular locus with sectional singularity of type A_1.

1.1. Main results and methods

The first result of this paper is a generalization of Example 1.2 to quasihomogeneous hypersurface singularities in $\mathbb{P}^n$. In particular we have obtained a new example of a smooth equisingular family of non-expected dimension: $V_{3,H}(S)$ where H is given by the local equation $x^3 + y^3 + z^3 + w^3 = 0$. For precise formulation see Theorem 3.1.

The next question that naturally arose was the behavior of these geometric properties of equianalytic families, with respect to the stabilization of the singularities. We found out that though stabilization preserves both the Tjurina algebra and $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(H))$, it can change the geometry of the equianalytic family radically. We have shown that for any hypersurface H of degree d satisfying $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(2d-2)) = 0$, after adding enough squares the obtained family has a reduced component of expected dimension. We have also shown that the condition $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(2d-2)) = 0$ always holds for plane curves. For precise formulation see Theorem 3.3.

The next result is concerned with deformation theory. Suppose that the germ of the equianalytic family has a reduced irreducible component of expected dimension. In this case the family is T-smooth at every of its regular points which lies in that component. It means that our singular hypersurface H has a deformation $\mathfrak{X} \to T$ such that for $t \neq t_0$ the deformation of $\mathfrak{X}_t$ induced by the linear system $|H|$ is versal. We show that this implies that the deformation of H induced by the linear system $|H|$ is 1-complete (see Section 2.2 for precise definition). For precise formulation see Theorem 3.5.

The methods that we use are the technique of cohomologies of ideal sheaves of zero-dimensional schemes associated with analytic types of singularities, methods for their calculation, and H^1-vanishing theorems. We also use the algorithms of computer algebra (see [11]) as a technical tool in the proof of the theorems.

The structure of the paper

This paper is organized in the following way. Section 2 is dedicated to the formulation of the necessary notions and background.

In Section 2.1 we introduce the notions of singularity theory such as **analytic singularity types**, **quasihomogeneous** singularities, **zero-dimensional schemes associated with singularities** and the **Castelnuovo function**. The main theorems of this section are the Mather–Yau theorem (Theorem 2.5), finite determinacy theorem (Theorem 2.8), Theorem 2.16 on quasihomogeneous singularities, Theorem 2.17 on semiquasihomogeneous polynomials, Theorem 2.21 which gives cohomological criteria of T-smoothness, and Lemmas 2.25 and 2.26 on the Castelnuovo function.

In Section 2.2 we introduce the notions of deformation theory such as **complete** and **versal** deformations. The most important statement for us in this section is Corollary 2.35 on versality of the deformation induced by a complete linear system.

In Section 2.3 we introduce notions and algorithms of computer algebra. The most important notions for us are the notion of **normal form** and the **RedNFBuchberger** algorithm for its computation.

In Section 3 we formulate our **main results**. The first result deals with equianalytic families of hypersurfaces with quasihomogeneous singularities of minimal obstructedness. The second result is about stable properties of obstructed equianalytic families. In Section 3.1 we give an application of the obtained results to the deformation theory.

In Section 4 we prove the theorem on families of hypersurfaces with quasihomogeneous singularities of minimal obstructedness.

In Section 5 we prove the theorem on stable properties of obstructed equianalytic families.

2. Preliminaries and notations

2.1. Notions of singularity theory

In this section we describe the types of isolated hypersurface singularities considered throughout the paper.

2.1.1. Analytic types of hypersurface singularities

Definition 2.1. Let Σ be an n-dimensional smooth projective variety. Two germs $(F, z) \subset (\Sigma, z)$ and $(G, w) \subset (\Sigma, w)$ of isolated hypersurface singularities are said to be *analytically equivalent* if there exists a local analytic isomorphism $(\Sigma, z) \to (\Sigma, w)$ mapping (F, z) to (G, w). The corresponding equivalence classes are called *analytic types*.

Notation 2.2. We denote by $\mathbb{C}\{x_1, \ldots, x_n\}$, or in short $\mathbb{C}\{x\}$, the algebra of convergent power series in n variables.

Definition 2.3. Series $f, g \in \mathbb{C}\{x\}$ are said to be contact equivalent if there exist an automorphism ϕ of $\mathbb{C}\{x\}$ and a unit $u \in \mathbb{C}\{x\}^*$ such that $f = u \cdot \phi(g)$. We denote $f \overset{c}{\sim} g$.

Note that polynomials f and g are contact equivalent if and only if the corresponding germs $(f^{-1}(0), 0)$ and $(g^{-1}(0), 0)$ are analytically equivalent.

Definition 2.4. Let S be an analytic type of reduced hypersurface singularities represented by $(H, z) \subset (\Sigma, z)$ and $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ be a local equation for (H, z). Define the jacobian of f by $j(f) = \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle$. The analytic algebras

$$M_f := \mathbb{C}\{x\}/j(f), \qquad T_f := \mathbb{C}\{x\}/\langle f, j(f) \rangle$$

are called the *Milnor* and *Tjurina* algebras of f, respectively, and the numbers

$$\mu(S) = \mu(H, z) := \dim_{\mathbb{C}} M_f, \qquad \tau(S) = \tau(H, z) := \dim_{\mathbb{C}} T_f$$

are called the *Milnor* and *Tjurina* numbers of S, respectively.

The following theorem shows that the Tjurina algebra is a complete invariant of an analytic singularity type.

Theorem 2.5 (*Mather–Yau*). *Let $f, g \in m \subset \mathbb{C}\{x\}$. The following are equivalent:*

(a) $f \overset{c}{\sim} g$;
(b) *for all* $b \geq 0$, $\mathbb{C}\{x\}/\langle f, m^b j(f)\rangle \cong \mathbb{C}\{x\}/\langle g, m^b j(g)\rangle$ *as* $\mathbb{C}$*-algebras;*
(c) *there is some* $b \geq 0$ *such that* $\mathbb{C}\{x\}/\langle f, m^b j(f)\rangle \cong \mathbb{C}\{x\}/\langle g, m^b j(g)\rangle$ *as* $\mathbb{C}$*-algebras.*

In particular, $f \overset{c}{\sim} g$ *iff* $T_f \cong T_g$.

Proof. See [12] for the case of an isolated singularity and $b = 0, 1$ or [13], Theorem 2.26 for the general case.

2.1.2. Finite determinacy

The aim of this section is to show that an isolated hypersurface singularity is already determined by its Taylor series expansion up to a sufficiently high order.

Definition 2.6. For $f \in \mathbb{C}\{x\}$ we define the *k-jet* of f by

$$jet(f, k) := f^{(k)} := \text{image of } f \text{ in } \mathbb{C}\{x\}/m^{k+1}.$$

We identify $f^{(k)}$ with the power series expansion of f up to (and including) order k.

Definition 2.7. $f \in \mathbb{C}\{x\}$ is called *contact k-determined* if for each $g \in \mathbb{C}\{x\}$ with $f^{(k)} = g^{(k)}$ we have $f \overset{c}{\sim} g$. The minimal such k is called *contact determinacy of f*.

Theorem 2.8 (*Finite Determinacy Theorem*). *Let $f \in m \subset \mathbb{C}\{x\}$.*
f is contact k-determined if $m^{k+1} \subset m^2 \cdot j(f) + m\langle f\rangle$.

Proof. This theorem is well known. See, for instance, [13], Theorem 2.23.

Corollary 2.9. *If $f \in m \subset \mathbb{C}\{x\}$, has an isolated singularity with Tjurina number τ, then f is contact $\tau + 1$-determined.*

2.1.3. Stable equivalence

Definition 2.10. Let $f \in \mathbb{C}\{x_1, \dots, x_l\}$ and $g \in \mathbb{C}\{x_1, \dots, x_k\}$. We say that f is *stably contact equivalent* to g if they become contact equivalent after addition with non-degenerate quadratic forms of additional variables. In other words,

$$f(x_1, \dots, x_l) + x_{l+1}^2 + \cdots + x_n^2 \overset{c}{\sim} g(x_1, \dots, x_k) + x_{k+1}^2 + \cdots + x_n^2.$$

Theorem 2.11. *Polynomials of the same number of variables are stably contact equivalent if and only if they are contact equivalent.*

Proof. See, for instance, [14], chapter II section 11.

We will use a more general lemma:

Lemma 2.12. (1) *Let $f \in \mathbb{C}\{x_1, \dots, x_n\}$. Let $g = f + x_{n+1}^2(1 + h)$ where $h \in m \subset \mathbb{C}\{x_1, \dots, x_{n+1}\}$. Then the Tjurina algebra T_f of f is isomorphic to the Tjurina algebra T_g of g.*

(2) *Let $f_1, f_2 \in \mathbb{C}\{x_1, \dots, x_n\}$. Let $g_i = f_i + x_{n+1}^2(1 + h_i)$, $i = 1, 2$ where $h_i \in m \subset \mathbb{C}\{x_1, \dots, x_{n+1}\}$. Then $f_1 \overset{c}{\sim} f_2$ if and only if $g_1 \overset{c}{\sim} g_2$.*

Proof. (1) In T_g,

$$0 = \frac{\partial g}{\partial x_{n+1}} = x_{n+1}\left(2 + 2h + x_{n+1}\frac{\partial h}{\partial x_{n+1}}\right).$$

Since $2 + 2h + x_{n+1}\frac{\partial h}{\partial x_{n+1}}$ is invertible in T_g, the latter implies $x_{n+1} = 0$. Hence T_f is isomorphic to T_g.

(2) Follows from (1) and from the Mather–Yau Theorem 2.5. □

2.1.4. Quasihomogeneous singularities

Definition 2.13. Let $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} a_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_n]$.

(i) The polynomial f is called *quasihomogeneous of type*

$$(w; d) = (w_1, \ldots, w_n; d)$$

if w_i, d are positive integers satisfying

$$\langle w, \alpha \rangle = w_1\alpha_1 + \cdots + w_n\alpha_n = d$$

for each $\alpha \in \mathbb{Z}^n_{\geq 0}$ with $a_\alpha \neq 0$.

(ii) An isolated hypersurface singularity $(H, x) \subset (\mathbb{C}^n, x)$ is called *quasihomogeneous* if there exists a quasihomogeneous polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ such that $\mathcal{O}_{H,x} \cong \mathbb{C}\{x\}/\langle f \rangle$.

Example 2.14. Let $f = \sum_{i=1}^n x_i^{\alpha_i}$. Then f is a quasihomogeneous polynomial of type $(1/\alpha_1, \ldots, 1/\alpha_n; 1)$. Such polynomials are called *canonical quasihomogeneous*.

Lemma 2.15. *Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be quasihomogeneous and $g \in \mathbb{C}\{x_1, \ldots, x_n\}$ be arbitrary. Then f and g are contact equivalent if and only if there exists an analytic local diffeomorphism ϕ that maps g to f.*

Proof. Let f be quasihomogeneous of type $((w_1, \ldots, w_n); d)$. If $\overset{c}{\sim} g$ then there exists a unit $u \in \mathbb{C}\{x\}^*$ and an automorphism $\psi \in \mathrm{Aut}\mathbb{C}\{x\}$ such that $u \cdot f = \psi(g)$. Choose a dth root $u^{1/d} \in \mathbb{C}\{x\}$. Now we take

$$\phi : \mathbb{C}\{x\} \to \mathbb{C}\{x\}, \qquad x_i \mapsto u^{w_i/d} \cdot x_i. \quad \square$$

A quasihomogeneous polynomial f of type $(w; d)$ obviously satisfies the relation

$$d \cdot f = \sum_{i=1}^{n} w_i x_i \frac{\partial f}{\partial x_i},$$

that implies that f is contained in $j(f)$, hence, for quasihomogeneous isolated hypersurface singularities $\mu = \tau$. For an isolated singularity the converse also holds. More precisely, K. Saito proved the following theorem.

Theorem 2.16 ([15]). *Let $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ and suppose that $f \in j(f)$. Then there exists a quasihomogeneous polynomial g and an analytic local diffeomorphism ϕ that maps g to f. Moreover, the normalized quasihomogeneity type $\frac{w}{d}$ of g is defined uniquely up to permutation.*

We will use the following important theorem:

Theorem 2.17. *Let f be a quasihomogeneous polynomial of weighted degree d and g be a polynomial such that the weighted degrees of all its terms are greater than d. Let $e_1, e_2, \ldots, e_\mu$ be a system of monomials that forms a basis of the Milnor algebra M_f of f. Then $e_1, e_2, \ldots, e_\mu$ form a basis of M_{f+g} as well. In particular, $\mu(f+g) = \mu(f)$.*

For a proof see [14], chapter II section 12.2.

Corollary 2.18. *Let H be a projective hypersurface defined by a quasihomogeneous polynomial f of weighted degree d. Let V_H denote the germ at H of the equisingular family of H. Let $H_F \in V_H$ be a hypersurface defined by a polynomial F. Suppose that all the terms of $F - f$ of weighted degree less than or equal to d are elements of a monomial basis of the Milnor algebra of f.*

Then $F - f$ has no terms of weighted degree less than or equal to d.

Proof. Decompose $F = f + g + \sum_{i=1}^r \lambda_i e_i$, where e_i are all the elements of a monomial basis of the Milnor algebra M_f of weighted degree less than or equal to d, and g has no terms which have weighted degree less than or equal to d. By the theorem, M_{f+g} and M_f have the same basis, and f belongs to the μ-constant stratum of $f+g$. Hence $F = (f+g) + \sum_{i=1}^r \lambda_i e_i$ belongs to the μ-constant stratum of $f+g$. On the other hand, by [16], the affine space $(f+g) + \mathrm{Span}\{e_i\}_{i=1}^r$ is transversal to the μ-constant stratum of $f+g$ and hence $\sum_{i=1}^r \lambda_i e_i = 0$. $\square$

2.1.5. Newton polytope

Definition 2.19. Let $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} a_\alpha x^\alpha \in \mathbb{C}\{x\}$, $a_0 = 0$. The convex hull in $\mathbb{R}^n$ of the support of f,

$$\Delta(f) := \mathrm{conv}\{\alpha \in \mathbb{Z}^n_{\geq 0} \mid a_\alpha \neq 0\},$$

is called *the Newton polytope of f*.

If f is quasihomogeneous with weight w of degree d then the Newton polytope $\Delta(f)$ will lie in the hyperplane $\{\alpha \in \mathbb{Z}^n_{\geq 0} \mid \sum \alpha_i w_i = d\}$. In particular, for a quasihomogeneous polynomial of two variables its Newton polygon is a line segment.

2.1.6. Zero-dimensional schemes associated with singularities

Definition 2.20. Let Σ be a smooth projective variety and $H \subset \Sigma$ a reduced hypersurface with singular locus $Sing(H) = \{z_1, \dots, z_r\}$. We define $Z^{ea}(H) \subset \Sigma$ to be the zero-dimensional scheme, concentrated at $Sing(H)$, given by the Tjurina ideals

$$\mathcal{J}_{Z^{ea}(H)/\Sigma,z_i} = \left\langle f_i, \frac{\partial f_i}{\partial x_1}, \dots, \frac{\partial f_i}{\partial x_n} \right\rangle \subset \mathcal{O}_{\Sigma,z_i},$$

where $f_i(x_1, \dots, x_n)$ is a local equation of H in a neighborhood of z_i. We denote by $\mathcal{J}_{Z^{ea}(H)/\Sigma} \subset \mathcal{O}_\Sigma$ the corresponding ideal sheaf.

The degree of $Z^{ea}(H)$ is

$$\deg Z^{ea}(H) = \sum_{z_i \in Sing(H)} \dim_{\mathbb{C}} \mathcal{O}_{\Sigma,z_i}/\mathcal{J}_{Z^{ea}(H)/\Sigma,z_i}.$$

Theorem 2.21. *Let $H \subset \mathbb{P}^n$ be a reduced hypersurface of degree d with precisely r singularities $z_1, \dots, z_r$ of analytic types $S_1, \dots, S_r$.*

(a) *$H^0(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(d))/H^0(\mathcal{O}_{\mathbb{P}^n})$ is isomorphic to the Zariski tangent space to $V_{|H|}(S_1, \dots, S_r)$ at H. Here, $H^0(\mathcal{O}_{\mathbb{P}^n})$ is embedded into $H^0(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(d))$ via multiplication by the equation of H.*

(b) *$h^0(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(d)) - h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(d)) - 1 \le dim(V_{|H|}(S_1, \dots, S_r), H) \le h^0(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(d)) - 1$.*

(c) *$H^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(d)) = 0$ if and only if $V_{|H|}(S_1, \dots, S_r)$ is T-smooth at H, that is, smooth of the expected dimension $\binom{d+n}{n} - 1 - \deg Z^{ea}(H)$.*

Proof. See [17].

Lemma 2.22. *Let $H \subset \mathbb{P}^n$ be a hypersurface and $Z \subset H$ be a zero-dimensional subscheme. Then $H^1(\mathcal{J}_{Z/\mathbb{P}^n}(H)) \cong H^1(\mathcal{J}_{Z/H}(H))$.*

Proof. The lemma follows from the exact sequence of sheaves

$$0 \to \mathcal{J}_{H/\mathbb{P}^n}(H) \to \mathcal{J}_{Z/\mathbb{P}^n}(H) \to \mathcal{J}_{Z/\mathbb{P}^n}(H) \otimes \mathcal{O}_H \to 0,$$

from the fact that $\mathcal{J}_{H/\mathbb{P}^n}(H) \cong \mathcal{O}_{\mathbb{P}^n}$, which implies $H^1(\mathcal{J}_{H/\mathbb{P}^n}(H)) = H^2(\mathcal{J}_{H/\mathbb{P}^n}(H)) = 0$, and from $\mathcal{J}_{Z/\mathbb{P}^n}(H) \otimes \mathcal{O}_H \cong \mathcal{J}_{Z/H}(H)$.

2.1.7. The Castelnuovo function of a zero-dimensional scheme in $\mathbb{P}^n$

Let $X \subset \mathbb{P}^n$ be a zero-dimensional scheme and $\mathcal{J}_{X/\mathbb{P}^n} \subset \mathcal{O}_{\mathbb{P}^n}$ the corresponding ideal sheaf.

Definition 2.23. The *Castelnuovo function* of X is defined as

$$\begin{array}{rcl} \mathcal{C}_X : \mathbb{Z}_{\ge 0} & \longrightarrow & \mathbb{Z}_{\ge 0} \\ d & \longmapsto & h^1(\mathcal{J}_{X/\mathbb{P}^n}(d-1)) - h^1(\mathcal{J}_{X/\mathbb{P}^n}(d)). \end{array}$$

Remark 2.24. Let $X \subset \mathbb{P}^n$ be a zero-dimensional scheme and $H \subset \mathbb{P}^n$ be a generic hyperplane not passing through the support of X. Then we have an exact reduction sequence

$$0 \longrightarrow \mathcal{J}_{X/\mathbb{P}^n}(d-1) \longrightarrow \mathcal{J}_{X/\mathbb{P}^n}(d) \longrightarrow \mathcal{O}_H(d) \longrightarrow 0,$$

respectively the corresponding exact cohomology sequence

$$H^0(\mathcal{J}_{X/\mathbb{P}^n}(d-1)) \xrightarrow{\pi_H} H^0(\mathcal{O}_H(d)) \longrightarrow H^1(\mathcal{J}_{X/\mathbb{P}^n}(d-1)) \longrightarrow H^1(\mathcal{J}_{X/\mathbb{P}^n}(d)) \longrightarrow 0.$$

In particular,

$$\mathcal{C}_X(d) = h^0(\mathcal{O}_H(d)) - \dim_{\mathbb{C}} \pi_H(H^0(\mathcal{J}_{X/\mathbb{P}^n}(d))).$$

We associate to X the numbers

$$a(X) = \min\{d \in \mathbb{Z} |\, h^0(\mathcal{J}_{X/\mathbb{P}^n}(d)) > 0\}$$

$$b(X) = \min\{d \in \mathbb{Z} |\, \mathbb{P}(H^0(\mathcal{J}_{X/\mathbb{P}^n}(d))) \text{ has no fixed component}\}$$

$$t(X) = \min\{d \in \mathbb{Z} | H^1(\mathcal{J}_{X/\mathbb{P}^n}(d)) = 0\}.$$

Here, a fixed component is a divisor D such that every element of the linear system $|H^0(\mathcal{J}_{X/\mathbb{P}^n}(d))|$ contains D as a component. We call the maximal divisor satisfying this property *the fixed component of* $|H^0(\mathcal{J}_{X/\mathbb{P}^n}(d))|$. The following lemma contains some basic properties of the Castelnuovo function.

Lemma 2.25. *Let $X \subset \mathbb{P}^n$ be a zero-dimensional scheme and $H \subset \mathbb{P}^n$ be a generic hyperplane not passing through the support of X. Then*

(a) *$\mathcal{C}_X(d) \ge 0$ for all d, and $\mathcal{C}_X(d) = 0$ for $d \gg 0$.*

(b) *$\mathcal{C}_X(d) \le h^0(\mathcal{O}_H(d))$, with equality if and only if $d < a(X)$.*

(c) $a(X) \leq b(X) \leq t(X) + 1$.
(d) $\mathcal{C}_X(d) = 0$ *if and only if* $d \geq t(X) + 1$.
(e) *If* $Y \subseteq X$ *then* $\mathcal{C}_Y(d) \leq \mathcal{C}_X(d)$.
(f) $\mathcal{C}_X(0) + \mathcal{C}_X(1) + \cdots + \mathcal{C}_X(d) = \deg X - h^1(\mathcal{J}_{X/\mathbb{P}^n}(d))$.

Proof. See [18] for curves or [19] for hypersurfaces.

Lemma 2.26. *Let* $Z = C_d \cap C_k$ *be the intersection of two plane curves* C_d, C_k *of degrees* d *and* k *without common components. Suppose* $k \leq d$. *Then*

$$\mathcal{C}_Z(i) \leq k \quad \text{for } i \geq 0, \quad \text{and} \quad \mathcal{C}_Z(d+k-j) = j-1 \quad \text{for } j = 1, \ldots, k+1.$$

Proof. See e.g. [20], Lemma 5.4.

2.2. Some notions of the local deformation theory

Definition 2.27. Let (X, x) and (S, s) be complex space germs. *A deformation of* (X, x) *over* (S, s) consists of a flat morphism $\phi : (\mathscr{X}, x) \to (S, s)$ of complex space germs together with an isomorphism from (X, x) to the fiber of ϕ, $(X, x) \to (\mathscr{X}_s, x) := (\phi^{-1}(s), x)$.

$(\mathscr{X}, x)$ is called the *total space*, (S, s) the *base space*, and $(\mathscr{X}_s, x) \cong (X, x)$ the *special fiber* of the deformation. We denote a deformation by

$$(i, \phi) : (X, x) \overset{i}{\hookrightarrow} (\mathscr{X}, x) \xrightarrow{\phi} (S, s),$$

or simply by $(\mathscr{X}, x) \xrightarrow{\phi} (S, s)$.

Definition 2.28. Let $(i, \phi) : (X, x) \overset{i}{\hookrightarrow} (\mathscr{X}, x) \xrightarrow{\phi} (S, s)$ and $(i', \phi') : (X, x) \overset{i}{\hookrightarrow} (\mathscr{X}', x') \xrightarrow{\phi'} (S', s')$ be two deformations of (X, x). A morphism of deformations from (i, ϕ) to (i', ϕ') consists of two morphisms (ψ, φ) such that the following diagram is commutative

$$\begin{array}{ccc} & (X,x) & \\ i' \swarrow & & \searrow i \\ (\mathscr{X}', x') & \xrightarrow{\psi} & (\mathscr{X}, x) \\ \phi' \downarrow & & \downarrow \phi \\ (S', s') & \xrightarrow{\varphi} & (S, s) \end{array}$$

Two deformations over the same base are isomorphic if there exists a morphism (ψ, φ) with ψ an isomorphism and φ the identity map.

Definition 2.29. Let $(i, \phi) : (X, x) \overset{i}{\hookrightarrow} (\mathscr{X}, x) \xrightarrow{\phi} (S, s)$ be a deformation of (X, x) and $\varphi : (T, t) \to (S, s)$ be a morphism of germs. Denote by $\varphi^*(\mathscr{X}, x)$ the fiber product $(\mathscr{X}, x) \times_{(S,s)} (T, t)$. We call

$$\varphi^*(i, \phi) := (\varphi^* i, \varphi^* \phi) : (X, x) \overset{\varphi^* i}{\hookrightarrow} \varphi^*(\mathscr{X}, x) \xrightarrow{\varphi^* \phi} (T, t)$$

the *deformation induced from* (i, ϕ) *by* φ, or just *pull-back*; φ is called the *base change map*.

Proposition 2.30. *Let* $(X, 0) \subset (\mathbb{C}^n, 0)$ *be a closed subgerm. Then any deformation* $(i, \phi) : (X, x) \overset{i}{\hookrightarrow} (\mathscr{X}, x) \xrightarrow{\phi} (S, s)$ *can be embedded. In other words, there exists a Cartesian diagram*

$$\begin{array}{ccc} (X, 0) & \overset{i}{\hookrightarrow} & (\mathscr{X}, x) \\ \downarrow & & \downarrow J \\ (\mathbb{C}^n, 0) & \overset{j}{\hookrightarrow} & (\mathbb{C}^n, 0) \times (S, s) \\ \downarrow & & \downarrow p \\ \{s\} & \hookrightarrow & (S, s) \end{array}$$

where J is a closed embedding, p is the second projection, j is the first inclusion and $\phi = p \circ J$. In particular, the embedding dimension is semicontinuous under deformations, that is, $\operatorname{edim}(\phi^{-1}(\phi(y)), y) \leq \operatorname{edim}(X, 0)$, *for all y in $\mathscr{X}$ sufficiently close to x.*

Proof. See [13], Corollary II.1.6.

2.2.1. Versal and complete deformations

A versal deformation of a complex space germ is a deformation which contains basically all information about any possible deformation of this germ. More precisely, we say that a deformation (i, ϕ) of (X, x) over (S, s) is *complete* if any other deformation over some base space (T, t) can be induced from (i, ϕ) by some base change $\varphi : (T, t) \to (S, s)$. A complete deformation is called *versal* if for any deformation of (X, x) over some subgerm $(T', t) \subset (T, t)$ induced by some base change $\varphi' : (T', t) \to (S, s)$, φ can be chosen in such a way that it extends φ'. We will now give the formal definitions.

Definition 2.31. (1) A deformation $(X, x) \overset{i}{\hookrightarrow} (\mathscr{X}, x) \overset{\phi}{\to} (S, s)$ of (X, x) is called *complete* if, for any deformation (j, ψ) : $(X, x) \overset{j}{\hookrightarrow} (\mathscr{Y}, y) \overset{\psi}{\to} (T, t)$ of (X, x) there exists a morphism $\varphi : (T, t) \to (S, s)$ such that (j, ψ) is isomorphic to the induced deformation $(\varphi^* i, \varphi^* \phi)$.

(2) A deformation (i, ϕ) is called *versal* if, for any deformation (j, ψ) as above the following hold: for any closed embedding $k : (T', t) \hookrightarrow (T, t)$ of complex space germs and any morphism $\varphi' : (T', t) \to (S, s)$ such that $(k^* j, k^* \psi)$ is isomorphic to the induced deformation $(\varphi'^* i, \varphi'^* \phi)$, there exists a morphism $\varphi : (T, t) \to (S, s)$ such that
 (i) $\varphi \circ k = \varphi'$ and
 (ii) (j, ψ) is isomorphic to the induced deformation $(\varphi^* i, \varphi^* \phi)$.
 This definition can be illustrated by the following commutative diagram:

$$\begin{array}{ccccc}
 & & (X, x) & & \\
 & k^* j \swarrow & \downarrow j & \searrow i & \\
k^*(\mathscr{Y}, y) & \hookrightarrow & (\mathscr{Y}, y) & \dashrightarrow & (\mathscr{X}, x) \\
k^* \psi \downarrow & & \psi \downarrow & & \downarrow \phi \\
(T', t) & \overset{k}{\hookrightarrow} & (T, t) & \overset{\varphi}{\dashrightarrow} & (S, s).
\end{array}$$

(3) A versal deformation is called *semiuniversal or miniversal* if the Zariski tangent map $T(\varphi) : T_{(T,t)} \to T_{(S,s)}$ is uniquely defined by (i, ϕ) and (j, ψ).

Our definition of versality is very restrictive. The deformations that we call complete are sometimes called versal in the literature. For example, in [14] the authors call our complete deformation "versal" and our versal deformation "infinitesimally versal".

Now we introduce a notion which is weaker than completeness, but still strong enough for many applications.

Definition 2.32. A deformation $(X, x) \overset{i}{\hookrightarrow} (\mathscr{X}, x) \overset{\phi}{\to} (S, s)$ of (X, x) is called *1-complete* if, for any one parametric deformation $(j, \psi) : (X, x) \overset{j}{\hookrightarrow} (\mathscr{Y}, y) \overset{\psi}{\to} (\mathbb{C}, 0)$ of (X, x) there exists a morphism $\varphi : (\mathbb{C}, 0) \to (S, s)$ such that (j, ψ) is isomorphic to the induced deformation $(\varphi^* i, \varphi^* \phi)$.

An arbitrary complex space germ may not have a versal deformation. It is a fundamental theorem of Grauert, that for isolated singularity a semiuniversal deformation exists.

Theorem 2.33 (*Grauert, 1972*). *Any complex space germ with isolated singularity has a semiuniversal deformation.*

Proof. See [21].

The following two statements describe the connection between equisingular families and versal deformations.

Theorem 2.34. *Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be an isolated singularity defined by $f \in \mathcal{O}_{\mathbb{C}^n,0}$ and $g_1, \ldots, g_\tau \in \mathcal{O}_{\mathbb{C}^n,0}$ be a basis of the Tjurina algebra T_f. If we set*

$$F(x, t) := f(x) + \sum_{j=1}^{\tau} t_j g_j(x), \quad (\mathscr{X}, 0) := V(F) \subset (\mathbb{C}^n \times \mathbb{C}^\tau, 0),$$

then $(X, 0) \hookrightarrow (\mathscr{X}, 0) \overset{\phi}{\to} (\mathbb{C}^\tau, 0)$, where ϕ is the second projection, is a semiuniversal deformation of $(X, 0)$.

Proof. See [13], corollary II.1.17.

Corollary 2.35. *Let (H, z) be a germ of a projective hypersurface with one isolated singularity z. Suppose that the equianalytic family of (H, z) is T-smooth at (H, z). Then the linear system $|H|$ induces a versal deformation of (H, z).*

We finish this section with a version of the classical curve selection lemma that will be used below.

Lemma 2.36. *Let X be an algebraic variety over $\mathbb{C}$. Let $U \subset X$ be a Zariski open subset and x be a point in the closure of U. Then there exists a morphism of analytic germs $\phi : (\mathbb{C}, 0) \to (X, x)$ such that $\phi(\mathbb{C} \setminus \{0\}) \subset U$.*

Proof. A basic lemma (see, for example, [22], Lemma 7.2.1) says that there exists a smooth curve C and a morphism $\nu : C \to \overline{U}$, such that $\nu^{-1}(U)$ is non-empty and x is contained in the image of ν. Denote $Z = \nu^{-1}(\overline{U} \setminus U)$. Then Z is a closed subset of C and hence consists of a finite number of points. Now take any point $z \in \nu^{-1}(\{x\}) \subset C$. It has a neighborhood which does not contain other points of Z. Since C is smooth, the analytic germ (C, z) is isomorphic to $(\mathbb{C}, 0)$. Hence ν defines the required morphism. □

2.3. Notions of computer algebra

In our computations we want to use the methods of computer algebra. Here we introduce the basic notions, that will be widely used in our proofs. A more detailed description of these notions can be found also in [11].

2.3.1. Monomial orderings

Definition 2.37. A *monomial ordering* is a total (or linear) ordering $>$ on the set of monomials $\mathrm{Mon}_n = \{x^\alpha \mid \alpha \in \mathbb{Z}^n_{\geq 0}\}$ in n variables satisfying

$$x^\alpha > x^\beta \Rightarrow x^\gamma x^\alpha > x^\gamma x^\beta$$

for all $\alpha, \beta, \gamma \in \mathbb{Z}^n_{\geq 0}$. We say also $>$ is a monomial ordering on $A[x_1, \ldots, x_n]$, where A is any ring, meaning that $>$ is a monomial ordering on Mon_n.

We identify Mon_n with $\mathbb{Z}^n_{\geq 0}$, and then a monomial ordering is a total ordering on $\mathbb{Z}^n_{\geq 0}$, which is compatible with the semigroup structure on $\mathbb{Z}^n_{\geq 0}$ given by addition. From a practical point of view, a monomial ordering $>$ allows us to write a polynomial $f \in K[x]$ in a unique ordered way as

$$f = a_\alpha x^\alpha + a_\beta x^\beta + \cdots + a_\gamma x^\gamma,$$

with $x^\alpha > x^\beta > \cdots > x^\gamma$, where no coefficient is zero.

The most important distinction is between global and local orderings.

Definition 2.38. Let $>$ be a monomial ordering on $\{x^\alpha \mid \alpha \in \mathbb{Z}^n_{\geq 0}\}$.

(1) $>$ is called a *global* ordering if $x^\alpha > 1$ for all $\alpha \neq (0, \ldots, 0)$,
(2) $>$ is called a *local* ordering if $x^\alpha < 1$ for all $\alpha \neq (0, \ldots, 0)$.

Important examples of monomial orderings are:

Example 2.39 (*Monomial Orderings*). In the following examples we fix an enumeration $x_1, \ldots, x_n$ of the variables, any other enumeration leads to a different ordering.

(1) Global orderings
 (i) *Lexicographical ordering* $>_{lp}$

$$x^\alpha >_{lp} x^\beta :\Leftrightarrow \exists 1 \leq i \leq n : \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i.$$

 (ii) *Degree lexicographical ordering* $>_{Dp}$

$$x^\alpha >_{Dp} x^\beta :\Leftrightarrow \deg x^\alpha > \deg x^\beta$$
$$\text{or} \quad (\deg x^\alpha = \deg x^\beta \text{ and } \exists 1 \leq i \leq n : \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i).$$

 (iii) *Weighted degree lexicographical ordering* $Wp(\omega_1, \ldots, \omega_n)$
 Given a vector $\omega = (\omega_1, \ldots, \omega_n)$ of integers, we define the *weighted degree* of x^α by $\deg_\omega(x^\alpha) := \langle \omega, \alpha \rangle := \omega_1\alpha_1 + \cdots + \omega_n\alpha_n$, that is, the variable x_i has degree ω_i. For a polynomial $f = \sum_\alpha a_\alpha x^\alpha$, we define the *weighted degree*,

$$\deg_\omega(f) := \max\{\deg_\omega(x^\alpha) \mid a_\alpha \neq 0\}.$$

 Using the weighted degree in (ii), with all $\omega_i > 0$, instead of the usual degree, we obtain the weighted degree lexicographical ordering, $Wp(\omega_1, \ldots, \omega_n)$.

(2) Local orderings
 (i) *Negative lexicographical ordering* $>_{ls}$

$$x^\alpha >_{ls} x^\beta :\Leftrightarrow \exists 1 \leq i \leq n : \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i.$$

(ii) *Negative degree lexicographical ordering* $>_{Ds}$:

$$x^\alpha >_{Ds} x^\beta :\Leftrightarrow \deg x^\alpha < \deg x^\beta$$
$$\text{or} \quad (\deg x^\alpha = \deg x^\beta \text{ and } \exists 1 \le i \le n : \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i).$$

(iii) *Negative weighted degree lexicographical ordering* $Ws(\omega_1, \ldots, \omega_n)$ is a weighted version of the last ordering.

Definition 2.40. Let $>$ be a fixed monomial ordering. Let $f \in K[x], f \neq 0$. Then f can be written in a unique way as a sum of non-zero terms

$$f = a_\alpha x^\alpha + a_\beta x^\beta + \cdots + a_\gamma x^\gamma, \qquad x^\alpha > x^\beta > \cdots > x^\gamma,$$

and $a_\alpha, a_\beta, \ldots, a_\gamma \in K$. We define:
(1) $LM(f) := x^\alpha$, the *leading monomial* of f,
(2) $LE(f) := \alpha$, the *leading exponent* of f,
(3) $LT(f) := a_\alpha x^\alpha$, the *leading term* of f,
(4) $LC(f) := a_\alpha$, the *leading coefficient* of f,
(5) $tail(f) := f - LT(f) = a_\beta x^\beta + \cdots + a_\gamma x^\gamma$, the tail of f.

Definition 2.41. For any monomial ordering $>$ on $\mathrm{Mon}(x_1, \ldots, x_n)$, we define the ring $K[x]_>$ associated to $K[x]$ and $>$ by

$$K[x]_> := \left\{ \frac{f}{u} \mid f, u \in K[x], LM(u) = 1 \right\}.$$

Note that $K[x]_> = K[x]$ if and only if $>$ is global and $K[x]_> = K[x]_{\langle x_1, \ldots, x_n \rangle}$ if and only if $>$ is local.

2.3.2. Normal form

Let $>$ be a monomial ordering and let $R = K[x_1, \ldots, x_n]_>$ (see Definition 2.41 above). For any subset $G \subset R$ define the ideal

$$L_>(G) := L(G) := \langle LM(g) | g \in G \setminus \{0\} \rangle_{K[x]}.$$

$L(G) \subset K[x]$ is called the *leading ideal* of G. Note that if I is an ideal, then $L(I)$ is the ideal generated by all leading monomials of all elements of I and not only by the leading monomials of a given set of generators of I.

Definition 2.42. Let $\mathcal{G}$ denote the set of all finite subsets $G \subset R$. A map

$$NF : R \times \mathcal{G} \to R, \quad (f, G) \mapsto NF(f \mid G),$$

is called a *normal form* on R if, for all $f \in R$ and $G \in \mathcal{G}$,
(0) $NF(0|G) = 0$;
(1) $NF(f|G) \neq 0 \Rightarrow LM(NF(f|G)) \notin L(G)$;
(2) if $G = \{g_1, \ldots, g_s\}$, then $r := f - NF(f|G)$ has a standard representation with respect to G, that is, either $r = 0$, or

$$r = \sum_{i=1}^{s} a_i g_i, \quad a_i \in R,$$

satisfying $LM(f) \geq LM(a_i g_i)$ for all i such that $a_i g_i \neq 0$.

NF is called a *reduced normal form*, if, moreover, $NF(f|G)$ is reduced with respect to G, i.e. no monomial of the power series expansion of $NF(f|G)$ is contained in $L(G)$.

As we can see from the definition, $NF(f|G) = 0$ if and only if $f \in \langle G \rangle$.

2.3.3. REDNFBUCHBERGER *algorithm for computation of normal form*

Algorithm 2.43 (REDNFBUCHBERGER *Algorithm*).
Assume that $>$ is a global monomial ordering.
Input: $f \in K[x]$, $G \in \mathcal{G}$
Output: $p \in K[x]$, a reduced normal form of f with respect to G.
1. $p := 0$; $h := f$;
2. while $(h \neq 0)$
  (a) while $(h \neq 0$ and $G_h := \{g \in G | LM(g)$ divides $LM(h)\} \neq \emptyset)$
    {choose any $g \in G_h$;
    $h := h - (LT(h)/LT(g)) \cdot g$}
  (b) if $(h \neq 0)$
    $\{p := p + LT(h)$;
    $h := tail(h)\}$;
3. return $p/LC(p)$;

The algorithm works in the following way: the inner loop (2) runs until it meets an "obstruction", i.e. the first monomial that is not divisible by the leading monomial of any member of G. When the inner loop (2) stops, h stores a normal form of f. To make this normal form reduced, we add the leading term of h, i.e. the "obstruction", to p and continue working with the tail of h in the same way.

Note that any specific choice of "any $g \in G_h$" can give a different normal form function. For proof of correctness of the algorithm see [11], section 1.6 algorithms 1.6.10 and 1.6.11.

2.3.4. Highest corner

Definition 2.44. Let $>$ be a monomial ordering on $\mathrm{Mon}(x_1, \ldots, x_n)$ and let $I \subset K[x_1, \ldots, x_n]_>$ be an ideal. A monomial $m \in \mathrm{Mon}(x_1, \ldots, x_n)$ is called the *highest corner* of I (with respect to $>$), denoted by $HC(I)$, if

(1) $m \notin L(I)$;
(2) $m' \in \mathrm{Mon}(x_1, \ldots, x_n), m' < m \Rightarrow m' \in L(I)$.

Lemma 2.45. *Let $>$ be a monomial ordering on $\mathrm{Mon}(x_1, \ldots, x_n)$ and let $I \subset K[x_1, \ldots, x_n]_>$ be an ideal. Let m be a monomial such that $m' < m$ implies $m' \in L(I)$. Let $f \in K[x_1, \ldots, x_n]$ such that $LM(f) < m$. Then $f \in I$.*

Proof. See [11] Lemma 1.7.13.

Lemma 2.46. *Let $>$ be a weighted degree ordering on $\mathrm{Mon}(x_1, \ldots, x_n)$. Moreover, let $f_1, \ldots, f_k$ be a set of generators of the ideal $I \subset K[x_1, \ldots, x_n]_>$ such that $J := \langle LM(f_1), \ldots, LM(f_k)\rangle$ has a highest corner $m := HC(J)$ and $f \in K[x_1, \ldots, x_n]_>$. If $LM(f) < HC(J)$ then $f \in I$.*

Proof. See [11] Lemma 1.7.17.

2.4. Affine coordinates and the stratum V_d^U

In this section we enter the notion of stratum V_d^U that will be used in all the proofs, in order to work in affine coordinates.

Definition 2.47. Let S be an analytic singularity type of projective hypersurfaces. Fix homogeneous coordinates on $\mathbb{P}^n$ and consider the open subset

$$U = \{(t_0 : \ldots : t_n) | t_0 \neq 0\} \subset \mathbb{P}^n.$$

We define $V_d^U(S)$ to be the space of all hypersurfaces of degree d that have a unique singular point inside U of singularity type S.

Note that both $V_d^U(S)$ and $V_d(S)$ are open subsets of the space of all hypersurfaces of degree d that have at least one isolated singular point of singularity type S. Hence for any hypersurface $H \in V_d^U(S) \cap V_d(S)$, the germs $V_{d,H} := (V_d(S), H)$ and $V_{d,H}^U := (V_d^U(S), H)$ coincide. Hence we will formulate statements on $V_{d,H}$ and prove them on $V_{d,H}^U$.

Remark 2.48. There exists a natural embedding $\nu : V_d^U(S) \hookrightarrow V_{d+1}^U(S)$ defined in the following way. Let $F(t_0, \ldots, t_n)$ be an equation of the hypersurface $H \in V_d^U(S)$. Then we define $\nu(H)$ to be the hypersurface defined by the equation $t_0 F(t_0, \ldots, t_n) = 0$. Note that in the coordinate system $x_i = \frac{t_i}{t_0}$ on U, H and $\nu(H)$ will be given by the same local equation.

Using the above embedding, the scheme theoretic structure on $V_d^U(S)$ can be computed in the following way. By Theorem 2.21 there exists N such that $V_{d+N}^U(S)$ is a smooth variety. Then $V_d^U(S)$ is equal to the scheme theoretic intersection $V_{d+N}^U(S) \cap |\mathcal{O}_{\mathbb{P}^n}(d)|$.

3. Main results

We start with a generalization of Example 1.2 to higher dimensions:

Theorem 3.1. *Let $H \subset \mathbb{P}^n$, $n \geq 3$ be the projective hypersurface given by the equation $\sum_{i=1}^n t_i^{\alpha_i} t_0^{d-\alpha_i} + \sum_{i=1}^n \lambda_i t_i^d = 0$ where $d = \sum_{i=1}^n \alpha_i - (2n+1)$ and the λ_i are complex numbers such that $z = (1, 0, \ldots, 0)$ is the unique singular point of H. Note that generic λ_i satisfy this condition.*

Let $V_{d,H}$ be the germ at H of the equianalytic family of H. Then for any $\{\alpha_i\}_{i=1}^n$ such that $d \geq \alpha_i \geq 2$ for all i, $V_{d,H}$ is non-T-smooth and $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(d)) = 1$.

Furthermore:

(i) *If $n = 4$ and $d = 3$ (i.e. $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 3$), the germ $V_{3,H}(S)$ is a smooth variety of non-expected codimension (one less than expected).*
(ii) *Otherwise, the germ $V_{d,H}(S)$ is a reduced irreducible non-smooth variety of expected codimension which has a smooth singular locus. Moreover, the germ $V_{d,H}(S)$ has the sectional singularity type A_1.*

(For proof see Section 4.)

Remark 3.2. It can be shown that the exceptional case of this theorem can be generalized in the following way:
Let $H \subset \mathbb{P}^n$ be the hypersurface given by the local equation $\sum_{i=1}^n t_i^d = 0$, $d \geq \max\{3, 7-n\}$. Then the germ $V_{d,H}$ is an orbit of PGL_{n+1} and hence is a smooth variety of non-expected codimension.

In Example 1.2 and Theorem 3.1 we have seen several examples of equianalytic strata of minimal obstructedness which are non-reduced, or reducible or have unexpected dimension. In these examples non-reduced families have smooth reduction, all components of reducible families are smooth and have expected codimension, and non-smooth families have smooth singular loci. The first statement follows from minimal obstructedness. We conjecture that the other two statements hold for general families of minimal obstructedness of Newton non-degenerate hypersurface singularities.

Also, we conjecture that if $h^1(\mathcal{J}_{Z^{ea}(H')/\mathbb{P}^n}(d))$ is constant along the equianalytic family of a unisingular projective hypersurface H, then the family has smooth reduction. This can be easily proven for reduced families (see [23], Proposition 2.4.1).

The next question that naturally arose was the behavior of the geometric properties of equianalytic families with respect to the stabilization of the singularities (see Section 2.1.3).

We found out that these phenomena are not stable. Namely, if we add a new variable x_{n+1} to the space and x_{n+1}^2 to the local equation of the hypersurface, the equianalytic stratum of the new hypersurface has the same h^1 and τ but is reduced irreducible of expected codimension. Apparently, the same is true for any singularity of minimal obstructedness, though sometimes more variables and their squares need to be added.

More generally, for any hypersurface singularity with $h^1 > 0$ but $h^1(2d-2) = 0$, the equianalytic stratum obtains an irreducible component which is reduced of expected dimension after adding $h^1 + 1$ squares. The condition $h^1(2d-2) = 0$ always holds for curves. For higher dimensions, $h^1(2d-2) = 0$ follows from the condition $h^1 < d-1$.

The following theorem summarizes all that was mentioned above.

Theorem 3.3. *Let $H \subset \mathbb{P}^n$ be hypersurface of degree $d \geq 3$ with the unique singular point $z = (1 : 0 : \ldots : 0)$. Let $h^1 := h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(d))$ and $\tau := \deg Z^{ea}(H)$. Let F_0 be the equation of H. Suppose $h^1 > 0$.*

For any $m \geq 1$ define $W^m \subset \mathbb{P}^{n+m}$ to be the hypersurface given by equation $F_0 + \sum_{j=1}^m t_{n+j}^2 t_0^{d-2}$ and z_m be the point $(1 : 0 : \ldots : 0)$. Denote by $V_{d,W^m}^{U_m}$ the germ at W^m of the family of all hypersurfaces of degree d that have a unique singular point inside $U_m = \{(t_0 : \ldots : t_{n+m},)|t_0 \neq 0\} \subset \mathbb{P}^{n+m}$, and are analytically equivalent to (W^m, z) near the singular point. Then:

(a) $h^1(\mathcal{J}_{Z^{ea}(W^m)/\mathbb{P}^{n+m}}(d)) = h^1$ *and* $\deg Z^{ea}(W^m) = \tau$.
(b) *If* $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(2d-2)) = 0$ *then the germs* $V_{d,W^m}^{U_m}$ *for* $m \geq h^1 + 1$ *have a reduced component of expected dimension.*
(c) *If H is a plane curve then already* $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^2}(2d-4)) = 0$ *and hence for* $m \geq h^1 + 1$ *the germs* $V_{d,W^m}^{U_m}$ *have a reduced component of expected dimension.*
(d) *If* $h^1 < d-1$ *then* $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(2d-2)) = 0$ *and hence for* $m \geq h^1 + 1$ *the germs* $V_{d,W^m}^{U_m}$ *have a reduced component of expected dimension.*
(e) *If* $h^1 = 1$ *then the germs* $V_{d,W^m}^{U_m}$ *are non-smooth of expected dimension for all* $m \geq 1$, *reduced for* $m \geq \max\{1, 5-d\}$ *and irreducible for* $m \geq \max\{1, 6-d\}$.

(For the proof see Section 5.)

Remark 3.4. (1) Statement (c) is not always true for $n \geq 3$. Consider, for example, H given by the local equation $\sum x_i^d = 0$. Then $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(k)) > 0$ for $k < n(d-2)$.
(2) It can be proven that if $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(2d-2)) = 0$ and $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(d+1)) = h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(d)) - 1$ then for $m \geq 2$ the germs $V_{d,W^m}^{U_m}$ have a reduced component of expected dimension.
(3) If $F_0 = \sum_{i=1}^n t_i^{\alpha_i} t_0^{d-\alpha_i}$ and $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(t)) = 0$ for some $d < t \leq 2d-2$, then for $m \geq \lceil h^1 \frac{(t-d+1)}{2(d-2)} \rceil + 1$ the germs $V_{d,W^m}^{U_m}$ have a reduced component of expected dimension. In particular, if $F_0 = \sum_{i=1}^n t_i^{\alpha_i} t_0^{d-\alpha_i}$ and $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(2d-3)) = 0$ then for $m \geq \lceil \frac{h^1}{2} \rceil + 1$ the germs $V_{d,W^m}^{U_m}$ have a reduced component of expected dimension. Together with (2) that implies that if F_0 is canonical quasihomogeneous and $h^1 < d-1$ then the germs $V_{d,W^m}(S^m)$ have a reduced component of expected dimension for any $m \geq \lceil \frac{h^1}{2} \rceil + 1$.

For proof see [23], Theorem 2.2.1(f).

3.1. Deformation theoretic meaning

In this subsection we give a deformation theoretic interpretation to our results. Since the proofs of the statements here are shorter and less technical, we give them right after the statements.

Theorem 3.5. *Let $H \in \mathbb{P}^n$ be a projective hypersurface of degree d with the unique singular point z. Suppose that the equianalytic stratum germ $V_{d,H}$ has a reduced component R of expected dimension. Then the deformation of H induced by the linear system $|H|$ is 1-complete.*

The proof of the theorem is based on the following observation: at every smooth point H' of R, the stratum $V_{d,H}$ is T-smooth. Hence the deformation of H' induced by the linear system $|H|$ is versal, and hence any 1-parametric deformation $(H, z) \hookrightarrow (\mathscr{X}, x) \to (\mathbb{C}, 0)$ of (H, z) can be induced from it by a map $\psi_{H'} : (\mathbb{C}, 0) \to (|H|, H')$. By the curve selection lemma, there exists a map $\phi : (\mathbb{C}, 0) \to (R, H)$ such that all points except 0 are mapped to non-singular points. Now we define the requested map $\varphi : (\mathbb{C}, 0) \to (|H|, H)$ by $\varphi(t) := \psi_{\phi(t)}(t)$ for $t \neq 0$ and $\varphi(0) = H$.

Now we give a precise proof, which includes the description how to choose the maps $\psi_{\phi(t)}$ analytically.

Proof. Denote $U = R \setminus Sing(V_d)$ where $Sing(V_d)$ is the singular locus of V_d. Let τ be the Tjurina number of (H, z). Consider the coincidence variety

$$Z := \{(H', W) | H' \in R, W \text{ is a } \tau\text{-dimensional affine subspace of } |H| \text{ and } H' \in W\}.$$

Let $Y \subset Z$ be the open subset defined by

$$Y := \{(H', W) \in Z | H' \in U \text{ and } W \text{ is transversal to } R \text{ at } H'\}.$$

By the curve selection lemma (Lemma 2.36), there exists a morphism of analytic germs $\phi : (\mathbb{C}, 0) \to (Z, (H, W))$ (for some τ-dimensional subspace W) such that $\phi(\mathbb{C} \setminus \{0\}) \subset Y$.

Now let $(H, z) \hookrightarrow (\mathscr{X}, x) \to (\mathbb{C}, 0)$ be a one-parametric deformation of (H, z). Let Ta_H denote the Tjurina algebra of (H, z) and let $(H, z) \hookrightarrow (\mathscr{Y}, y) \to (Ta_H, 0)$ be the semiuniversal deformation over it described in Theorem 2.34. Since this deformation is semiuniversal, there exists a morphism $\psi : (\mathbb{C}, 0) \to (Ta_H, 0)$ such that $\psi^*(\mathscr{Y}, y) = (\mathscr{X}, x)$. Note that the monomial basis of the Tjurina algebra of (H, z) is also a basis of Tjurina algebras in a neighborhood of H. Thus we identify those Tjurina algebras as vector spaces.

For any point $(H', W) \in Y$, the factor morphism $T_{H'}|H| \to Ta_{H'} \cong Ta_H$ defines an isomorphism $p_{H',W} : W \cong Ta_H$, since W is transversal to the kernel of the factor morphism, which is the tangent space to R at H'. This defines a morphism $\Psi : (\mathbb{C}, 0) \times Y \to |H|$ by

$$\Psi(t, (H', W)) := p^{-1}_{H',W}(\psi(t)).$$

Now, we define $\varphi : (\mathbb{C}, 0) \setminus 0 \to (|H|, H)$ by $\varphi := \Psi \circ (Id \times \phi)$, and extend it to 0 by $\varphi(0) := H$. $\square$

Corollary 3.6. *Let $H \in \mathbb{P}^n$ be a unisingular hypersurface of degree $d \geq 3$ defined by the equation $\sum_{i=1}^n t_i^{\alpha_i} t_0^{d-\alpha_i} + \sum_{i=1}^n \lambda_i t_i^d = 0$. Suppose that H has one isolated singularity and $d+1 = \sum_{i=1}^n (\alpha_i - 2)$. Then, unless $n = 2, d \leq 6$ or $n = 4, d = 3$, the deformation of H induced by the linear system $|H|$ is 1-complete.*

Corollary 3.7. *Let $H \subset \mathbb{P}^n$ be a hypersurface of degree $d \geq 3$ with the unique singular point $z = (1 : 0 : \ldots : 0)$. Let F_0 be the equation of H. For any $m \geq 1$ define $W^m \subset \mathbb{P}^{n+m}$ to be the hypersurface given by equation $F_0 + \sum_{j=1}^m t_{n+j}^2 t_0^{d-2}$ and z_m be the point $(1 : 0 : \ldots : 0)$.*

Suppose that $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(2d-2)) = 0$. Then for $m \geq h^1 + 1$ the deformation of W^m induced by the linear system $|W^m|$ is 1-complete.

Let us now demonstrate one known application of 1-completeness. Suppose that we want to construct a hypersurface of degree d having m isolated singular points of prescribed analytic singularity types $S_1, \ldots, S_m$. Suppose that we can construct a hypersurface H with unique more complicated singularity that splits to singularities of the given types S_i after a one-parameter deformation by hypersurfaces of higher degrees. If the deformation of H induced by the linear system $|H|$ is 1-complete, there exists a deformation of H by hypersurfaces from $|H|$ which contains the desired hypersurfaces.

Proposition 3.8. *Let $H \subset \mathbb{P}^n$ be a hypersurface of degree d with one isolated singular point z of the analytic singularity type S. Let $(H, z) \overset{i}{\hookrightarrow} (\mathscr{X}, x) \overset{\phi}{\to} (|H|, H)$ be the deformation of (H, z) induced by the linear system $|H|$. Suppose that it is 1-complete. Let $S_1, \ldots, S_m$ be analytic singularity types. Suppose also that there exists a one-parameter deformation (j, ψ) : $(H, z) \overset{j}{\hookrightarrow} (\mathscr{Y}, y) \overset{\psi}{\to} (\mathbb{C}, 0)$ of (H, z) that includes hypersurfaces having m singularities of types $S_1, \ldots, S_m$. Then there exists a one-parameter deformation of (H, z) consisting of hypersurfaces of degree d that includes hypersurfaces having m singularities of types $S_1, \ldots, S_m$.*

Proof. Since the deformation of H induced by $|H|$ is 1-complete, there exists a morphism $\varphi : (\mathbb{C}, 0) \to (|H|, H)$ such that (j, ψ) is isomorphic to the induced deformation $(\varphi^* i, \varphi^* \phi)$. Hence the induced deformation $(\varphi^* i, \varphi^* \phi)$ includes hypersurfaces having m singularities of types $S_1, \ldots, S_m$. On the other hand, the deformation (i, ϕ) consists of hypersurfaces of degree d, and hence the induced deformation $(\varphi^* i, \varphi^* \phi)$ also consists of hypersurfaces of degree d. $\square$

4. Proof of the theorem on quasihomogeneous hypersurface singularities

This section is dedicated to the proof of Theorem 3.1.

4.1. The structure of the proof

First of all we pass to affine coordinates $x_i = \frac{t_i}{t_0}$. In these coordinates, H is given by the local equation $f = \sum_{i=1}^n x_i^{\alpha_i} + \sum_{i=1}^n \lambda_i x_i^d = 0$. It is a semiquasihomogeneous polynomial with non-degenerate quasihomogeneous part $g = \sum_{i=1}^n (1 + \delta_{\alpha_i,d}\lambda_i) x_i^{\alpha_i}$. It is easy to see that $f \in j(f)$ and $j(f) = j(g)$. Hence f and g have the same Tjurina ideal and Tjurina algebra, $f \overset{c}{\sim} g$ and $T_f = M_f$.

Then we prove the equality $h^1(\mathcal{J}_{Z^{ea}(H,z)/\mathbb{P}^n}(d)) = 1$ (see Section 4.2). Next (in Section 4.3) we switch to substratum germ $V_{d,H}^{0,0}$ of $V_{d,H}$ consisting of hypersurfaces given by polynomials of the form

$$F(x_1, \dots, x_n) = \sum_{i=1}^n x_i^{\alpha_i} + \sum_{i=1}^n \lambda_i x_i^d + \sum_{I \in \mathcal{D}} a_I x^I,$$

$$\text{where } \mathcal{D} = \left(\{I \in \mathbb{Z}_{\geq 0}^n | \alpha_n \leq |I| \leq d\} \setminus \bigcup_{1 \leq i \neq j \leq n} \{(0, \dots, 0, \alpha_i - 1, 0, \dots, 0, \underset{j}{1}, 0, \dots, 0)\} \right) \setminus \bigcup_{1 \leq i \leq n} \{(0, \dots, 0, \alpha_i, 0, \dots, 0)\}.$$

We prove that this substratum is transversal to the orbits of the group of affine transformations of $\mathbb{C}^n$.

First, we consider the case $\alpha_1 < 2\alpha_n$ (Section 4.4). In this case for any hypersurface H which lies in the stratum germ there exists an affine coordinate change s.t. the equation of H in the new coordinates does not include any terms that lie below the Newton polytope $\Delta(f)$, and has the same terms laying on $\Delta(f)$ as f.

Let $F = f + f_1$ where f_1 is a polynomial which has no terms below and on $\Delta(f)$. We claim that F is contact equivalent to f if and only if $F \in j(F) = \langle F_{x_1}, \dots, F_{x_n} \rangle$.

One direction is obvious: if they are equivalent then they have the same Milnor and Tjurina numbers and hence $\mu(F) = \tau(F)$, i.e. $F \in j(F)$. To prove the other direction we use Saito theorem (Theorem 2.16). It says that if $F \in j(F)$ then there exists a quasihomogeneous polynomial h and a coordinate change ϕ that maps h to F. Then the linear part of ϕ will map h to the quasihomogeneous part of F, which is g. Therefore, h and g are contact equivalent and hence F and f are contact equivalent. So the hypersurface H_F belongs to $V_{d,H}^{0,0}$ if and only if $F \in \langle F_{x_1}, \dots, F_{x_n} \rangle$.

We check that condition using a computer algebra algorithm (Algorithm 4.2). In this way we obtain a system of equations on $V_{d,H}^{0,0}$. In case $d = 3,\ n = 4$ the substratum germ consists of one point. We show that otherwise the system consists of a subsystem having a diagonal linear part, and one more equation with quadratic principle part of rank ≥ 3.

In the case of $\alpha_1 \geq 2\alpha_n$, there are hypersurfaces H_F in $V_{d,H}^{0,0}$ whose equations include some terms below or on the Newton polytope $\Delta(f)$. For every such polynomial F, we pass to new coordinates in which F has no terms below and on $\Delta(f)$, write equations on the coefficients of F in the new coordinates and express new coefficients through the old ones.

Again we check that the obtained system consists of a subsystem having diagonal linear part, and one more equation with quadratic principle part of rank ≥ 3. This is done in Section 4.5.

We show that in both cases the last equation lies in the ideal generated by elements that appear in its quadratic part. We deduce from this fact the smoothness of the singular locus.

4.2. Proof that $h^1(\mathcal{J}_{Z^{ea}(H,z)/\mathbb{P}^n}(d)) = 1$

Suppose, for convenience, $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n \geq 2$.

First, we pass to affine coordinates $x_i = \frac{t_i}{t_0}$. In these coordinates, H is given by local equation $f = \sum x_i^{\alpha_i} + \sum \lambda_i x_i^d$. It is a semiquasihomogeneous polynomial with non-degenerate quasihomogeneous part $g = \sum_{i=1}^n (1 + \delta_{\alpha_i,d}\lambda_i) x_i^{\alpha_i}$. It is easy to see that $f \in j(f)$ and $j(f) = j(g)$. Hence f and g have the same Tjurina ideal and Tjurina algebra, which also coincides with their Milnor algebras. By Mather–Yau theorem this implies that f and g are contact equivalent and hence belong to the same stratum.

The polynomial g is quasihomogeneous of type $(1/\alpha_1, \dots, 1/\alpha_n; 1)$ hence (H, z) is a quasihomogeneous hypersurface singularity. The Newton polytope of g is

$$\Delta(g) = \left\{ I \in \mathbb{Z}_{\geq 0}^n \,\middle|\, \sum_{j=1}^n \frac{I_j}{\alpha_j} = 1 \right\}.$$

We will now show that $V_{d,H}$ is non-T-smooth at H and $h^1(\mathcal{J}_{Z^{ea}(H,z)/\mathbb{P}^n}(d)) = 1$.

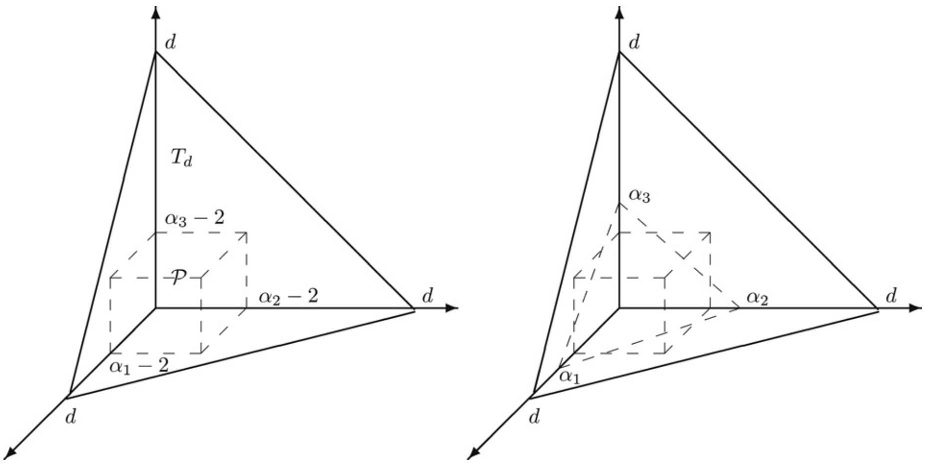

Fig. 1. Newton polytope.

The Tjurina algebra of f has a basis $\{x^I, I \in \mathcal{P}\}$, where $\mathcal{P}$ is the parallelepiped $\mathcal{P} = \{I \in \mathbb{Z}^n_{\geq 0} | I_j \leq \alpha_j - 2 \text{ for all } 0 \leq j \leq n\}$. So $\tau(H, z) = |\mathcal{P}| = \prod_{i=1}^n (\alpha_i - 1)$, where by $|\mathcal{P}|$ we denote the number of integer points in $\mathcal{P}$.

Hence $h^0(\mathcal{O}_{Z^{ea}(H,z)}) = \tau(H, z) = |\mathcal{P}|$,

$$H^0(\mathcal{J}_{Z^{ea}(H,z)/\mathbb{P}^n}(d)) = \left\{ \sum_{I \in T_d \setminus \mathcal{P}} a_I x^I \right\}$$

where T_d is the simplex $\{I \in \mathbb{Z}^n_{\geq 0} | |I| \leq d\}$ (see Fig. 1). This means that

$$h^0(\mathcal{J}_{Z^{ea}(H,z)/\mathbb{P}^n}(d)) = |T_d| - |T_d \cap \mathcal{P}|.$$

From the exact sequence

$$0 \to H^0(\mathcal{J}_{Z^{ea}(H,z)/\mathbb{P}^n}(d)) \to H^0(\mathcal{O}_{P^n}(d)) \to H^0(\mathcal{O}_{Z^{ea}(H,z)}) \to H^1(\mathcal{J}_{Z^{ea}(H,z)/\mathbb{P}^n}(d)) \to 0$$

we conclude that

$$\begin{aligned} h^1(\mathcal{J}_{Z^{ea}(H,z)/\mathbb{P}^n}(d)) &= h^0(\mathcal{J}_{Z^{ea}(H,z)/\mathbb{P}^n}(d)) - h^0(\mathcal{O}_{P^n}(d)) + h^0(\mathcal{O}_{Z^{ea}(H,z)}) \\ &= |T_d| - |T_d \cap \mathcal{P}| - |T_d| + |\mathcal{P}| = |\mathcal{P} \setminus T_d| = 1. \end{aligned}$$

The same argument shows that $h^1(\mathcal{J}_{Z^{ea}(H,z)/\mathbb{P}^n}(d+1)) = |\mathcal{P} \setminus T_{d+1}| = 0$. Thus by Theorem 2.21 the germ $V_{d+1,H}$ is T-smooth and the germ $V_{d,H}$ is non-T-smooth.

Because of minimal obstructedness ($h^1 = 1$), $V_{d,H}$ may be either non-smooth of expected codimension or smooth of non-expected codimension.

Now we would like to find out when it is non-smooth and when it has non-expected codimension. First we will pass to a more convenient substratum, which has the same geometric properties.

4.3. Switch to substratum and notations

First of all let us shift the singularity to the origin. Let S be the singularity type of (H, z). The family $V_d^U(S)$ is invariant under the action of affine transformations of $\mathbb{C}^n$. Consider the subgroup generated by translations. We switch to the section $V_d'(S)$ of $V_d^U(S)$ transversal to orbits of this group and given by the conditions that the singularity is in the origin.

In the same way, using the subgroup GL_n consisting of the linear coordinate changes, we want to reduce to the substratum $V_d^{0,0}(S)$ of $V_d'(S)$ consisting of all hypersurfaces H_F given by polynomials F which also do not include the monomials $x_i^{\alpha_i - 1} x_j$ for $i \neq j$, and include the monomials $x_i^{\alpha_i}$ with coefficient 1 if $\alpha_i \neq d$ or with coefficient $1 + \lambda_i$ if $\alpha_i = d$. For this purpose we will prove the following lemma.

Lemma 4.1. (i) *$T_H V_d'(S) = T_H V_d^{0,0}(S) \oplus T_H GL_n H$*

(ii) *$GL_n V_d^{0,0}(S) = V_d'(S)$ in a neighborhood of H and $GL_n H \cap V_d^{0,0}(S) = \{H\}$ in a neighborhood of H.*
The same is true for $V_{d+1}'(S)$:

(iii) *$T_H V_{d+1}'(S) = T_H V_{d+1}^{0,0}(S) \oplus T_H GL_n H$*

(iv) *$GL_n V_{d+1}^{0,0}(S) = V_{d+1}'(S)$ in a neighborhood of H and $GL_n H \cap V_{d+1}^{0,0}(S) = \{H\}$ in a neighborhood of H.*

Proof. (i): For any point $H_F \in V'_d(S)$ and any $0 \le i, j \le n$ denote by c_{ij} the coefficient of the monomial $x_i^{\alpha_i - 1} x_j$ in the polynomial $F - f$. The tangent space to $V_d^{0,0}(S)$ at H is given inside $T_H V'_d(S)$ by the equations $c_{ij} = 0$. On the other hand $T_H GL_n H = Span\{\alpha_i x_i^{\alpha_i - 1} x_j + \lambda_i d x_i^{d-1} x_j\}$. Hence $T_H V'_d(S) = T_H V_d^{0,0}(S) \oplus T_H GL_n H$.

(iii) is proven in the same way.

(iv) follows from (iii) since $V_{d+1}^{0,0}(S)$ is smooth (see Section 4.2).

(ii) follows from (iv) since GL_n preserves degree:

$$\begin{aligned} GL_n V_d^{0,0}(S) &= GL_n(V_{d+1}^{0,0}(S) \cap |\mathcal{O}_{\mathbb{P}^n}(d)|) = (GL_n V_{d+1}^{0,0}(S)) \cap |\mathcal{O}_{\mathbb{P}^n}(d)| \\ &= V'_{d+1}(S) \cap |\mathcal{O}_{\mathbb{P}^n}(d)| = V'_d(S). \end{aligned}$$

Here, $|\mathcal{O}_{\mathbb{P}^n}(d)| = |H|$ is the linear system of hypersurfaces of degree d. Also $GL_n H \cap V_d^{0,0}(S) \subset GL_n H \cap V_{d+1}^{0,0}(S) = \{H\}$ in a neighborhood of H. □

From this lemma we see that it is enough to prove our statement for the germ $V_{d,H}^{0,0}$ of $V_d^{0,0}(S)$ at H.

Consider now arbitrary hypersurface $H_F \in V_{d,H}^{0,0}$ given by a polynomial equation $F = 0$. Since H_F is obtained from H by a local analytic diffeomorphism and both have their only singularity at the origin, F has no terms of degree less than α_n.

So we will work with substratum germ $V_{d,H}^{0,0}$ of $V_{d,H}$ consisting of hypersurfaces given by polynomials of the form

$$\begin{aligned} F(x_1, \ldots, x_n) &= \sum_{i=1}^{n} x_i^{\alpha_i} + \sum_{i=1}^{n} \lambda_i x_i^d + \sum_{I \in \mathcal{D}} a_I x^I, \\ &\text{where } \mathcal{D} = \left(\{I \in \mathbb{Z}^n_{\ge 0} | \alpha_n \le |I| \le d\} \setminus \bigcup_{1 \le i \ne j \le n} \{(0, \ldots, 0, \alpha_i - 1, 0, \ldots, 0, \underset{j}{1}, 0, \ldots, 0)\} \right) \\ &\setminus \bigcup_{1 \le i \le n} \{(0, \ldots, 0, \alpha_i, 0, \ldots, 0)\}. \end{aligned} \tag{4.3.1}$$

For convenience, we introduce the following notations:

(a) let b_I be the coefficients of basis monomials above the Newton polytope $\Delta(f)$, i.e. $b_I := a_I$ for $I \in \mathcal{D}$ such that $I_j \le \alpha_j - 2$ for all j and $w(I) > 1$;

(b) let e_I be the coefficients of basis monomials below the Newton polytope $\Delta(f)$ but of degree at least α_n, i.e. $e_I := a_I$ for $I \in \mathcal{D}$ such that $I_j \le \alpha_j - 2$ for all j and $w(I) \le 1$;

(c) let $g_I := a_I$ for $I \in E$ where $E = \{I \in \mathcal{D} | I_j = \alpha_j - 1$ for some j, $I_k \le \alpha_k - 2$ for all $k \ne j$ and $I_k > 0$ for some $k \ne j\}$;

(d) let $u_I := a_I$ for $I \in \mathcal{D}$ such that ($I_j \ge \alpha_j$ for some j) or ($I_j = \alpha_j - 1$ and $I_k = \alpha_k - 1$ for some $k \ne j$);

(e) let $q_I := a_I$ for $I = (0, \ldots, 0, \alpha_j - 1, 0, \ldots, 0)$ for some j;

(f) for $I = (i_1, \ldots, i_{k-1}, \alpha_k - 1, i_{k+1}, \ldots, i_n) \in E$ denote

$$dual(I) := (\alpha_1 - 2 - i_1, \ldots, \alpha_{k-1} - 2 - i_{k-1}, \alpha_k - 1, \alpha_{k+1} - 2 - i_{k+1}, \ldots, \alpha_n - 2 - i_n).$$

Note that $dual(I)$ also lies in E and $dual(dual(I)) = I$.

Let $A = \mathbb{C}[a_I]$ be the algebra of polynomials generated by a_I, $I \in \mathcal{D}$. Let $m = \langle a_I \rangle$ be the maximal ideal in A generated by all a_I, $G = \langle g_I \rangle$ be the ideal in A generated by all g_I and $B = \langle b_I \rangle$ be the ideal in A generated by all b_I.

4.4. Proof of the theorem for the case $\alpha_1 < 2\alpha_n$

In this case the Newton polytope $\Delta(f)$ lies below the hyperplane $|I| = 2\alpha_n$.

We want to find out for which $\{a_I\}$ H_F lies in $V_{d,H}^{0,0}$. Our F does not include monomials x^I for I below and on the Newton polytope and satisfying $I_j = \alpha_j - 1$ for some j. Hence, by Corollary 2.18, in order to belong to our substratum, F should include no terms below and on the Newton polytope except of $x_i^{\alpha_i}$.

Let $F = f + f_1$ where f_1 is a polynomial which has no terms below and on $\Delta(f)$. We claim that F is contact equivalent to f if and only if $F \in j(F) = \langle F_{x_1}, \ldots, F_{x_n} \rangle$.

One direction is obvious: if they are equivalent then they have the same Milnor and Tjurina numbers and hence $\mu(F) = \tau(F)$, i.e. $F \in j(F)$. To prove the other direction we use Saito theorem (Theorem 2.16). It says that if $F \in j(F)$ then there exists a quasihomogeneous polynomial h and a coordinate change ϕ that maps h to F. Then the linear part of ϕ will map h to the quasihomogeneous part of F, which is g. Therefore, h and g are contact equivalent and hence F and f are contact equivalent. So the hypersurface H_F belongs to $V_{d,H}^{0,0}$ if and only if $F \in \langle F_{x_1}, \ldots, F_{x_n} \rangle$.

In order to check whether $F(x_1, \ldots, x_n) \in \langle F_{x_1}, \ldots, F_{x_n} \rangle$ we use the REDNFBUCHBERGER algorithm (Algorithm 2.43). We refer to a neighborhood of the origin, hence we consider $F(x_1, \ldots, x_n)$ and $\langle F_{x_1}, \ldots, F_{x_n} \rangle$ in the local ring $R = \mathbb{C}[x_1, \ldots, x_n]_{\langle x_1, \ldots, x_n \rangle}$. To compute in this ring, we define a local monomial ordering on $\mathbb{C}[x_1, \ldots, x_n]$ such that the ring associated to $\mathbb{C}[x_1, \ldots, x_n]$ and this ordering will be $\mathbb{C}[x_1, \ldots, x_n]_{\langle x_1, \ldots, x_n \rangle}$. We choose the negative weighted degree lexicographical ordering with $w = (1/\alpha_1, \ldots, 1/\alpha_n)$ (see Example 2.39, ordering (2)(iii)).

In general, the REDNFBUCHBERGER algorithm does not stop for local orderings. However, in our case we can stop it manually when the leading monomial of the tail is less than $x_1^{\alpha_1-2}\cdot\dots\cdot x_n^{\alpha_n-2}$. We are allowed to do that by Lemma 2.46, for $x_1^{\alpha_1-2}\cdot\dots\cdot x_n^{\alpha_n-2}$ is the highest corner of $\langle LM(F_{x_1}),\dots,LM(F_{x_n})\rangle = \langle x_1^{\alpha_1-1},\dots,x_n^{\alpha_n-1}\rangle$. Indeed, any monomial smaller than $x_1^{\alpha_1-2}\cdot\dots\cdot x_n^{\alpha_n-2}$ has degree of x_j bigger than or equal to α_j-1 for some j and hence lies in $\langle LM(F_{x_1}),\dots,LM(F_{x_n})\rangle$ and $x_1^{\alpha_1-2}\cdot\dots\cdot x_n^{\alpha_n-2}\notin\langle LM(F_{x_1}),\dots,LM(F_{x_n})\rangle$.

There is another explanation why we can stop the algorithm at this point. Consider $V_{d+1,H}$. It is smooth of expected codimension (see Section 4.2). Therefore $V^{0,0}_{d+1,H}$ is also smooth and has expected codimension which is equal to the number of basis elements which lie above the Newton polytope. Since we have exactly this number of independent equations on this stage, there will be no more equations. Also since $V^{0,0}_{d+1,H}$ is smooth, all the equations on it will have independent linear parts. When we return to $V^{0,0}_{d,H}$, the linear part of only one of them may vanish.

So we rewrite the algorithm in the following way:

Algorithm 4.2 (*Modified* REDNFBUCHBERGER *Algorithm*).

1. $p:=0, h:=F$;
2. while $(h\neq 0$ and $LM(h)\geq x_1^{\alpha_1-2}\cdot\dots\cdot x_n^{\alpha_n-2})$
   (a) while $(h\neq 0$ and $LM(h)\geq x_1^{\alpha_1-2}\cdot\dots\cdot x_n^{\alpha_n-2}$
   and exists i such that $LM(F_{x_i})$ divides $LM(h))$
   $\{h:=h-(LT(h)/LT(F_{x_i}))\cdot F_{x_i}\}$
   (b) if $(h\neq 0$ and $LM(h)\geq x_1^{\alpha_1-2}\cdot\dots\cdot x_n^{\alpha_n-2})$
   $\{p:=p+LT(h);$
   $h=tail(h)\}$;
3. return p;

As a result, we obtain the normal form

$$NF(F|\langle F_{x_1},\dots,F_{x_n}\rangle)=\sum R_I(a_J)x^I,$$

where x^I, $I_k\leq\alpha_k-2$, for all k are elements of the basis of algebra $\mathbb{C}[x_1,\dots,x_n]/\langle f_{x_1},\dots,f_{x_n}\rangle$ which lie above $\Delta(f)$ and $R_I(a_J)$ are polynomials in a_J. Hence F belongs to the ideal $\langle F_{x_1},\dots,F_{x_n}\rangle$ if and only if all coefficients $R_I(a_J)=0$.

Thus we obtain a system of equations on a_J:

$$R_I(a_J)=0. \tag{4.4.2}$$

Let us now analyze the R_I.

Lemma 4.3. *Denote* $\psi_I(g_J,u_K,b_L):=R_I-(1-w(I))b_I$ *for* $|I|\leq d$ *and* $\psi_I(g_J,u_K,b_L)=-R_I$ *for* $|I|=d+1$, *i.e.* $I=(\alpha_1-2,\dots,\alpha_n-2)$. *Then*

(i) *All* b_L *that appear in* ψ_I *satisfy* $w(L)<w(I)$.
(ii) *All* ψ_I *are polynomials from* G^2+Bm. *Recall that* $G=\langle g_I\rangle$ *and* $B=\langle b_I\rangle$.
(iii) $\psi_{(\alpha_1-2,\dots,\alpha_n-2)}-\sum_{J\in E}A_k\cdot g_J\cdot g_{dual(J)}\in Bm+m^3$ *where* A_k *are positive rational numbers.*

Proof. Let us trace the changes of the coefficients of h and p during the algorithm. Denote the coefficient of x^I in h by c_I. In the first step we eliminate the monomials $x_i^{\alpha_i}$ for all i. As a result we obtain $h:=F-\sum_{i=1}^n(1/\alpha_i)\cdot x_i\cdot F_{x_i}$. After this step $c_I=(1-w(I))a_I$. Note that c_I is non-zero iff a_I is non-zero, since $w(I)>1$ unless $a_I=0$.

Now let S be the coefficient of the leading monomial of h. If it is a basic monomial, we add the leading term of h to p and subtract it from h. Otherwise, $S_m\geq\alpha_m-1$ for some m and hence we can eliminate this term using F_{x_m}. After such elimination step the change of h is expressed by the formula

$$(h)^{new}=(h)^{old}-\frac{1-w(S)}{\alpha_m}a_Ix_1^{S_1}\cdot\dots\cdot x_m^{S_m-\alpha_m+1}\cdot\dots\cdot x_n^{S_n}F_{x_m}.$$

Hence the c_I after this step is

$$c_I^{new}=(1-w(I))a_I-\frac{I_m-S_m+\alpha_m}{\alpha_m}(1-w(S))a_S\cdot a_{I-S+J^m},$$

where $J^m:=(0,\dots 0,\alpha_m,0,\dots,0)$. If some coordinate of $I-S+J^m$ is negative, then $a_{I-S+J^m}=0$ and hence c_I did not change.

Suppose that I is a basic index. Let us show that after this step $c_I-(1-w(I))b_I\in G^2+Bm$. We know that $S_m\geq\alpha_m-1$. If $S_k>I_k$ for some $k\neq m$ then $(I-S+J^m)_k<0$ and hence c_I did not change. Hence we can assume $S_k\leq I_k$ for $k\neq m$. Let us consider several cases.

1. If $S_m \geq \alpha_m$ then $a_{I-S+J^m} \in B$ and hence $a_S \cdot a_{I-S+J^m} \in mB$.
2. If $S_m = \alpha_m - 1$ and $I_m < \alpha_m - 2$ then $a_{I-S+J^m} \in B$ and hence $a_S \cdot a_{I-S+J^m} \in mB$.
3. If $S_m = \alpha_m - 1$ and $I_m = \alpha_m - 2$ then $a_S \in G$ and $a_{I-S+J^m} \in G$ and hence $a_S \cdot a_{I-S+J^m} \in G^2$.

Let us now consider any elimination step of the algorithm at which we eliminate some x^S using F_{x_m} for some m. This is only possible if $S_m \geq \alpha_m - 1$. The change of h in this step is expressed by the formula

$$(h)^{new} = (h)^{old} - \frac{1}{\alpha_m} c_S^{old} x_1^{S_1} \cdot \dots \cdot x_m^{S_m-\alpha_m+1} \cdot \dots \cdot x_n^{S_n} F_{x_m}.$$

Hence the change of c_I in this step is expressed by

$$c_I^{new} = c_I^{old} - \frac{I_m - S_m + \alpha_m}{\alpha_m} c_S^{old} \cdot a_{I-S+J^m}. \tag{4.4.3}$$

For any basic index S, the R_S is equal to the coefficient of x^S in p after the termination of the algorithm.

Note that $w(I - S + J^m) = w(I) - w(S) + 1$, and recall that $a_J = 0$ if $w(J) < 1$. Hence c_I is influenced only if $w(S) < w(I)$ and $w(I - S + J^m) < w(I)$. This proves (i).

Let us now prove by induction that at every step of the algorithm, $c_I - (1 - w(I))b_I \in Bm + G^2$ for any basic coefficient I, and $c_I \in B + G$ for any $I \in E$. After the first step of the algorithm these statements clearly hold. We suppose that they hold before a step in which we eliminate x^S using F_{x_m}, and show that they still hold after this step.

First let I be a basic index.

Let us consider several cases.

1. If $S_m > I_m + 1$ then $a_{I-S+J^m} \in B$ and hence $a_{I-S+J^m} \cdot c_S^{old} \in Bm$.
2. If $S_m \leq I_m$ then S is a basic index and by induction hypothesis $c_S^{old} \in B + G^2$ and hence $a_{I-S+J^m} \cdot c_S^{old} \in Bm + G^2$.
3. If $S_m = I_m + 1$ and $I_m < \alpha_m - 2$ then S is a basic index and by induction hypothesis $c_S^{old} \in B + G^2$ and hence $a_{I-S+J^m} \cdot c_S^{old} \in Bm + G^2$.
4. If $S_m = I_m + 1$ and $I_m = \alpha_m - 2$ then $S \in E$ and $I - S + J^m \in E$ and hence by induction hypothesis $a_{I-S+J^m} \cdot c_S^{old} \in (B + G)(B + G) \subset Bm + G^2$.

Now let $I \in E$. Then there exists k such that $I_k = \alpha_k - 1$ and $I_j \leq \alpha_j - 2$ for $j \neq k$. We know that $S_m \geq \alpha_m - 1$, $S_p \leq I_p$ for $p \neq m$ and $I_m - S_m + \alpha_m \geq 0$.

Let us consider several cases.

1. If $S_m > I_m + 1$ then $a_{I-S+J^m} \in B + G$.
2. If $S_m < I_m + 1$ then $c_S^{old} \in B + G$.
3. If $S_m = I_m + 1$ and $S_k > 0$ then $a_{I-S+J^m} \in B + G$.
4. If $S_m = I_m + 1$ and $S_k = 0$ then $c_S^{old} \in B + G$.

This proves (ii).

Substituting $I = (\alpha_1 - 2, \dots, \alpha_n - 2)$ in 4.4.3 and arguing in the same way we obtain (iii). $\square$

Corollary 4.4. *The system (4.4.2) is equivalent to the system consisting of equations*

$$b_I = \widetilde{\psi}_I(g_J, u_K) \quad \text{for } I \neq (\alpha_1 - 2, \dots, \alpha_n - 2), \tag{4.4.4}$$

where $\widetilde{\psi}_I \in G^2$ are polynomials in g_J and u_K, and the last equation

$$R := \sum_{I \in E} A_k \cdot g_I \cdot g_{dual(I)} + \Theta(g_J, u_K) = 0, \tag{4.4.5}$$

where A_k are positive rational numbers and $\Theta(g_J, u_K)$ is a polynomial from G^2m.

Now we see that our substratum germ is isomorphic to the germ at 0 of the affine variety given in the affine space with coordinates $\{g_J, u_K\}$ by the last equation (4.4.5). The quadratic part Q of this equation is a non-degenerate quadratic form in $\{g_I | I \in E\}$. We will now show that unless $n = 4$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 3$, the quadratic form Q has rank at least 3. For this it is enough to find 3 points $I \in E$. Indeed for any such point $I = (j_1, \dots, j_{k-1}, \alpha_k - 1, j_{k+1}, \dots, j_n)$ the dual point $I' = (\alpha_1 - 2 - j_1, \dots, \alpha_{k-1} - 2 - j_{k-1}, \alpha_k - 1, \alpha_{k+1} - 2 - j_{k+1}, \dots, \alpha_n - 2 - j_n)$ also satisfies these conditions. Consider several cases separately:

(0) For $n = 4$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 3$, $d = 12 - 9 = 3$, there are no g_I. Moreover, $\mathcal{D}$ is empty. So $V_{d,H}^{0,0}$ consists of one point. Hence $V_{d,H}$ is also a smooth algebraic variety of dimension $n^2 - 1 + n = 19$. The expected dimension is $\binom{n+d}{n} - 1 - \prod_{i=1}^{n}(\alpha_i - 1) = \binom{7}{4} - 1 - 2^4 = 18$. So, $V_{d,H}$ has non-expected codimension. Till the end of the proof we assume that this is not the case.

(1) $\alpha_1 \geq 4$. In this case we have the points $(1, 1, 0, \ldots, 0, \alpha_n - 1)$ and $(2, 0, 0, \ldots, 0, \alpha_n - 1)$ in I. Since $d \geq \alpha_1, \alpha_2 \geq 3$.
(1.1) $\alpha_2 = 3$. In this case $n \geq 4$ and $\alpha_3 \geq 3$ since $d \geq \alpha_1$. Hence we have one more point $(1, 0, 1, 0, \ldots, 0, \alpha_n - 1)$ in I.
(1.2) $\alpha_2 \geq 4$. In this case we have point $(0, 2, 0, \ldots, 0, \alpha_n - 1)$ in I.

(2) $\alpha_1 = 3$. In this case $n \geq 4$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 3$ for the same reason. So unless $n = 4$ we have points $(1, 1, 0, \ldots, 0, \alpha_n - 1)$, $(0, 1, 1, 0, \ldots, 0, \alpha_n - 1)$ and $(1, 0, 1, 0, \ldots, 0, \alpha_n - 1)$ in I.

Note that for all mentioned points in cases 1 and 2, $|I| = \alpha_n + 1$ which is less than or equal to d. Note also that the weight of all these points is at least $\frac{2}{\alpha_1} + \frac{\alpha_n - 1}{\alpha_n} > \frac{2}{2\alpha_n} + \frac{\alpha_n - 1}{\alpha_n} = 1$.

Since the quadratic form is non-degenerate of rank ≥ 3, our substratum germ is a reduced irreducible non-smooth variety of expected codimension and of order two.

Now we are going to prove that the singular locus Y of our substratum germ coincides with the germ X_0 at H of the affine subspace $X = Z(G)$. Since Eq. (4.4.5) lies in G^2, all its first order partial derivatives lie in G, and hence the singular locus includes X_0. Let Z be the variety given by the equations

$$\frac{\partial R}{\partial g_I} = 0$$

for $I \in E$. Since the linear part of this system of equations is non-degenerate, the germ Z_0 of Z at f is smooth and hence irreducible. Clearly $Y \subset Z_0$ and hence $X_0 \subset Z_0$. They have the same dimension and Z_0 is irreducible hence $X_0 = Z_0$ which implies $X_0 = Y$.

So our substratum germ is a reduced irreducible non-smooth variety of expected codimension which has a smooth singular locus.

4.5. Proof of the theorem for the case $\alpha_1 \geq 2\alpha_n$.

We will make now a series of coordinate changes so that in new coordinates F (see (4.3.1)) will not have terms of the form $x_i^{\alpha_i - 1} x^J$ lying below and on the Newton polytope, except $x_i^{\alpha_i}$. The first one will be $x_i \mapsto x_i$ for $i < n$ and $x_n \mapsto x_n - \frac{1}{\alpha_n} a_{\langle 2,0,\ldots,0,\alpha_n - 1\rangle} x_1^2$. The coefficients of the polynomial F in the new coordinates are expressed through the coefficients in the old coordinates by the formula

$$a^{new}_{\langle i_1,\ldots,i_n\rangle} = a_{\langle i_1,\ldots,i_n\rangle} + \sum_{s=1}^{[i_1/2]} (-1)^s \binom{k+s}{s} a_{\langle i_1 - 2s, i_2, \ldots, i_{n-1}, i_n + s\rangle} \left(\frac{a_{\langle 2,0,\ldots,0,\alpha_n - 1\rangle}}{\alpha_n} \right)^s. \tag{4.5.6}$$

After this coordinate change the coefficient $a^{new}_{\langle 2,0,\ldots,0,\alpha_n - 1\rangle}$ will vanish.

Note that in the new coordinates F might get terms of degree more than d. In fact, those terms might have very high degrees. However, by finite determinacy theorem (Theorem 2.8) f is $d+3$-determined. This means that after each coordinate change we may (and will) erase all the terms of F of degree more than $d + 3$.

In the same way, we get rid of all the coefficients of the form $a_{\langle j_1,\ldots,j_{n-1},\alpha_n - 1\rangle}$ in ascending order of the corresponding monomials. We recall that the monomial ordering we use is the negative weighted degree lexicographical ordering with $w = (1/\alpha_1, \ldots, 1/\alpha_n)$ (see Example 2.39). The coordinate change indexed $J = (j_1, \ldots, j_{n-1}, \alpha_n - 1)$ will be $x_i \mapsto x_i$ for $i < n$ and $x_n \mapsto x_n - \frac{1}{\alpha_n} a_{\langle j_1,\ldots,j_{n-1},\alpha_n - 1\rangle} x_1^{j_1} \cdot \cdots \cdot x_{n-1}^{j_{n-1}}$. The coefficients of the polynomial F in the new coordinates are expressed through the coefficients in the old coordinates by the formula

$$a^{new}_{\langle i_1,\ldots,i_n\rangle} = a^{prev}_{\langle i_1,\ldots,i_n\rangle} + \sum_{s=1}^{\min\{[i_l/j_l] | j_l \neq 0, 0 \leq l \leq n-1\}} (-1)^s \binom{k+s}{s} a^{prev}_{\langle i_1 - sj_1, \ldots, i_{n-1} - sj_{n-1}, i_n + s\rangle} \left(\frac{a^{prev}_J}{\alpha_n} \right)^s. \tag{4.5.7}$$

As can be seen from formula (4.5.7), g_I can be affected only during a coordinate change whose index does not exceed I by any coordinate, and hence has lower weight. Thus after all these coordinate changes, all coefficients $g_{\langle j_1,\ldots,j_{n-1},\alpha_n - 1\rangle}$ will be zero. Denote by a'_I the coefficient of the monomial x^I after the coordinate changes. It is easy to see that $a'_I = a_I + \phi'_I(a_J)$ where $\phi'_I(a_J) \in mG$. Note that

$$a'_{\langle \alpha_1 - 2,\ldots,\alpha_n - 2\rangle} = \sum A_J \cdot g_{\langle j_1,\ldots,j_{n-1},\alpha_n - 1\rangle} \cdot g_{\langle \alpha_1 - 2 - j_1,\ldots,\alpha_{n-1} - 2 - j_{n-1},\alpha_n - 1\rangle} + \Phi'(a_I)$$

where $A_J \in \mathbb{R} \setminus \{0\}$ for all J, and $\Phi'(a_I) \in mG^2$.

We continue with the coordinate changes and, in the same way as before, we get rid of the coefficients of the form $g'_{\langle j_1,\ldots,j_{n-2},\alpha_{n-1} - 1,0\rangle}$ starting from $g'_{\langle 1,0,\ldots,0,\alpha_{n-1} - 1,0\rangle}$. This time it might be non-zero since $\phi'_{\langle 1,0,\ldots,0,\alpha_{n-1} - 1,0\rangle}(a_J)$ may be non-zero. However, it lies in the ideal G.

We do the same for $g'_{\langle j_1,\ldots,j_{k-1},\alpha_k - 1,0,\ldots,0\rangle}$ for all $k \geq 2$ in the descending order of k.

Denote by $\widetilde{a}_I$ the coefficient of the monomial x^I after all these coordinate changes. Again, $\widetilde{a}_I = a_I + \phi_I(a_J)$ where $\phi_I(a_J) \in mG$ and

$$\widetilde{a}_{\langle \alpha_1 - 2,\ldots,\alpha_n - 2\rangle} = \sum_{k=1}^{n} \sum A_J \cdot g_{\langle j_1,\ldots,j_{k-1},\alpha_k - 1,0,\ldots,0\rangle} \cdot g_{\langle \alpha_1 - 2 - j_1,\ldots,\alpha_{k-1} - 2 - j_{k-1},\alpha_k - 1,\alpha_{k+1} - 2,\ldots,\alpha_n - 2\rangle} + \Phi(a_I)$$

where $A_J \neq 0$ for all J and $\Phi(a_J) \in mG^2$.

Now we want to find out for which $\{a_I\}$ H_F lies in $V^{0,0}_{d,H}$. Let $\widetilde{F}(x_1,\ldots,x_n)$ be the polynomial F in new coordinates. $\widetilde{F}$ does not include monomials x^I for I below and on the Newton polytope and satisfying $I_j=\alpha_j-1$ for some j. Hence, by Corollary 2.18, in order to belong to our substratum $\widetilde{F}$ should include no terms below and on the Newton polytope except of $x_i^{\alpha_i}$.

In other words, we have the following equations on $\widetilde{a}_I$:

$$\widetilde{e}_I=0 \tag{4.5.8}$$
$$\widetilde{q}_I=0. \tag{4.5.9}$$

As explained in the previous subsection (and also Section 4.1), $F\overset{c}{\sim}f$ iff $M_F\simeq T_F$ i.e. $F(x_1,\ldots,x_n)\in\langle F_{x_1},\ldots,F_{x_n}\rangle$.

As in case one, in order to check that we use the redNFBuchberger algorithm with the negative weighted degree lexicographical ordering with $w=(1/\alpha_1,\ldots,1/\alpha_n)$ (see Example 2.39).

Again, we can stop the algorithm manually when the leading monomial of the tail is less than $x_1^{\alpha_1-2}\cdot\cdots\cdot x_n^{\alpha_n-2}$ (see Algorithm 4.2).

As a result, we obtain the normal form

$$NF(\widetilde{F}|\langle\widetilde{F}_{x_1},\ldots,\widetilde{F}_{x_n}\rangle)=\sum R_I(\widetilde{a}_J)x^I,$$

where x^I are elements of the basis of algebra $\mathbb{C}[x_1,\ldots,x_n]\,/_{\langle f_{x_1},\ldots,f_{x_n}\rangle}$ which lie above $\Delta(f)$ and $R_I(\widetilde{a}_J)$ are polynomials in $\widetilde{a}_J$. Hence $\widetilde{F}$ belongs to the ideal $\langle\widetilde{F}_{x_1},\ldots,\widetilde{F}_{x_n}\rangle$ if and only if all the coefficients $R_I(\widetilde{a}_J)$ are 0.

Thus we obtain a system of equations on $\widetilde{a}_J$:

$$R_I(\widetilde{a}_J)=0 \tag{4.5.10}$$

where R_I has the form $R_I=\widetilde{b}_I\prod_{i=1}^n\widetilde{u}_i^{\gamma_{I,i}}+\psi_I(\widetilde{g}_J,\widetilde{u}_K,\widetilde{b}_L)$, where all L have weight less than that of I. Thus we can, as in case 1, express $\widetilde{b}_I$ and obtain an equivalent system of equations:

$$\widetilde{b}_I=\frac{\widetilde{\psi}_I(\widetilde{g}_J,\widetilde{u}_K)}{\prod\limits_{i=1}^n\widetilde{u}_i^{\gamma_{I,i}}} \tag{4.5.11}$$

where $\widetilde{\psi}_I\in\widetilde{G}^2$ for $\widetilde{G}=\langle\widetilde{g}_J\rangle$ and $\widetilde{u}_i=\widetilde{a}_{\langle 0,\ldots,0,\alpha_i,0,\ldots,0\rangle}$.

The number of equations in system (4.5.11) is equal to the number of basis coefficients above the Newton polytope $\Delta(f)$.

Now we express new coefficients through the old ones. Recall that $\widetilde{u}_i=1+\phi_{\langle 0,\ldots,0,\alpha_i,0,\ldots,0\rangle}(a_J)$, $\widetilde{a}_I=a_I+\phi_I(a_J)$ for other $I\in\mathcal{D}$, and $\widetilde{a}_I=\phi_I(a_J)$ for $I\notin\mathcal{D}$, where $\phi_I(a_J)\in mG$.

Consider system (4.5.8). After substituting old coefficients it will be $e_I=-\phi_I(a_J)$, where $\phi_I\in m^2$. The same with the system (4.5.9).

Consider system (4.5.11) except the last equation i.e. the equation on $\widetilde{b}_{\langle\alpha_1-2,\ldots,\alpha_n-2\rangle}$. After substituting the old coefficients and multiplying by denominators it will become

$$b_I=\widetilde{\psi}_I(a_J), \tag{4.5.12}$$

where $\widetilde{\psi}_I\in m^2$.

The last equation is of particular interest. After passing to the old coordinates and multiplying by the denominator its right-hand side will become

$$\widetilde{\psi}_{\langle\alpha_1-2,\ldots,\alpha_n-2\rangle}(\widetilde{g}_J,\widetilde{u}_K)=\sum_{J\in S}A_J\cdot g_J\cdot g_{dual(J)}+\Theta'(a_K),$$

where $S\subset E$ is the set of indices of terms that did not vanish during the coordinate changes, $\Theta'(a_K)$ lies in mG^2 and the A_J are positive rational numbers. The left-hand side will be

$$\widetilde{b}_{\langle\alpha_1-2,\ldots,\alpha_n-2\rangle}\cdot\prod_{i=1}^n\widetilde{u}_i^{\gamma_{I,i}}=(0+\phi_{\langle\alpha_1-2,\ldots,\alpha_n-2\rangle})\left(\prod_{i=1}^n(1+\phi_i)^{\gamma_{I,i}}\right)=\sum_{J\in T}A_J\cdot g_J\cdot g_{dual(J)}+\Psi(a_K),$$

where $T=\{J\in E|J\notin S$, and $dual(J)\notin S\}$, $\Psi(a_K)\in mG^2$ and A_J are positive rational numbers. Moving the right-hand side to the left we obtain the equation

$$P:=\sum_{J\in E}B_J\cdot g_J\cdot g_{dual(J)}+\Theta(a_K)=0, \tag{4.5.13}$$

where $\Theta(a_K)\in mG^2$, the B_J are non-zero rational numbers and $sign(B_J)=sign(B_{dual(J)})$.

So our substratum germ $V^{0,0}_{d,H}$ is isomorphic to the germ at the origin of the affine variety given in the affine space with coordinates $\{a_I\}$ by the system of equations

$$\begin{aligned} &e_I + \phi_I(a_J) = 0 \\ &q_I + \phi_I(a_J) = 0 \\ &b_I - \widetilde{\psi}_I(a_J) = 0 \end{aligned} \tag{4.5.14}$$

and the last Eq. (4.5.13).

We have to prove that $V^{0,0}_{d,H}$ is a reduced irreducible non-smooth variety of expected codimension which has a smooth singular locus.

The system (4.5.14) has a diagonal linear part. Hence all the equations in it are independent and the variety W, defined by it, is smooth at 0. The last Eq. (4.5.13) does not depend on the preceding ones and does not have a linear part. Hence $V^{0,0}_{d,H}$ is non-smooth of expected codimension.

The quadratic part of Eq. (4.5.13) is

$$Q := \sum_{J\in E} B_J \cdot g_J \cdot g_{dual(J)}.$$

Since the B_J are non-zero, it is a quadratic non-degenerate form of rank r equal to the number of integer points in E.

Now we will show that $r \geq 3$. We do that by exhibiting 3 integer points in E: $(2, 0, \dots, 0, \alpha_n - 1)$, $(1, 1, 0, \dots, 0, \alpha_n - 1)$ and $(2, 1, 0, \dots, 0, \alpha_n - 1)$. Since $\alpha_1 \leq d$ we have $\alpha_2 \geq 3$, and hence $I_j \leq \alpha_j - 2$ for $j < n$ for all the 3 points. Hence, it is enough to show that their degrees do not exceed d. Indeed, their maximal degree is $\alpha_n + 2 \leq 2\alpha_n \leq \alpha_1 \leq d$. So $r \geq 3$. Hence the variety defined by the principle part of our system of equations on $V^{0,0}_{d,H}$ is reduced and irreducible and hence our germ is reduced and irreducible. Since the quadratic form Q is non-zero $V^{0,0}_{d,H}$ has order two.

Now we are going to prove that the singular locus Y of our substratum germ coincides with the germ X_0 at the origin of the affine subvariety X given in W by the ideal G.

Since Eq. (4.5.13) lies in G^2, all its first order partial derivatives lie in G, and hence X_0 lies in the singular locus. Consider the jacobian of the system obtained by merging system (4.5.14) with the last Eq. (4.5.13). Fix $I \in E$. Let $M(I)$ be the minor of the jacobian given by columns that include partial derivatives by g_I and all e_J, b_J and q_J. The linear part of $M(I)$ is $A_I g_{dual(I)}$. Let Z_0 be the germ at the origin of subvariety of W given by the system of equations $M(I) = 0$ for all $I \in E$. Since the linear part of this system is diagonal, Z_0 is irreducible and has the same dimension as X_0. As $X_0 \subset Y \subset Z_0$, it implies $X_0 = Y = Z_0$.

So the substratum germ $V^{0,0}_{d,H}$ is reduced irreducible non-smooth variety of expected codimension which has a smooth singular locus.

4.6. Proof that the sectional singularity type is A_1 for all cases

The sectional singularity type of a variety germ with smooth singular locus is the singularity type of transversal intersection of the singular locus with a linear space.

Consider the linear subspace L spanned by e_I, q_I, b_I and g_I. We have seen that it is transversal to the singular locus of $V^{0,0}_{d,H}$ in the affine space with coordinates $\{a_I\}$. Hence the sectional singularity type of $V^{0,0}_{d,H}$ is the singularity type of the scheme theoretic intersection $L \cap V^{0,0}_{d,H}$. $L \cap V^{0,0}_{d,H}$ is given in L by a system of Eqs. (4.5.14) with linear part non-degenerate in e_I, q_I and b_I and one more equation with quadratic principle part which is not degenerate in the variables g_I. Hence the singularity type of $L \cap V^{0,0}_{d,H}$ is A_1. □

5. Proof of the theorem on stability properties of obstructed equianalytic families

This section is dedicated to the proof of Theorem 3.3.

5.1. The structure of the proof

First of all, we pass to local coordinates $x_i = \frac{t_i}{t_0}$, and denote $f_0(x_1, \dots, x_n) := F_0(1, x_1, \dots, x_n)$.

It is known (Lemma 2.12) that polynomials $g_1 = f_1 + x^2_{n+1}(1 + h_1)$ and $g_2 = f_2 + x^2_{n+1}(1 + h_2)$, where $f_i \in \mathbb{C}\{x_1, \dots, x_n\}$ and $h_i \in m \subset \mathbb{C}\{x_1, \dots, x_{n+1}\}$, are contact equivalent if and only if the f_i are contact equivalent.

For any polynomial $F = f_0 + x^2_{n+1} + \sum a_{I,j} x^I x^j_{n+1} \in m^2 \subset \mathbb{C}\{x_1, \dots, x_{n+1}\}$ there exists an analytic diffeomorphism of $(\mathbb{C}^{n+1}, 0)$ that brings F to the form $f_0 + x^2_{n+1}(1 + h)$. One can write explicit formulas for this diffeomorphism that depend polynomially on the coefficients of F (see Section 5.2). Now we build a map of germs $\phi : (|\mathcal{O}_{\mathbb{P}^{n+1}}(d)|, W^1) \to (|\mathcal{O}_{\mathbb{P}^n}(\tau + 1)|, H)$ in the following way: take the equation of the hypersurface which includes x^2_{n+1} with coefficient 1, bring it to the form $f_0 + x^2_{n+1}(1 + h)$ (using the above diffeomorphism) and take the hypersurface defined by the $(\tau + 1)$-jet of f_0. By Lemma 2.12 and the finite determinacy theorem (Theorem 2.8), the preimage of $V_{\tau+1,H}$ will be V^U_{d,W^1}.

In Section 5.2 we obtain an explicit formula for ϕ (formula (5.2.16)). The linear part of ϕ is the identity and the quadratic part depends only on the coefficients of monomials which include x_{n+1} with degree 1.

Since $V_{\tau+1,H}$ is T-smooth, it is locally defined by a system (*) of τ equations having non-degenerate linear part. Using ϕ we obtain a system (**) of equations on V^U_{d,W^1}. This is done in Section 5.3. Since the linear part of ϕ is the identity, the linear part of (**) is obtained from the linear part (*) by substituting zeros for the coefficients of monomials of degree more than d. Since the tangent space to $V_{d,H}$ has codimension $\tau - h^1$, the rank of the linear part of (**) will be $\tau - h^1$. This proves statement (a).

The quadratic part of (**) is the sum of two systems. First is obtained from quadratic part of (*) by substituting zeros for coefficients of monomials of degree more than d. The second summand is obtained from the linear part of (*) by composing it with the quadratic part of ϕ and then substituting zeros for the coefficients of the monomials of degree more than d. We show that if $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(2d-2)) = 0$ then the second summand is non-zero.

In case $h^1 = 1$, (**) has $\tau - 1$ equations with independent linear parts and one more equation with quadratic principal part which is independent of the linear parts of the previous equations. In Section 5.5 we show that it has rank at least 2 for $m \geq \max\{1, 5-d\}$ and at least 3 for $m \geq \max\{1, 6-d\}$. This finishes the proof of statement (e).

In Section 5.4 we analyze the system (**) in the general case and prove statement (b).

In Sections 5.6 and 5.7 we prove statements (c) and (d) using Lemmas 2.25 and 2.26 on the Castelnuovo function.

5.2. *Coordinate changes*

First of all, we pass to local coordinates $x_i = \frac{t_i}{t_0}$, and denote $f_0(x_1, \ldots, x_n) := F_0(1, x_1, \ldots, x_n)$.

Let $F(x_1, \ldots, x_{n+1}) = f_0 + x_{n+1}^2 + \sum a_{I,j} x^I x_{n+1}^j$ be a polynomial of degree $\leq d$. We want to find equations on the coefficients $a_{I,j}$ that F should satisfy in order to define a hypersurface that belongs to V^U_{d,W^1}. First of all we may suppose that the coefficient of monomial x_{n+1}^2 is one.

Now we want to get rid of the coefficients of monomials $x^I x_{n+1}$. In order to do that we make the following series of coordinate changes: $x_i \mapsto x_i$ for $1 \leq i \leq n$, $x_{n+1} \mapsto x_{n+1} - 1/2 a_{I,1} x^I$. After this coordinate change the new coefficients will be expressed from the previous ones by the following formula:

$$a^{(I)}_{J,k} = \sum_{s=0}^{\max_l\{[J_l/I_l]\}} \binom{s+k}{k} \left(-\frac{1}{2}\right)^s a^{prev}_{J-sI,s+k} (a^{prev}_{I,1})^s. \tag{5.2.15}$$

We start from I having smallest degree, and continue in increasing order of degrees. All the coefficients of the form $a_{J,1}$ influenced during the coordinate change indexed I have degree more than degree of $a_{I,1}$ (except $a_{I,1}$ which vanishes). Hence we can continue making such coordinate changes until F has no coefficients $a_{I,1}$ of degree less than $\tau + 2$. Denote the final coefficients by $a'_{I,j}$. From the formula (5.2.15) we see that they can be expressed through the original coefficients by

$$a'_{I,0} = a_{I,0} - 1/2 \sum a_{J,1} a_{I-J,1} - 1/4 a^2_{I/2,1} + \phi_I(a_{K,k}) \tag{5.2.16}$$

where $\phi_I \in \langle a_J \rangle^3$.

5.3. *Equations defining V^U_{d,W^1} and proof of statement (a)*

We want to find equations on the coefficients $a_{I,j}$ that F should satisfy in order to be analytically equivalent to f_0. By the finite determinacy theorem (Theorem 2.8) we can suppose that all coefficients of F of degree more than $\tau + 1$ are zero. Then we can present F in the form

$$F = f_0 + \sum a'_{I,0} x^I + x_{n+1}^2 \left(1 + \sum a'_{I,j} x^I x_{n+1}^{j-2}\right).$$

Then, by Lemma 2.12 $F \overset{c}{\sim} f_0$ if and only if $f_0 \overset{c}{\sim} f_0 + \sum a'_{I,0} x^I$. Therefore, in order for F to lie in $V^U_{\tau+1,W^1}$, $f_0 + \sum a'_{I,0} x^I$ should lie in $V_{\tau+1,H}$. Hence in order to find the needed equations on $a_{I,j}$ we have to take equations that define $V_{\tau+1,H}$ inside the linear system $|\mathcal{O}_{\mathbb{P}^n}(\tau+1)|$, substitute there $a'_{I,0}$ and then using formulas (5.2.16) to express $a'_{I,0}$ through the old coefficients $a_{I,j}$.

Let us now investigate the equations on $V_{\tau+1,H}$. Since $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(2d-2)) = 0$, both $V_{\tau+1,H}$ and $V_{2d-2,H}$ are smooth and have expected codimension τ. That means that there is a system of τ local equations on $V_{\tau+1,H}$ with a non-degenerate linear part which remains non-degenerate after substituting zeroes in place of all coefficients of monomials of degree bigger than $2d-2$.

Replacing the system of equations by an equivalent one, we can suppose that the linear part of this system has echelon form such that every row has a special element a_{L_i} which appears only in the linear part of this row and satisfies $|L_i| \leq 2d-2$.

When we substitute in these equations zeros instead of the coefficients of degree $> d$ the rank of the linear part of the system drops by h^1. That means that h^1 rows of the linear part include only coefficients of degrees $> d$. Renumber the rows so that those will be the last h^1 equations. Denote the index of the special element of equation number i by L_i.

As can be seen from formula (5.2.16), the linear part of $a'_{I,0}$ is $a_{I,0}$, and the quadratic part is $-1/2\sum a_{J,1}a_{I-J,1} - 1/4a^2_{I/2,1}$. Hence the linear parts of the last h^1 equations remain 0 when we express the new coefficients through the old ones.

Therefore the stratum germ V^U_{d,W^1} is locally defined by a system of τ equations, of which only $\tau - h^1$ have a linear part, which is non-degenerate. Hence $h^1(\mathcal{J}_{Z^{ea}(W^1)/\mathbb{P}^{n+1}}(d)) = h^1$. By induction the same is true for all V^U_{d,W^m} for any $m \geq 1$. By Lemma 2.12 $\deg Z^{ea}(W^m) = \tau$. Statement (a) is now proven.

5.4. Proof of statement (b)

We have to show that V^U_{d,W^m} has an irreducible component which is reduced of expected dimension for $m \geq h^1 + 1$. Consider W^2. It is obtained from W^1 by the same procedure of adding a square. Hence we know the form of the equations on V^U_{d,W^2}. The system consisting of the first $\tau - h^1$ equations has a non-degenerate linear part, and it is the same linear part as in the equations on $V_{d,H}$. The last h^1 equations start from quadratic forms.

Let us analyze the quadratic form of equation number $\tau - h^1 + j$. Let w^j_0 be the quadratic form appearing in the equation number $\tau - h^1 + j$ on $V_{d,H}$ (it may be zero). The quadratic part of the corresponding equation on V^U_{d,W^1} is $w^j_0 + w^j_1$ where w^j_1 depends only on coefficients $a_{I,1}$, and its form was described above. Hence the quadratic part of the corresponding equation on V^U_{d,W^2} is $w^j_0 + w^j_1 + w^j_2$ where w^j_1 depends only on coefficients $a_{I,1,0}$, and w^j_2 is the same as w^j_1 but with variables $a_{I,0,1}$.

Continuing in the same way we see that for general m, the quadratic part of equation number $\tau - h^1 + j$ will be $w^j_0(a_{\langle I,0,\dots,0\rangle}) + w^j_1(a_{\langle I,1,0,\dots,0\rangle}) + w^j_2(a_{\langle I,0,1,0,\dots,0\rangle}) + \cdots + w^j_m(a_{\langle I,0,\dots,0,1\rangle})$. For convenience, we denote $0^m := (0,\dots,0) \in \mathbb{Z}^m$ and $e_i := (0,\dots,0,1,0,\dots,0) \in \mathbb{Z}^m$.

Let X^m be the variety defined by the principle part of the system of equations on V^U_{d,W^m}. In order to show that V^U_{d,W^m} has a reduced component of expected dimension, it is enough to show that X^m has a reduced component of expected dimension. To do that we will prove that there is a point in X^m in which the jacobian of the principle part of the system of equations on V^U_{d,W^m} has maximal rank.

For every $1 \leq j \leq h^1$ we choose J_j and K_j (not necessary different) such that $J_j + K_j = L_{\tau-h^1+j}$. Here, L_i is the index of the special element of the linear part of equation number i on $V_{\tau,H}$. Let A be the linear subspace spanned by all $\{a_{K_j,e_j}\}_{j=1}^{h^1}$ and by all $\{a_{I,e_m}\}_{1\leq|I|\leq d-1}$.

Consider the minor defined by derivatives with respect to $a_{L_i,0^m}$ for $1 \leq i \leq \tau - h^1$ and to a_{J_j,e_j} for $1 \leq j \leq h^1$. We claim that the restriction of this minor on the linear subspace A is $C \cdot a_{K_1,e_1} \cdot \ldots \cdot a_{K_{h^1},e_{h^1}}$ where C is non-zero real number. This minor is a determinant of a block matrix in which the first block is the identity matrix $Id_{(\tau-h^1)\times(\tau-h^1)}$. It is left to show that the second block is a diagonal matrix with $C_j \cdot a_{K_j,e_j}$ on the diagonal.

Indeed, consider, for example, equation number $\tau - h^1 + 1$. The derivative of w^1_0 w.r.t. a_{J_1,e_1} is 0 since w^1_0 does not depend on it at all. The same is true about $w^1_{\geq 2}$. The derivative of w^1_1 w.r.t. a_{J_1,e_1} contains a_{K_1,e_1} with non-zero coefficient C_1. It may also contain other a_{I,e_1}, but they vanish on our subspace A. Consider now the derivative of equation number $\tau - h^1 + j$ (for $j > 1$) w.r.t. a_{J_1,e_1}. Again, the derivatives of w_0 and $w^j_{\geq 2}$ is zero. The derivative of w^j_1 does not contain a_{K_1,e_1} since the coefficient $a_{J_1+K_1} = a_{L_1}$ does not appear in the linear part of equation number $\tau - h^1 + j$ on $V_{\tau,H}$. Hence this derivative also vanishes on the subspace A spanned by all $\{a_{K_j,e_j}\}_{j=1}^{h^1}$ and by all $\{a_{I,e_m}\}_{1\leq|I|\leq d-1}$.

By the same reason, the restriction to A of the derivative of equation number $\tau - h^1 + j$ with respect to a_{J_i,e_i} is equal to 0 if $j \neq i$ and $C_i \cdot a_{K_i,e_i}$ if $j = i$.

Hence on A the minor is a determinant of the diagonal matrix with entries $(C_j a_{K_j,e_j})$ and hence the minor is $Ca_{K_1,e_1} \cdot \ldots \cdot a_{K_{h^1},e_{h^1}}$. Clearly, every neighborhood of 0 in $X^m \cap A$ contains a point in which $Ca_{K_1,e_1} \cdot \ldots \cdot a_{K_{h^1},e_{h^1}} \neq 0$. At this point, a maximal minor of the jacobian is non-zero, hence the jacobian has maximal rank. Therefore the tangent space to X^m at this point has codimension τ and hence X^m has a reduced component of expected codimension τ. Hence V^U_{d,W^m} also has a reduced component of expected codimension for $m \geq h^1 + 1$.

5.5. Proof of statement (e)

Now we have $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(d)) = 1$. By Lemma 2.25, (d) on the Castelnuovo function $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(d+1)) = 0$. Hence from Section 5.3 V^U_{d,W^1} is defined by a system of $\tau - 1$ equations with non-degenerate linear part and one more equation without linear part (since $h^1 = 1$). Hence V^U_{d,W^1} is non-smooth.

In order to show that V^U_{d,W^m} has expected codimension (respectively is reduced, respectively irreducible) it is enough to show that the scheme defined by the principle parts of the above system of equations has expected codimension (respectively is reduced, respectively irreducible).

For $1 \le i \le \tau - 1$, we can just express a_{L_i} from equation number i. Therefore, this scheme is isomorphic to the scheme X^m defined by the quadratic part w of the last equation in the affine space of coefficients $\{a_{I,J} | 2 \le |I+J| \le d\} \setminus \{a_{L_i,0}\}_{i=1}^{\tau-1}$. Since w is non-zero, X^m has expected codimension. X^m is reduced if $rank(w) \ge 2$ and irreducible if $rank(w) \ge 3$.

As in Section 5.4, $w = w_0(a_{\langle I,0,\dots,0\rangle}) + w_1(a_{\langle I,1,0,\dots,0\rangle}) + w_2(a_{\langle I,0,1,0,\dots,0\rangle}) + \dots + w_m(a_{\langle I,0,\dots,0,1\rangle})$ where $rank(w_1) = \dots = rank(w_m) \ge 1$. Hence X^m is reduced for $m \ge 2$ and irreducible for $m \ge 3$. It is left to deal with $m = 1$ and $m = 2$. It is enough to show that for $d = 4$ we have $rank(w_1) \ge 2$ and for $d \ge 5$ we have $rank(w_1) \ge 3$.

We start with the case $d \ge 5$. The quadratic form w_1 can be expressed by

$$w_1 = \sum_{(I,J)\text{ s.t. }|I+J|=d+1} C_{I,J} a_{I,e_1} a_{J,e_1}.$$

Note that if $I + J = L_\tau$ then $C_{I,J} \ne 0$. Since $|L_\tau| = d + 1$, it can be presented as $L_\tau = I_1 + I_2$ where $|I_1| = 2$ and $|I_2| = d - 1$ and also as $L_\tau = I_3 + I_4$, where $|I_3| = 3$ and $|I_4| = d - 2$. Note that I_1, I_2 and I_3 are different since they have different degrees since $d \ge 5$. If $d = 5$, it is possible that $I_3 = I_4$. Consider the reduction w_1' of w_1 on the (3 or 4 dimensional) linear subspace spanned by $a_{I_1,e_1}, a_{I_2,e_1}, a_{I_3,e_1}, a_{I_4,e_1}$. The only sums of pairs of those multiindices whose degrees are $d + 1$ are $I_1 + I_2$ and $I_3 + I_4$. Hence $w_1' = C_{L_\tau}(a_{I_1,e_1} a_{I_2,e_1} + a_{I_3,e_1} a_{I_4,e_1})$ which has rank 3 or 4. Hence the rank of w_1 is at least 3. Hence V^U_{d,W^m} is reduced and irreducible for $m \ge 1$.

Consider now $d = 4$. As in the previous case, we can find I_1 and I_2 such that $|I_1| = 2$, $|I_2| = d - 1 = 3$ and $I_1 + I_2 = L_\tau$. By reducing w_1 on the subspace spanned by a_{I_1,e_1} and a_{I_2,e_1} we see that the rank of w_1 is at least 2. Hence V^U_{d,W^m} is reduced for $m \ge 1$ and irreducible for $m \ge 2$.

5.6. *Proof of statement (c)*

By Lemma 2.25(d), it is enough to show that $\mathcal{C}_{Z^{ea}(H)}(2d-3) = 0$. Let $Z(j(f_0))$ be zero-dimensional scheme defined by the jacobian $j(f_0)$ where f_0 is the local equation of H. Then $Z^{ea}(H)$ is a subscheme of $Z(j(f_0))$ and hence by Lemma 2.25(e) it is enough to show that $\mathcal{C}_{Z(j(f_0))}(2d-3) = 0$. Let $C_1 = Z(\frac{\partial f_0}{\partial x_1})$ and $C_2 = Z(\frac{\partial f_0}{\partial x_2})$ and let Z' be the complete intersection $C_1 \cap C_2$. Then $Z(j(f_0))$ is a subscheme of Z' and hence by Lemma 2.25(e) it is enough to show that $\mathcal{C}_{Z'}(2d-3) = 0$. Let k be the degree of $\frac{\partial f_0}{\partial x_1}$ and l be the degree of $\frac{\partial f_0}{\partial x_2}$. By Lemma 2.26, $\mathcal{C}_{Z'}(l + k - 1) = 1 - 1 = 0$. Since $k, l \le d - 1$ we obtain $\mathcal{C}_{Z'}(2d - 3) = 0$. Therefore $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^2}(2d-4)) = 0$ and hence by (b) the germs V^U_{d,W^m} have a reduced component of expected dimension for $m \ge h^1 + 1$.

5.7. *Proof of statement (d)*

Suppose $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(2d-2)) > 0$. Then $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(l)) > 0$ for all $d \le l \le 2d - 2$. Hence by Lemma 2.25,(d) the Castelnuovo function $C_{Z^{ea}(H)}(l) > 0$. Hence $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(d)) - h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(2d-2)) = \sum_{l=d+1}^{2d-2} C_{Z^{ea}(H)}(l) \ge d - 2$. Hence $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(2d-2)) \le h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(d)) - (d-2) \le 0$. So $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^n}(2d-2)) = 0$ and by (b) the germs V^U_{d,W^m} have a reduced component of expected dimension for $m \ge h^1 + 1$. □

Remark 5.1. If one wishes W^m to be unisingular, they can be defined by $F_0 + \sum_{j=1}^m t_{n+j}^2 t_0^{d-2} + \sum_{j=1}^m \lambda_j t_{n+j}^d$, where the λ_j are generic. The same proof shows that an analogous theorem will hold about the equisingular family germ V_{d,W^m}.

Remark 5.2. Statement (c) can be strengthened as follows: let H be a projective plane curve of degree d which is not a union of d lines through the same point. Then $h^1(\mathcal{J}_{Z^{ea}(H)/\mathbb{P}^2}(2d-5)) = 0$.

For a proof see [23], Remark 3.4.4.

Acknowledgements

First of all we would like to thank Prof. Eugenii Shustin for supervising this work.

We would also like to thank Prof. Joseph Bernstein for teaching us basics of Algebraic Geometry, Prof. Gert-Martin Greuel for fruitful discussions and Dr. Dmitry Kerner for useful remarks.

Finally, we wish to thank the referee for very careful reading and useful remarks.

The first-named author was supported by Hermann-Minkowski-Minerva Center for Geometry at the Tel-Aviv University, by the grant no. 465/04 from the Israel Science Foundation. The second-named author was supported by GIF grant no. 861/05 and ISF grant no. 1438/06.

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