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The similarity problem for J-nonnegative Sturm–Liouville operators

Illya M. Karabash [a,c], Aleksey S. Kostenko [b,*], Mark M. Malamud [c]

[a] *Department of Mathematics and Statistics, University of Calgary, 2500 University Drive NW, Calgary T2N 1N4, Alberta, Canada*

[b] *Department of Nonlinear Analysis, Institute of Applied Mathematics and Mechanics, NAS of Ukraine, R. Luxemburg Str., 74, Donetsk 83114, Ukraine*

[c] *Department of Partial Differential Equations, Institute of Applied Mathematics and Mechanics, NAS of Ukraine, R. Luxemburg Str., 74, Donetsk 83114, Ukraine*

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Abstract

Sufficient conditions for the similarity of the operator $A := \frac{1}{r(x)}(-\frac{d^2}{dx^2} + q(x))$ with an indefinite weight $r(x) = (\operatorname{sgn} x)|r(x)|$ are obtained. These conditions are formulated in terms of Titchmarsh–Weyl m-coefficients. Sufficient conditions for the regularity of the critical points 0 and ∞ of J-nonnegative Sturm–Liouville operators are also obtained. This result is exploited to prove the regularity of 0 for various classes of Sturm–Liouville operators. This implies the similarity of the considered operators to self-adjoint ones. In particular, in the case $r(x) = \operatorname{sgn} x$ and $q \in L^1(\mathbb{R}, (1 + |x|)\,dx)$, we prove that A is similar to a self-adjoint operator if and only if A is J-nonnegative. The latter condition on q is sharp, i.e., we construct $q \in \bigcap_{\gamma<1} L^1(\mathbb{R}, (1 + |x|)^\gamma\,dx)$ such that A is J-nonnegative with the singular critical point 0. Hence A is not similar to a self-adjoint operator. For periodic and infinite-zone potentials, we show that J-positivity is sufficient for the similarity of A to a self-adjoint operator. In the case $q \equiv 0$, we prove the regularity of the critical point 0 for a wide class of weights r. This yields new results for "forward–backward" diffusion equations.

Keywords: J-self-adjoint operator; Sturm–Liouville operator; Titchmarsh–Weyl m-function; Similarity; Spectral function of J-nonnegative operators; Critical points

* Corresponding author.

E-mail addresses: karabashi@yahoo.com (I.M. Karabash), duzer80@mail.ru (A.S. Kostenko), mmm@telenet.dn.ua (M.M. Malamud).

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1. Introduction

Consider the Sturm–Liouville equation

$$-y''(x) + q(x)y(x) = \lambda r(x)y(x), \quad x \in \mathbb{R}, \tag{1.1}$$

with a real potential $q \in L^1_{\mathrm{loc}}(\mathbb{R})$ and an indefinite weight $r \in L^1_{\mathrm{loc}}(\mathbb{R})$. We assume that $|r(x)| > 0$ a.e. on $\mathbb{R}$ and r has only one turning point $x = 0$, i.e., $r(x) = (\operatorname{sgn} x)|r(x)|$.

Consider the operator $L = \frac{1}{|r(x)|}(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x))$ defined on its maximal domain $\mathfrak{D}$ in the Hilbert space $L^2(\mathbb{R}, |r|\,\mathrm{d}x)$. If $L = L^*$ ($L \geqslant 0$), then the operator

$$A := \frac{(\operatorname{sgn} x)}{|r(x)|}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x)\right), \quad \operatorname{dom}(A) = \mathfrak{D}, \tag{1.2}$$

associated with (1.1) is called *J-self-adjoint* (*J-nonnegative*). This means that A is self-adjoint (nonnegative) with respect to the indefinite inner product $[f, g] := (Jf, g) = \int_{\mathbb{R}} f\bar{g}r\,\mathrm{d}x$, where the operator J is defined by

$$(Jf)(x) = (\operatorname{sgn} x)f(x), \quad f \in L^2\big(\mathbb{R}, \big|r(x)\big|\,\mathrm{d}x\big). \tag{1.3}$$

In this paper, we will always assume that $L = L^*$, i.e.,

$$\textit{the differential expression } (1.1) \textit{ is limit point at } +\infty \textit{ and } -\infty. \tag{1.4}$$

So the operator A is J-self-adjoint. However, it is easy to see that A is nonself-adjoint in $L^2(\mathbb{R}, |r|\,\mathrm{d}x)$ (see Section 2.1).

The main problem we are concerned with is *the similarity* of a J-nonnegative operator (1.2) to a self-adjoint operator. Recall that closed operators T_1 and T_2 in a Hilbert space $\mathfrak{H}$ are called *similar* if there exist a bounded operator S with the bounded inverse S^{-1} in $\mathfrak{H}$ such that $S\operatorname{dom}(T_1) = \operatorname{dom}(T_2)$ and $T_2 = ST_1S^{-1}$.

Ordinary and partial differential operators with indefinite weights have intensively been investigated during the last two decades (see [1,6,7,9,10,14,15,18–20,22–25,33,36,38,41–44,51–54, 57,60] and references therein).

The similarity of the operator A to a self-adjoint one is essential for the theory of forward–backward parabolic equations arising in certain physical models and in the theory of random processes (see [6,13,21,25–27,37] and references therein). Theorem 1.3 of this paper yields new results for "forward–backward" diffusion equations (see e.g. [37, Section 5.3]).

Spectral theory of $\mathcal{J}$-nonnegative operators was developed by M.G. Krein and H. Langer [29,46] (see Section 2.3). If the resolvent set $\rho(\mathcal{A})$ of a $\mathcal{J}$-nonnegative operator $\mathcal{A}$ is nonempty, then the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ is real. Moreover, $\mathcal{A}$ has a *spectral function* $E_{\mathcal{A}}(\cdot)$ with properties similar to that of a spectral function of a self-adjoint operator. The main difference is the occurrence of *critical points*. Significantly different behavior of the spectral function $E_{\mathcal{A}}(\cdot)$ occurs at a *singular critical point* in any neighborhood of which $E_{\mathcal{A}}(\cdot)$ is unbounded. A critical point is *regular* if it is not singular. It should be stressed that only 0 and ∞ may be critical points of $\mathcal{J}$-nonnegative operators. Furthermore, $\mathcal{A}$ is similar to a self-adjoint operator if and only if 0 and ∞ are not singular and $\ker\mathcal{A} = \ker\mathcal{A}^2$ (see Proposition 2.5).

If the operator A has a discrete spectrum, the similarity of A to a self-adjoint operator is equivalent to the *Riesz basis property* of eigenvectors. For this case, R. Beals [6] showed that the eigenfunctions of Sturm–Liouville problems of type (1.1) form a Riesz basis if $r(x)$ behaves like $(\operatorname{sgn} x)|x|^{\beta}$, $\beta > -1/2$, at $x = 0$. Improved versions of Beals' condition were provided in [14,22,51–54,57]. In [14,22], differential operators with nonempty essential spectrum were considered and the regularity of the critical point ∞ was proved for a wide class of indefinite weight functions. For J-nonnegative operators of the form (1.2), the result of B. Ćurgus and H. Langer [14, Section 3] is formulated in Proposition 2.7. In particular, it implies the regularity of ∞ if there exist constants $\delta > 0$, $\beta_{\pm} > -1$, and positive functions $p_+ \in C^1[0,\delta]$, $p_- \in C^1[-\delta,0]$ such that

$$r(x) = (\operatorname{sgn} x)\, p_{\pm}(x)|x|^{\beta_{\pm}}, \quad \pm x \in (0,\delta). \tag{1.5}$$

The existence of Sturm–Liouville operators of type (1.2) with the singular critical point ∞ was established by H. Volkmer [57] in 1996. Corresponding examples were constructed later (see [1,9,22,23,51] and references therein).

It turned out that the question of regularity of 0 is more complicated. Several abstract similarity criteria may be found in [3,11,12,34,48,50,56], but it is not easy to apply them to operators of the form (1.2). First results of this type were obtained for the operators $(\operatorname{sgn} x)|x|^{-\alpha}\frac{d^2}{dx^2}$, $\alpha > -1$, by B. Ćurgus, B. Najman, and A. Fleige (see [15] for the case $\alpha = 0$, and [24] for arbitrary $\alpha > -1$). Their approach was based on the abstract regularity criterion [12, Theorem 3.2]. Another approach based on the resolvent criterion of similarity (see Theorem 3.1) was used by the authors of the present paper [35,36,41–43] as well as by M.M. Faddeev and R.G. Shterenberg [18,19]. Namely, in [35,36], the result of [15] was reproved (see also [34]). It was shown in [18] that *if* $r(x) = \operatorname{sgn} x$, $\int_{\mathbb{R}}(1+x^2)|q(x)|\,dx < \infty$ *and* $\sigma(A) \subset \mathbb{R}$, *then* A *is similar to a self-adjoint operator*. The case when $q \equiv 0$ and $r(x) \approx \pm|x|^{\alpha_{\pm}}$, $\alpha_{\pm} > -1$, as $x \to \pm\infty$, was considered in [19,43]. A complete analysis for the case of a finite-zone potential was done in [42].

Our main aim is to present a simple and efficient regularity condition for the critical point 0 of operator (1.2) and then to apply it to various classes of potentials (decaying, periodic, and quasi-periodic) as well as to the case when $r(\cdot)$ is nontrivial. In particular, we show that restrictions imposed in [18,19] are superfluous (see Remarks 4.12, 7.4) and give simple proofs for [24, Theorem 2.7] and [42, Corollary 7.4].

Our method is based on two ideas of [38,42]. Namely, the resolvent criterion (Theorem 3.1) was used in [42] to reduce the similarity problem to a two weight norm inequality for the Hilbert transform and to obtain similarity conditions in terms of Titchmarsh–Weyl m-coefficients. In particular, [42, Theorem 5.9] states that A is similar to a self-adjoint operator if

$$\sup_{\lambda \in \mathbb{C}_+} \left| \frac{M_+(\lambda) + M_-(\lambda)}{M_+(\lambda) - M_-(\lambda)} \right| < \infty, \tag{1.6}$$

where $M_{\pm}(\lambda)$ are the Titchmarsh–Weyl m-coefficients associated with (1.2) on $\mathbb{R}_{\pm}$ (explicit definitions are given in Section 2.2).

In this paper we show that a weaker form of (1.6) (see Theorem 3.4) remains still sufficient for similarity, and obtain also its local version using the Krein space approach of [38]. Namely, if the operator A is J-nonnegative and

$$\sup_{\lambda\in\Omega_R^0}\left|\frac{M_+(\lambda)+M_-(\lambda)-c}{M_+(\lambda)-M_-(\lambda)}\right|<\infty,\qquad \Omega_R^0:=\bigl\{\lambda\in\mathbb{C}_+\colon\ |\lambda|<R\bigr\},\tag{1.7}$$

for certain constants $R>0$ and $c\in\mathbb{R}$, then 0 is not a singular critical point of A. Combining conditions (1.7) and Proposition 2.7, we obtain all (sufficient) similarity results of this paper. However the verification of (1.7) requires deep analysis of the m-coefficients.

Condition (1.6) is not necessary [42, Remark 8.1]. Generally, it is violated for operators considered in Sections 6 and 4, thought (1.7) can be applied (we do not know whether (1.7) is necessary). Note that the spectral analysis of the finite-zone case [42, Theorem 7.2] was based on the similarity criterion (Theorem 3.1) and Muckenhoupt weights rather than on condition (1.6). The proof of [42, Theorem 7.2] does not require J-nonnegativity of operators, but it is quite complicated and it is difficult to extend this proof to the operators considered in Sections 6 and 7.

It was proved in [38] that a condition slightly weaker than (1.7) is necessary for the similarity. Also, its local version was given (see Theorem 3.6). This result was used to show that the critical point 0 of operator A may be singular even if $q=0$ (a corresponding example was constructed). On the other hand, it was proved that there exists a continuous potential $q\in L^2(\mathbb{R})$ such that the operator $(\operatorname{sgn}x)(-\mathrm{d}^2/\mathrm{d}x^2+q)$ is J-nonnegative and 0 is its singular critical point. *The second aim of this paper is to present an explicit potential with the above property* (see Theorem 5.2).

The paper is organized as follows.

In Section 2, we collect necessary definitions and statements from the spectral theory of Sturm–Liouville operators and from the spectral theory of $\mathcal{J}$-nonnegative operators in Krein spaces.

The local regularity condition (1.7) is obtained in Section 3.

In Section 4, we investigate the J-self-adjoint operator A with $r(x)=\operatorname{sgn}x$ and q satisfying

$$\int_{\mathbb{R}}\bigl(1+|x|\bigr)\bigl|q(x)\bigr|\,\mathrm{d}x<\infty.\tag{1.8}$$

For such operators, we obtain the following criterion.

Theorem 1.1. *Let A be an operator of the form* $(\operatorname{sgn}x)(-\mathrm{d}^2/\mathrm{d}x^2+q(x))$. *If the potential q satisfies* (1.8)*, then the following statements are equivalent*:

(i) *A is similar to a self-adjoint operator,*
(ii) *A is J-nonnegative* (*i.e., $L\geqslant 0$*)*,*
(iii) *the spectrum of A is real.*

Under condition (1.8), $\sigma(L)\cap(-\infty,0)$ may be nonempty but is finite. For this case, we provide a complete spectral analysis of the operator A. Namely, it is shown that $\sigma_{\mathrm{ess}}(A)=\mathbb{R}$, A has no real eigenvalues, and the discrete spectrum $\sigma_{\mathrm{disc}}(A)$ consists of a finite number of nonreal eigenvalues; we use results of [14] and [42] to describe their algebraic and geometric multiplicities both in terms of definitizing polynomials and in terms of Titchmarsh–Weyl m-coefficients.

In Section 5, it is shown that Theorem 1.1 is sharp in the sense that condition (1.8) cannot be weaken to $q \in L^1(\mathbb{R}, (1+|x|)^\gamma\, \mathrm{d}x)$ with $\gamma < 1$. Actually, we construct a potential q_0 such that

(i) $q_0(x) \approx 2(1+|x|)^{-2}$ *as* $|x| \to \infty$,
(ii) *the operator* $A = (\operatorname{sgn} x)(-\mathrm{d}^2/\mathrm{d}x^2 + q_0(x))$ *is* J*-nonnegative*,
(iii) 0 *is a singular critical point of* A.

Note that if $r(x) = (\operatorname{sgn} x)|r(x)|$, the regularity of the critical point ∞ of a J-nonnegative operator of the form (1.2) depends only on local behavior of the weight r in a neighborhood of $x = 0$ (see [53, Theorem 4.1]). It appears that the latter is not true for the critical point 0. We show that the regularity of the critical point 0 depends not only on behavior of the weight r at ∞ (see [39, Example 1]) but also on local behavior of the potential q. This gives an answer to a one question posed by B. Ćurgus (see Section 5.2).

In Section 6, condition (1.7) is applied to operators with periodic potentials.

Theorem 1.2. *Assume that the potential* $q \in L^1_{\mathrm{loc}}(\mathbb{R})$ *is* $\mathcal{T}$*-periodic,* $q(x+\mathcal{T}) = q(x)$ *a.e.,* $\mathcal{T} > 0$*. If the operator* $L = -\mathrm{d}^2/\mathrm{d}x^2 + q(x)$ *is nonnegative, then the operator* $A = (\operatorname{sgn} x)L$ *is similar to a self-adjoint operator.*

This theorem can easily be extended to a more general class of Sturm–Liouville operators with periodic coefficients (see Remark 6.2). Also, a similar result is obtained for the class of infinite-zone potentials. This class includes smooth periodic potentials. Generally, infinite-zone potentials are almost-periodic [47]. For J-nonnegative operators with finite-zone potentials, the similarity to a self-adjoint operator was obtained in [42, Corollary 7.4]. We present a simple proof for this result (see Section 6.2).

In Section 7, the following theorem is proved.

Theorem 1.3. *Let* $q \equiv 0$ *and* $r(x) = \pm p(x)|x|^{\alpha_\pm}$, $x \in \mathbb{R}_\pm$, *where* $\alpha_\pm > -1$ *are constants and the function* p *is positive a.e. on* $\mathbb{R}$*. Assume also that*

$$\pm \int_{\pm 1}^{\pm\infty} |x|^{\alpha_\pm/2} \big| p(x) - c_\pm \big|\, \mathrm{d}x < \infty, \tag{1.9}$$

with certain constants $c_\pm > 0$*. Then*:

(i) 0 *is a regular critical point of the operator* $A = -\frac{(\operatorname{sgn} x)}{|r(x)|}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$;
(ii) *if the weight* r *also satisfies the assumptions of Proposition* 2.7(i)*, then the operator* A *is similar to a self-adjoint one.*

Note that the results of A. Fleige, B. Najman [24, Theorem 2.7] and M.M. Faddeev, R.G. Shterenberg [19, Theorems 3 and 7] are particular cases of Theorem 1.3. Moreover, we give a short proof of [24, Theorem 2.7].

Some results of the present paper were announced without proofs in brief communications [41,44]. Preliminary version of this paper was published as a preprint [40].

Notation. Let T be a linear operator in a Hilbert space $\mathfrak{H}$. In what follows, $\mathrm{dom}(T)$, $\ker(T)$, $\mathrm{ran}(T)$ are the domain, kernel, range of T, respectively; $\sigma(T)$ and $\rho(T)$ denote the spectrum and the resolvent set of T; $R_T(\lambda) := (T - \lambda I)^{-1}$, $\lambda \in \rho(T)$, is the resolvent of T; $\sigma_{\mathrm{p}}(T)$ stands for the set of eigenvalues; the discrete spectrum $\sigma_{\mathrm{disc}}(T)$ is the set of isolated eigenvalues of finite algebraic multiplicity; $\sigma_{\mathrm{ess}}(T) := \sigma(T) \setminus \sigma_{\mathrm{disc}}(T)$ is the essential spectrum.

We put $\mathbb{C}\pm := \{\lambda \in \mathbb{C}\colon\ \pm\operatorname{Im}\lambda > 0\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := [0, +\infty)$, $\mathbb{R}_- := (-\infty, 0]$. Denote by $\chi_{\mathcal{S}}(\cdot)$ the indicator function of a set $\mathcal{S} \subset \mathbb{R}$, and $\chi_\pm(t) := \chi_{\mathbb{R}_\pm}(t)$. We write $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ ($\in AC_{\mathrm{loc}}(\mathbb{R})$) if the function f is Lebesgue integrable (absolutely continuous) on every bounded interval in $\mathbb{R}$; $f(x) \asymp g(x)$ $(x \to x_0)$ if both f/g and g/f are bounded functions in a certain neighborhood of x_0; $f(x) \approx g(x)$ $(x \to x_0)$ means that $\lim_{x\to x_0} f(x)/g(x) = 1$. We write $f(x) = O(g(x))$ $(f(x) = o(g(x)))$ as $x \to x_0$ if $f(x) = h(x)g(x)$ and $h(x)$ is bounded in a certain neighborhood of x_0 (resp., $\lim_{x\to x_0} h(x) = 0$).

2. Preliminaries

2.1. Differential operators

Consider the differential expressions

$$\ell[y] := \frac{1}{|r|}(-y'' + qy) \quad \text{and} \quad a[y] := \frac{1}{r}(-y'' + qy), \tag{2.1}$$

assuming that $q, r \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $xr(x) > 0$ for a.a. $x \in \mathbb{R}$. Let $\mathfrak{D}$ be the maximal linear manifold in $L^2(\mathbb{R}, |r(x)|\,\mathrm{d}x)$ on which $\ell[\cdot]$ and $a[\cdot]$ have a natural meaning:

$$\mathfrak{D} := \bigl\{ f \in L^2\bigl(\mathbb{R}, \bigl|r(x)\bigr|\,\mathrm{d}x\bigr)\colon\ f, f' \in AC_{\mathrm{loc}}(\mathbb{R}),\ \ell[f] \in L^2\bigl(\mathbb{R}, \bigl|r(x)\bigr|\,\mathrm{d}x\bigr) \bigr\}. \tag{2.2}$$

Define the operators L and A by

$$\mathrm{dom}(L) = \mathrm{dom}(A) = \mathfrak{D}, \quad Lf = \ell[f] \quad \text{and} \quad Af = a[f].$$

The operators A and L are closed in $L^2(\mathbb{R}, |r(x)|\,\mathrm{d}x)$. In the sequel, (1.4) is supposed, i.e., $L = L^*$. It is clear that $A = JL$, where $J^* = J^{-1} = J$ is defined by (1.3). Thus, the operator A is J-self-adjoint. But A is nonself-adjoint since $A^* = LJ$ and $\mathrm{dom}(A^*) = J\mathfrak{D} \neq \mathrm{dom}(A)$.

It is obvious that the following restrictions of the operators L and A

$$\begin{gathered} L_{\min} := L \upharpoonright \mathfrak{D}_{\min}, \qquad A_{\min} := A \upharpoonright \mathfrak{D}_{\min}, \\ \mathfrak{D}_{\min} := \bigl\{ f \in \mathfrak{D}\colon\ f(0) = f'(0) = 0 \bigr\} \end{gathered} \tag{2.3}$$

are closed densely defined symmetric operators with equal deficiency indices $(2, 2)$. By $\mathfrak{D}^*_{\min}$ we denote the domain of the adjoint operator $L^*_{\min}$ of $L_{\min}$. Note that $\mathfrak{D}_{\min} = \mathfrak{D} \cap J\mathfrak{D}$. This implies $\mathrm{dom}(A^*_{\min}) = \mathrm{dom}(L^*_{\min}) = \mathfrak{D}^*_{\min}$ and $A_{\min} = JL_{\min}$ (see e.g. [42]). The extensions A_0 and L_0 defined by

$$\begin{gathered} A_0 := A^*_{\min} \upharpoonright \mathfrak{D}_0, \qquad L_0 := L^*_{\min} \upharpoonright \mathfrak{D}_0, \\ \mathfrak{D}_0 := \bigl\{ f \in \mathfrak{D}^*_{\min}\colon\ f'(+0) = f'(-0) = 0 \bigr\}, \end{gathered} \tag{2.4}$$

are self-adjoint operators and $A_0 = JL_0 = L_0J$.

2.2. Titchmarsh–Weyl m-coefficients

Let $c(x,\lambda)$ and $s(x,\lambda)$ denote solutions of the initial-value problems

$$-y''(x)+q(x)y(x)=\lambda\left|r(x)\right|y(x),\quad x\in\mathbb{R}, \tag{2.5}$$

$$c(0,\lambda)=s'(0,\lambda)=1,\qquad c'(0,\lambda)=s(0,\lambda)=0. \tag{2.6}$$

Since Eq. (2.5) is limit-point at $+\infty$, there exists a unique holomorphic function $m_+(\cdot):\mathbb{C}\setminus\mathbb{R}\to\mathbb{C}$, such that the solution $s(x,\lambda)-m_+(\lambda)c(x,\lambda)$ belongs to $L^2(\mathbb{R}_+,|r(x)|\,\mathrm{d}x)$ (see e.g. [55]). Similarly, the limit point case at $-\infty$ yields the fact that there exists a unique holomorphic function $m_-(\cdot):\mathbb{C}\setminus\mathbb{R}\to\mathbb{C}$ such that $s(x,\lambda)+m_-(\lambda)c(x,\lambda)\in L^2(\mathbb{R}_-,|r(x)|\,\mathrm{d}x)$. If $\lambda\in\mathbb{C}\setminus\mathbb{R}$ and $f_\pm(\cdot,\lambda)$ are nontrivial $L^2(\mathbb{R}_\pm,|r|\,\mathrm{d}x)$-solutions of Eq. (2.5) (which are unique up to a multiplicative constant), then

$$m_+(\lambda)=-\frac{f_+(+0,\lambda)}{f'_+(+0,\lambda)},\qquad m_-(\lambda)=\frac{f_-(-0,\lambda)}{f'_-(-0,\lambda)},\quad \lambda\notin\mathbb{R}. \tag{2.7}$$

The functions $f_\pm(\cdot,\lambda)$ and $m_\pm(\cdot)$ are called *the Weyl solutions* and *the Titchmarsh–Weyl m-coefficients* (*or Titchmarsh–Weyl functions*) *for* (2.5) on $\mathbb{R}_\pm$, respectively. We put

$$M_\pm(\lambda):=\pm m_\pm(\pm\lambda),\qquad \psi_\pm(x,\lambda)=\bigl(s(x,\pm\lambda)-M_\pm(\lambda)c(x,\pm\lambda)\bigr)\chi_\pm(x). \tag{2.8}$$

It is easily seen that $a[\psi_\pm(x,\lambda)]=\lambda\psi_\pm(x,\lambda)$, where $a[\cdot]$ is defined by (2.1). By definition of $m_\pm$, $\psi_\pm(\cdot,\lambda)\in L^2(\mathbb{R},|r(x)|\,\mathrm{d}x)$ for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$. The functions $M_\pm(\cdot)$ are said to be *the Titchmarsh–Weyl m-coefficients for Eq.* (1.1) *on* $\mathbb{R}_\pm$ (associated with the Neumann boundary condition $y'(\pm 0)=0$).

It is known (see e.g. [55]) that the functions $\psi_\pm$ and $M_\pm$ are connected by

$$\int\limits_{\mathbb{R}_\pm}\left|\psi_\pm(x,\lambda)\right|^2\left|r(x)\right|\mathrm{d}x=\frac{\operatorname{Im}M_\pm(\lambda)}{\operatorname{Im}\lambda}\quad\text{for all }\lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{2.9}$$

This implies that M_+ and M_- (as well as m_+ and m_-) belong to the class (R), i.e., *they are holomorphic in* $\mathbb{C}\setminus\mathbb{R}$, $M_\pm(\bar{\lambda})=\overline{M_\pm(\lambda)}$, *and* $\operatorname{Im}\lambda\cdot\operatorname{Im}M_\pm(\lambda)\geqslant 0$ *for* $\lambda\in\mathbb{C}_+\cup\mathbb{C}_-$.

Definition 2.1. (See [31].) An R-function M belongs to

(i) the Krein–Stieltjes class (S) if M is holomorphic on $\mathbb{C}\setminus\mathbb{R}_+$ and $M(\lambda)\geqslant 0$ for $\lambda<0$;
(ii) the Krein–Stieltjes class (S^{-1}) if M is holomorphic on $\mathbb{C}\setminus\mathbb{R}_+$ and $M(\lambda)\leqslant 0$ for $\lambda<0$.

If $M\in(S)$ then it admits the integral representation (see [31, Section 5])

$$M(\lambda)=c+\int\limits_{-0}^{+\infty}\frac{\mathrm{d}\tau(s)}{s-\lambda},\quad\text{where } c\geqslant 0,\qquad \int\limits_{0}^{+\infty}(1+s)^{-1}\,\mathrm{d}\tau(s)<+\infty,$$

and $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function. This representation yields that an S-function M is increasing on $(-\infty, 0)$, and $M(\lambda_0) = 0$ for certain $\lambda_0 < 0$ exactly when $M \equiv 0$. Note also that $M \in (S^{-1})$ if and only if $(-1/M) \in (S)$.

The nonnegativity of the self-adjoint operator L can be described in terms of the m-coefficients $m_\pm$.

Proposition 2.2. *The operator $L = \frac{1}{|r|}(-\frac{d^2}{dx^2} + q)$ is nonnegative if and only if $(-1/m_+ - 1/m_-) \in (S^{-1})$. If, in addition, $r(x) = -r(-x)$ and $q(x) = q(-x)$ for a.a. $x \in \mathbb{R}$, then $L \geqslant 0$ exactly when $m_+ \in (S)$.*

Proof. Let $L^D_\pm$ be the self-adjoint operators (in $L^2(\mathbb{R}_\pm, |r(x)|\,dx)$) associated with the Dirichlet boundary value problems

$$-y''(x) + q(x)y(x) = \lambda\left|r(x)\right|y(x), \quad x \in \mathbb{R}_\pm, \qquad y(\pm 0) = 0. \tag{2.10}$$

Recall that the functions $\widetilde{m}_\pm := -1/m_\pm$ are the Titchmarsh–Weyl m-coefficients associated with the problems (2.10). In particular, $\widetilde{m}_\pm \in (R)$ and

$$c(x,\lambda) \pm \widetilde{m}_\pm(\lambda)s(x,\lambda) \in L^2\big(\mathbb{R}_\pm, \left|r(x)\right|dx\big) \quad \text{whenever } \lambda \in \rho\big(L^D_\pm\big). \tag{2.11}$$

If $L \geqslant 0$, then its symmetric restriction $L_{\min}$ defined by (2.3) is nonnegative too. Moreover, the extension $L^D_+ \oplus L^D_-$ of $L_{\min}$ corresponding to the Dirichlet boundary condition at 0 is a Friedrichs extension, i.e., a maximal nonnegative self-adjoint extension of $L_{\min}$ (see [45] and [16, Proposition 4]). So $L_{\min} \geqslant 0$ if and only if $L^D_+ \geqslant 0$ and $L^D_- \geqslant 0$. This implies that both $\widetilde{m}_+$ and $\widetilde{m}_-$ are analytic on $\mathbb{C} \setminus \mathbb{R}_+$ and real on $(-\infty, 0)$. It follows from (2.11) that

$$\big\{\lambda < 0\colon \widetilde{m}_+(\lambda) = -\widetilde{m}_-(\lambda)\big\} = \sigma_p(L) \cap (-\infty, 0) = \sigma(L) \cap (-\infty, 0).$$

Since $L \geqslant 0$, we see that $\widetilde{m}_+(\lambda) + \widetilde{m}_-(\lambda) \neq 0$ if $\lambda < 0$. Moreover, $\widetilde{m}_+(\lambda) + \widetilde{m}_-(\lambda) < 0$ for $\lambda < 0$ since $\widetilde{m}_\pm(-\infty) = -\infty$. Thus, $\widetilde{m}_+ + \widetilde{m}_- \in (S^{-1})$.

If q and $|r|$ are even, then $m_-(\cdot) = m_+(\cdot)$. Hence, $\widetilde{m}_+ + \widetilde{m}_- = 2\widetilde{m}_+(\cdot) = -2/m_+ \in (S^{-1})$ or, equivalently, $m_+ \in (S)$. $\square$

Remark 2.3. In the recent paper [7], the number of negative squares of self-adjoint operators in Krein spaces were investigated in terms of abstract Weyl functions (cf. [41, Theorem 2]). In particular, Proposition 2.2 was proved under additional assumptions (see Proposition 4.4 and Theorem 4.7 in [7]).

2.3. Spectral functions of J-nonnegative operators

Let $\mathfrak{H}$ be a Hilbert space with a scalar product $(\cdot,\cdot)_\mathfrak{H}$. Let $\mathfrak{H}_+$ and $\mathfrak{H}_-$ be closed subspaces of $\mathfrak{H}$ such that $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$. Denote by $P_\pm$ the orthogonal projections from $\mathfrak{H}$ onto $\mathfrak{H}_\pm$. Put $\mathcal{J} = P_+ - P_-$ and $[\cdot,\cdot] := (\mathcal{J}\,\cdot,\cdot)_\mathfrak{H}$. Then the pair $\mathcal{K} = (\mathfrak{H}, [\cdot,\cdot])$ is called a *Krein space* (see e.g. [5,46] for the original definition). The form $[\cdot,\cdot]$ is called *an inner product* in the Krein space $\mathcal{K}$ and the operator $\mathcal{J}$ is called *a fundamental symmetry*.

Let T be a densely defined operator in $\mathfrak{H}$. By $T^{[*]}$ denote the adjoint of T with respect to $[\cdot,\cdot]$. The operator T is called *$\mathcal{J}$-self-adjoint* (*$\mathcal{J}$-nonnegative*) if $T = T^{[*]}$ (resp., $[Tf, f] \geqslant 0$ for

$f \in \mathrm{dom}(T)$). It is easy to see that $T^{[*]} := \mathcal{J} T^* \mathcal{J}$ and T is $\mathcal{J}$-self-adjoint ($\mathcal{J}$-nonnegative) if and only if $\mathcal{J} T$ is self-adjoint (resp., nonnegative).

Let $\mathfrak{S}$ be the semiring consisting of all bounded intervals with endpoints different from 0 and $\pm\infty$ and their complements in $\overline{\mathbb{R}} := \mathbb{R} \cup \infty$.

Theorem 2.4. *(See [46].) Let T be a $\mathcal{J}$-nonnegative $\mathcal{J}$-self-adjoint operator in $\mathfrak{H}$ such that $\rho(T) \neq \emptyset$. Then $\sigma(T) \subset \mathbb{R}$ and there exists a mapping $\Delta \to E(\Delta)$ from $\mathfrak{S}$ into the set of bounded linear operators in $\mathfrak{H}$ such that the following properties hold* ($\Delta, \Delta' \in \mathfrak{S}$):

(E1) $E(\Delta \cap \Delta') = E(\Delta) E(\Delta')$, $E(\emptyset) = 0$, $E(\overline{\mathbb{R}}) = I$, $E(\Delta) = E(\Delta)^{[*]}$;
(E2) $E(\Delta \cup \Delta') = E(\Delta) + E(\Delta')$ *if* $\Delta \cap \Delta' = \emptyset$;
(E3) *the form* $\pm[\cdot,\cdot]$ *is positive definite on* $E(\Delta)\mathfrak{H}$ *if* $\Delta \subset \mathbb{R}_\pm$;
(E4) $E(\Delta)$ *is in the double commutant of the resolvent* $R_T(\lambda)$ *and* $\sigma(T \restriction E(\Delta)\mathfrak{H}) \subset \overline{\Delta}$;
(E5) *if* Δ *is bounded, then* $E(\Delta)\mathfrak{H} \subset \mathrm{dom}(T)$ *and* $T \restriction E(\Delta)\mathfrak{H}$ *is a bounded operator.*

According to [46, Proposition II.4.2], $s \in \{0, \infty\}$ is called *a critical point* of T if, for each $\Delta \in \mathfrak{S}$ such that $s \in \Delta$, the form $[\cdot,\cdot]$ is indefinite on $E(\Delta)\mathfrak{H}$ (the latter means that there exist $h_\pm \in E(\Delta)\mathfrak{H}$ such that $\pm[h_\pm, h_\pm] > 0$). The set of critical points is denoted by $c(T)$. If $\alpha \notin c(T)$, then for arbitrary $\lambda_0, \lambda_1 \in \mathbb{R} \setminus c(T)$, $\lambda_0 < \alpha$, $\lambda_1 > \alpha$, the limits $\lim_{\lambda \uparrow \alpha} E([\lambda_0, \lambda])$ and $\lim_{\lambda \downarrow \alpha} E([\lambda, \lambda_1])$ exist in the strong operator topology; here in the case $\alpha = \infty$, $\lambda_1 > \alpha$ ($\lambda \downarrow \alpha$) means $\lambda_1 > -\infty$ ($\lambda \downarrow -\infty$). If $\alpha \in c(T)$ and the above limits do still exist, then α is called *regular critical point* of T, otherwise α is called *singular*.

The following proposition is well known (cf. [46, Section 6]).

Proposition 2.5. *Let T be a $\mathcal{J}$-nonnegative and $\mathcal{J}$-self-adjoint operator in the Hilbert space $\mathfrak{H}$. Assume that $\rho(T) \neq \emptyset$ and* $\ker T = \ker T^2$. *Then the following assertions are equivalent*:

(i) *T is similar to a self-adjoint operator,*
(ii) 0 *and ∞ are not singular critical points of T.*

Proposition 2.6. *(See [14], see also [38].) If the* (J*-self-adjoint*) *operator A defined by* (1.2) *is J-nonnegative, then its spectrum $\sigma(A)$ is real.*

So any J-nonnegative operator of type (1.2) has a spectral function $E_A(\cdot)$. Note that ∞ is always a critical point of A, and 0 may be its critical point.

Proposition 2.7. *(See [14].) Assume that the* (J*-self-adjoint*) *operator A defined by* (1.2) *is J-nonnegative.*

(i) *Assume that there exist $\delta > 0$ and constants $s_\pm > 0$, $s_\pm \neq 1$, such that $r \in AC_{\mathrm{loc}}(\mathcal{I}_\delta^\pm)$, $(r(x)/r(s_\pm x))' \in L^\infty(\mathcal{I}_\delta^\pm)$, where $\mathcal{I}_\delta^+ = (0, \delta]$, $\mathcal{I}_\delta^- = [-\delta, 0)$. If there exist limits $\lim_{x \to \pm 0}(r(x)/r(s_\pm x)) \neq s_\pm$, then ∞ is a regular critical point of A.*
(ii) *If $L \geqslant \varepsilon > 0$ and the assumptions of statement* (i) *are satisfied, then A is similar to a self-adjoint operator.*

Proposition 2.7 and the slightly stronger condition (1.5) follows directly from [14, Theorem 3.6(i)], [14, Lemma 3.5(iii)], and the remarks in the last two paragraphs of [14, Section 3.2].

3. Sufficient conditions for regularity of critical points

Let A, L, J, and $A_{\min}$ be the operators defined in Section 2.1, and M_+, M_- the Titchmarsh–Weyl m-coefficients for (1.1) (see Section 2.2).

3.1. Our approach to the similarity problem is based on the resolvent similarity criterion obtained in [48,50] (a resolvent similarity criterion, somewhat different from the one given below, was obtained in [11]).

Theorem 3.1. *(See [48,50].) A closed operator T in a Hilbert space $\mathfrak{H}$ is similar to a self-adjoint operator if and only if $\sigma(T) \subset \mathbb{R}$ and the inequalities*

$$\sup_{\varepsilon>0} \varepsilon \int_{\mathbb{R}} \left\| \mathcal{R}_T(\eta + \mathrm{i}\varepsilon) f \right\|^2 \mathrm{d}\eta \leqslant K_1 \|f\|^2, \quad f \in \mathfrak{H}, \tag{3.1}$$

$$\sup_{\varepsilon>0} \varepsilon \int_{\mathbb{R}} \left\| \mathcal{R}_{T^*}(\eta + \mathrm{i}\varepsilon) f \right\|^2 \mathrm{d}\eta \leqslant K_{1*} \|f\|^2, \quad f \in \mathfrak{H}, \tag{3.2}$$

hold with constants K_1 and K_{1*} independent of f.

Remark 3.2. If $\mathcal{J} = \mathcal{J}^* = \mathcal{J}^{-1}$ and T is a $\mathcal{J}$-self-adjoint operator, then $T^* = \mathcal{J} T \mathcal{J}$. So (3.2) is equivalent to (3.1) since in this case $\|\mathcal{R}_{T^*}(\lambda) f\| = \|\mathcal{R}_T(\lambda) f\|$ for all $f \in \mathfrak{H}$, $\lambda \in \rho(T)$.

3.2. For constants $b, c \in \mathbb{R}$, consider the operator $\mathcal{A}_{b,c} := A_{\min}^* \restriction \mathrm{dom}(\mathcal{A}_{b,c})$,

$$\mathrm{dom}(\mathcal{A}_{b,c}) = \left\{ f \in \mathrm{dom}\left(A_{\min}^*\right):\ f(+0) - f(-0) = c f'(-0),\ f'(+0) = b f'(-0) \right\}. \tag{3.3}$$

The operator A defined by (1.2) coincides with $\mathcal{A}_{1,0}$. Note also that the formal differential expression $\frac{1}{r}(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x) + c\delta'(x))$, where δ is the Dirac function, may be associated with the operator $A_{1,c}$ (see e.g. [4,43]).

Proposition 3.3.

(i) *$\mathcal{A}_{b,c} = \mathcal{A}_{b,c}^*$ if and only if $b = -1$ and $c \in \mathbb{R}$.*
(ii) *$\sigma(\mathcal{A}_{b,c}) \setminus \mathbb{R} = \{\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-:\ M_-(\lambda) - bM_+(\lambda) - c = 0\}$.*
(iii) *If $\lambda \notin \mathbb{R}$ and $\lambda \in \rho(\mathcal{A}_{b,c})$, then for all $f \in L^2(\mathbb{R}, |r|\,\mathrm{d}x)$,*

$$(\mathcal{A}_{b,c} - \lambda)^{-1} f = (A_0 - \lambda)^{-1} f + \frac{\mathcal{F}_-(f,\lambda) - \mathcal{F}_+(f,\lambda)}{M_-(\lambda) - bM_+(\lambda) - c}\bigl(b\psi_+(\cdot,\lambda) + \psi_-(\cdot,\lambda)\bigr), \tag{3.4}$$

where $\mathcal{F}_\pm(f,\lambda) := \int_{\mathbb{R}_\pm} f(x)\psi_\pm(x,\lambda)|r(x)|\,\mathrm{d}x$.

Proof. (i) can be obtained by direct calculation. On the other hand, it follows from the proof of [42, Proposition 5.8]. Indeed, for the operator $\mathcal{A}_{b,c}$, the matrix B defined by [42, formula (5.24)] equals $\left(\begin{smallmatrix} 0 & b \\ -1 & c \end{smallmatrix}\right)$, and $\mathcal{A}_{b,c} = \mathcal{A}_{b,c}^*$ exactly when $B = B^*$. The proofs of (ii)–(iii) are similar to that of [42, Proposition 4.4] (see also [38, Lemma 4.1]). $\square$

Theorem 3.4. *If there exists a constant $c \in \mathbb{R}$ such that the function*

$$\frac{|M_+(\lambda) + M_-(\lambda) - c|}{|M_+(\lambda) - M_-(\lambda)|} \tag{3.5}$$

is bounded on $\mathbb{C}_+$, then the operator A is similar to a self-adjoint one.

Proof. The proof is similar to that of [42, Theorem 5.9]. We present a sketch.

Let $c \in \mathbb{R}$. Note that $A_0 = A_0^*$ (see (2.4)) and $\mathcal{A}_{-1,c} = \mathcal{A}_{-1,c}^*$. Hence inequality (3.1) holds for the resolvents of both the operators A_0 and $\mathcal{A}_{-1,c}$. Therefore (3.4) implies that for $f \in L^2(\mathbb{R}, |r|\,\mathrm{d}x)$

$$\sup_{\varepsilon>0} \varepsilon \int\limits_{\mu=-\infty}^{+\infty} \left\| \frac{\mathcal{F}_\pm(f, \mu + \mathrm{i}\varepsilon)\, \psi_\pm(x, \mu + \mathrm{i}\varepsilon)}{M_+(\mu + \mathrm{i}\varepsilon) + M_-(\mu + \mathrm{i}\varepsilon) - c} \right\|^2_{L^2(|r|\,\mathrm{d}x)} \mathrm{d}\mu \leqslant C_1 \|f\|^2. \tag{3.6}$$

The same arguments and Remark 3.2 show that the operator $A = \mathcal{A}_{1,0}$ is similar to a self-adjoint one exactly when

$$\sup_{\varepsilon>0} \varepsilon \int\limits_{\mu=-\infty}^{+\infty} \left\| \frac{\mathcal{F}_\pm(f, \mu + \mathrm{i}\varepsilon) \psi_\pm(x, \mu + \mathrm{i}\varepsilon)}{M_+(\mu + \mathrm{i}\varepsilon) - M_-(\mu + \mathrm{i}\varepsilon)} \right\|^2_{L^2(|r|\,\mathrm{d}x)} \mathrm{d}\mu \leqslant C_2 \|f\|^2. \tag{3.7}$$

Combining (3.6) with the assumption of the theorem, we get (3.7). $\square$

Theorem 3.4 is valid for J-self-adjoint (not necessary J-nonnegative) operators of the form (1.2). If $c = 0$, this result coincides with [42, Theorem 5.9].

Theorem 3.5. *Assume that the operator A is J-nonnegative. If ratio* (3.5) *is bounded on the set $\Omega_R^0 := \{\lambda \in \mathbb{C}_+\colon\ |\lambda| < R\}$ (on the set $\Omega_R^\infty := \{\lambda \in \mathbb{C}_+\colon\ |\lambda| > R\}$) for certain constants $R > 0$ and $c \in \mathbb{R}$, then the point* 0 *(resp., the point* ∞*) is not a singular critical point of A.*

Proof. By Proposition 2.6 and Theorem 2.4, A has a spectral function $E_A(\Delta)$. Therefore $\mathcal{P}_R := E_A([-R/2, R/2])$ is a bounded J-orthogonal projection. Using properties (E1), (E2), and (E4) of $E_A(\Delta)$, we obtain the decomposition

$$A = A^0 \dotplus A^\infty, \qquad A^0 := A \upharpoonright \mathfrak{H}_0, \qquad A^\infty := A \upharpoonright \mathfrak{H}_\infty, \qquad L^2\big(\mathbb{R}, |r|\,\mathrm{d}x\big) = \mathfrak{H}_0 \dotplus \mathfrak{H}_\infty,$$

where $\mathfrak{H}_0 := \operatorname{ran}(\mathcal{P}_R)$ and $\mathfrak{H}_\infty := \operatorname{ran}(I - \mathcal{P}_R)$. Moreover,

$$\sigma\big(A^0\big) \subset [-R/2, R/2], \qquad \sigma\big(A^\infty\big) \subset (-\infty, -R/2] \cup [R/2, +\infty).$$

Obviously, A^0 is a J-self-adjoint J-nonnegative operator. Note that A^0 has the singular critical point 0 if and only if so does A.

Let us prove that the resolvent of A^0 satisfies (3.1) if the function (3.5) is bounded on Ω_R^0. Indeed, using the last assumption, formula (3.4), and arguing as in proof of Theorem 3.4, we obtain

$$\varepsilon \int_{\mathcal{I}_\varepsilon} \left\| \left(A^0 - (\mu + \mathrm{i}\varepsilon)\right)^{-1} f \right\|^2 \mathrm{d}\mu = \varepsilon \int_{\mathcal{I}_\varepsilon} \left\| \left(A - (\mu + \mathrm{i}\varepsilon)\right)^{-1} f \right\|^2 \mathrm{d}\mu \leqslant C_1 \|f\|^2, \tag{3.8}$$

where $\mathcal{I}_\varepsilon := [-\sqrt{R^2 - \varepsilon^2}, \sqrt{R^2 - \varepsilon^2}]$ if $\varepsilon < R$, and $\mathcal{I}_\varepsilon = \emptyset$ if $\varepsilon \geqslant R$. Further, (E5) yields that A^0 is bounded. From this and $\sigma(A^0) \subset [-R/2, R/2]$, one gets $\|(A^0 - \lambda)^{-1}\| \leqslant C_2 |\lambda|^{-1}$ for $|\lambda| > R$. Hence,

$$\varepsilon \int_{\mathbb{R} \setminus \mathcal{I}_\varepsilon} \left\| \left(A^0 - (\mu + \mathrm{i}\varepsilon)\right)^{-1} f \right\|^2 \mathrm{d}\mu \leqslant C_2 \|f\|^2 \int_{\mathbb{R} \setminus \mathcal{I}_\varepsilon} \varepsilon |\mu + \mathrm{i}\varepsilon|^{-2} \,\mathrm{d}\mu \leqslant C_2 \pi \|f\|^2. \tag{3.9}$$

Combining (3.8) and (3.9) with Remark 3.2, we see that A^0 is similar to a self-adjoint operator. Thus 0 is not a singular critical point of A^0. The proof for the case of the critical point ∞ is similar. □

3.3. In Section 5, we will use the following necessary condition for regularity.

Theorem 3.6. *(See [38,39].) Assume that the operator A is J-nonnegative.*

(i) *If* 0 *is not a singular critical point of A and* $\ker A = \ker A^2$*, then*

$$\sup_{\lambda \in \Omega_R^0} \left| \frac{\operatorname{Im}(M_+(\lambda) + M_-(\lambda))}{M_+(\lambda) - M_-(\lambda)} \right| = C_R < \infty, \quad R > 0. \tag{3.10}$$

(ii) *If ∞ is not a singular critical point of A, then the function in* (3.10) *is bounded on Ω_R^∞ for all $R > 0$.*

Remark 3.7. If $\operatorname{Re}(M_+(\lambda) + M_-(\lambda)) - c = O(\operatorname{Im}(M_+(\lambda) - M_-(\lambda)))$ as $\lambda \to 0$, $\lambda \in \mathbb{C}_+$, the necessary conditions of Theorem 3.6 imply the sufficient conditions of Theorem 3.5. The results of the following sections show that this is the case for several classes of coefficients.

4. Operators with decaying potentials and regular critical point 0

In this section, we consider the operator

$$A = (\operatorname{sgn} x)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x)\right), \quad \operatorname{dom}(A) = \mathfrak{D}, \tag{4.1}$$

with the potential $q \in L^1(\mathbb{R})$ having a finite first moment. That is we consider the case when $r(x) = \operatorname{sgn} x$ and q satisfies (1.8).

4.1. The asymptotic behavior of the Titchmarsh–Weyl m-coefficient

Since $|r| \equiv 1$, Eq. (2.5) becomes

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in \mathbb{R}. \tag{4.2}$$

Note that condition (1.8) implies that (4.2) is limit point at both $+\infty$ and $-\infty$. Let $c(\cdot,\lambda)$, $s(\cdot,\lambda)$, and $m_\pm(\cdot)$ be the solutions and the Titchmarsh–Weyl m-coefficients of (4.2) defined as in Section 2.2. Denote by $\sqrt{z}$, $z \in \mathbb{C} \setminus \mathbb{R}_+$, the branch of the multifunction $z^{1/2}$ with cut along the positive semi-axis $\mathbb{R}_+$ singled out by $\sqrt{-1} = \mathrm{i}$.

Lemma 4.1. *Let*

$$\int\limits_{\mathbb{R}_+} \bigl(1+|x|\bigr)\bigl|q(x)\bigr|\,\mathrm{d}x < \infty. \tag{4.3}$$

Let $s(\cdot,0)$ be the solution of (4.2) with $\lambda = 0$.

(i) *If $s(\cdot,0)$ is unbounded on $\mathbb{R}_+$, then for certain constants $a_+ > 0$ and $b_+ \in \mathbb{R}$,*

$$m_+(\lambda) = \frac{a_+}{b_+ - \mathrm{i}\sqrt{\lambda}}\bigl[1+o(1)\bigr], \quad \lambda \to 0,\ \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{4.4}$$

(ii) *If $s(\cdot,0)$ is bounded on $\mathbb{R}_+$, then for a certain constant $k_+ > 0$,*

$$m_+(\lambda) = \mathrm{i}k_+\sqrt{\lambda}\bigl[1+o(1)\bigr], \quad \lambda \to 0,\ \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{4.5}$$

Proof. First note that it suffices to prove (4.4) and (4.5) for $\lambda \in \mathbb{C}_+$ since m_+ is an R-function and hence $\overline{m_+(\lambda)} = m_+(\bar\lambda)$.

(i) In the case $q \in L^1(\mathbb{R}_+)$, the Titchmarsh–Weyl m-coefficient admits another representation (see [55, Chapter V, §3]), which is distinct from (2.7). Namely,

$$m_+(\lambda) = \frac{a(\lambda)}{b(\lambda)}, \quad \lambda \in \mathbb{C}_+, \tag{4.6}$$

$$a(\lambda) = \frac{\mathrm{i}}{2\sqrt{\lambda}} + \frac{\mathrm{i}}{2\sqrt{\lambda}} \int\limits_0^{+\infty} q(x)\mathrm{e}^{\mathrm{i}\sqrt{\lambda}x} s(x,\lambda)\,\mathrm{d}x,$$

$$b(\lambda) = \frac{1}{2} + \frac{\mathrm{i}}{2\sqrt{\lambda}} \int\limits_0^{+\infty} q(x)\mathrm{e}^{\mathrm{i}\sqrt{\lambda}x} c(x,\lambda)\,\mathrm{d}x, \tag{4.7}$$

where the functions a, b are analytic in $\mathbb{C}_+$.

In order to estimate $c(x,\lambda)$ and $s(x,\lambda)$, we use transformation operators preserving initial conditions at the point $x = 0$. Indeed, it follows from [49, formulas (1.2.9)–(1.2.11)] (see also [47]) that $c(x,\lambda)$ and $s(x,\lambda)$ admit the following representations

$$c(x,\lambda) = \cos x\sqrt{\lambda} + \int_{-x}^{x} K(x,t)\cos t\sqrt{\lambda}\,\mathrm{d}t, \tag{4.8}$$

$$s(x,\lambda) = \frac{\sin x\sqrt{\lambda}}{\sqrt{\lambda}} + \int_{-x}^{x} K(x,t)\frac{\sin t\sqrt{\lambda}}{\sqrt{\lambda}}\,\mathrm{d}t, \tag{4.9}$$

where the kernel $K(x,t)$ satisfies the estimates (see [49, formulas (1.2.20), (1.2.21)] and also [47])

$$\left|K(x,t)\right| \leqslant \frac{1}{2}w_0\left(\frac{x+t}{2}\right)\mathrm{e}^{w_1(x)-w_1(\frac{x+t}{2})-w_1(\frac{x-t}{2})}, \quad 0 \leqslant |t| < x, \tag{4.10}$$

$$w_0(x) := \int_0^x \left|q(y)\right|\mathrm{d}y, \qquad w_1(x) := \int_0^x w_0(y)\,\mathrm{d}y. \tag{4.11}$$

Under assumption (1.8), one can simplify (4.10) as follows

$$\left|K(x,t)\right| \leqslant \frac{1}{2}w_0\left(\frac{x+t}{2}\right)\mathrm{e}^{\widetilde{w}_1(\frac{x+t}{2})}, \quad \widetilde{w}_1(x) := \int_0^x \int_y^{+\infty} \left|q(t)\right|\mathrm{d}t\,\mathrm{d}y \tag{4.12}$$

since inequality (1.8) implies $\widetilde{w}_1(+\infty) = C_0 < \infty$. Hence (1.8), (4.12), and (4.11) implies $|K(x,t)| \leqslant C_1 < \infty$ for all $0 \leqslant |t| < x$. Combining this fact with (4.8) and (4.9), one obtains

$$\left|c(x,\lambda)\right| \leqslant (1+C_1x)\mathrm{e}^{x|\operatorname{Im}\sqrt{\lambda}|}, \qquad \left|\sqrt{\lambda}s(x,\lambda)\right| \leqslant (1+C_1x)\mathrm{e}^{x|\operatorname{Im}\sqrt{\lambda}|}, \tag{4.13}$$

for all $x \in \mathbb{R}_+$ and $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$. We also need the following inequality (see [49, formulas (3.1.28′), (3.1.23)])

$$\left|s(x,\lambda)\right| \leqslant x\mathrm{e}^{x|\operatorname{Im}\sqrt{\lambda}|}\mathrm{e}^{\widetilde{w}_1(x)} \leqslant C_2x\mathrm{e}^{x|\operatorname{Im}\sqrt{\lambda}|}, \quad C_2 := e^{C_0}, \tag{4.14}$$

which holds for all $x \in \mathbb{R}_+$, $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$, and is better than (4.13) as $\lambda \to 0$.

Further, we put

$$\widetilde{a}(\lambda) = 1 + \int_0^{+\infty} q(t)\mathrm{e}^{\mathrm{i}\sqrt{\lambda}t}s(t,\lambda)\,\mathrm{d}t, \qquad \widetilde{b}(\lambda) = \int_0^{+\infty} q(t)\mathrm{e}^{\mathrm{i}\sqrt{\lambda}t}c(t,\lambda)\,\mathrm{d}t, \tag{4.15}$$

$$a_+ := \widetilde{a}(0) = 1 + \int_0^{+\infty} q(t)s(t,0)\,\mathrm{d}t, \qquad b_+ := \widetilde{b}(0) = \int_0^{+\infty} q(t)c(t,0)\,\mathrm{d}t. \tag{4.16}$$

By (1.8), (4.13), and (4.14), the integrals in (4.15) and (4.16) exist and are finite for all $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$. Note also that $a_+, b_+ \in \mathbb{R}$ since $q(\cdot)$, $c(\cdot,0)$, and $s(\cdot,0)$ are real functions.

Let us show that $a_+ = 0$ if and only if $s(x,0)$ is bounded on $\mathbb{R}_+$. Indeed, integrating the equation

$$-y''(x) + q(x)y(x) = 0, \quad x > 0, \tag{4.17}$$

and using $s'(0,0) = 1$, we get

$$s'(x,0) = 1 + \int_0^x q(t)s(t,0)\,\mathrm{d}t, \quad x \geqslant 0.$$

By (4.16), $a_+ = 0$ exactly when $s'(x,0) = o(1)$ as $x \to +\infty$. On the other hand, Eq. (4.17) with $q(\cdot)$ satisfying (1.8) has two linearly independent solutions $y_1(x)$ and $y_2(x)$ such that (see [28, Theorem X.17.1])

$$y_1(x) \approx 1, \qquad y_1'(x) = o(1), \qquad y_2(x) \approx x, \qquad y_2'(x) \approx 1, \tag{4.18}$$

as $x \to +\infty$. Hence, $s(x,0) = c_1 y_1(x) + c_2 y_2(x)$. So we conclude that $s'(x,0) = o(1)$ as $x \to +\infty$ if and only if $s(\cdot,0) = c_1 y_1(\cdot) \in L^\infty(\mathbb{R}_+)$.

Note that $c(x,\lambda)$ and $s(x,\lambda)$ are entire functions of λ for every $x \in \mathbb{R}_+$. Combining this fact with (4.13), (4.14), and first Helly's theorem, we obtain that functions (4.15) are continuous on $\mathbb{C}_+ \cup \mathbb{R}$. Due to the assumption $s(\cdot,0) \notin L^\infty(\mathbb{R}_+)$, we have $a_+ \neq 0$. Therefore,

$$a(\lambda) = \mathrm{i}\frac{a_+}{2\sqrt{\lambda}}\big(1 + o(1)\big), \qquad b(\lambda) = \frac{1}{2} + \mathrm{i}\frac{b_+}{2\sqrt{\lambda}}\big(1 + o(1)\big), \tag{4.19}$$

as $\lambda \to 0$ and (4.4) easily follows from (4.6) and (4.19).

To complete the proof of (i), it remains to note that $a_+ > 0$ since $m_+ \in (R)$.

(ii) Let the solution $s(x,0)$ be bounded, i.e., $s(\cdot,0) \in L^\infty(\mathbb{R}_+)$.

Under condition (4.3), for every λ in the closed upper half-plane $\overline{\mathbb{C}_+}$ Eq. (4.2) has a solution that admits the representation by means of a transformation operator preserving asymptotic behavior at infinity (see [49, Lemma 3.1.1], and also [47, Chapter I, §4])

$$e(x,\lambda) := \mathrm{e}^{\mathrm{i}\sqrt{\lambda}x} + \int_x^{+\infty} \widetilde{K}(x,t)\mathrm{e}^{\mathrm{i}\sqrt{\lambda}t}\,\mathrm{d}t, \quad x > 0,\ \lambda \in \overline{\mathbb{C}_+}, \tag{4.20}$$

where the kernel $\widetilde{K}(x,t)$ satisfies the following estimates for $x, t \geqslant 0$

$$\big|\widetilde{K}(x,t)\big| \leqslant \frac{1}{2}\widetilde{\omega}_0\left(\frac{x+t}{2}\right)\mathrm{e}^{\widetilde{\omega}(x)},$$

$$\widetilde{\omega}_0(x) := \int_x^\infty \big|q(t)\big|\,\mathrm{d}t, \qquad \widetilde{\omega}(x) = \int_x^\infty \widetilde{\omega}_0(t)\,\mathrm{d}t. \tag{4.21}$$

Note that, $e(x,\lambda) = \mathrm{e}^{\mathrm{i}\sqrt{\lambda}x}(1 + o(1))$ as $x \to +\infty$. In particular, $e(\cdot,\lambda)$ is the Weyl solution of (4.2) if $\lambda \in \mathbb{C}_+$. Moreover,

$$e(x,0) = 1 + \int_x^\infty \widetilde{K}(x,t)\,\mathrm{d}t$$

is a nontrivial bounded solution of (4.17). Therefore (cf. (4.18)),

$$e(x,0) = c_0' s(x,0) \quad \text{with } (0 \neq)\ c_0' = -\widetilde{K}(0,0) + \int_0^\infty K_x'(0,t)\,\mathrm{d}t,$$

and $e(0,0) = c_0' s(0,0) = 0$. Hence [49, formula (3.2.26)], $e(0,\lambda)$ has the form

$$e(0,\lambda) = \mathrm{i}\sqrt{\lambda}\widehat{K}_1(-\sqrt{\lambda}), \qquad K_1(x) = \int_x^\infty \widetilde{K}(0,t)\,\mathrm{d}t, \tag{4.22}$$

where $\widehat{K}_1(\lambda) := \int_0^\infty K_1(t)\mathrm{e}^{-\mathrm{i}\lambda t}\mathrm{d}t$. Moreover, $\widehat{K}_1$ is continuous at zero since $K_1 \in L^1(\mathbb{R}_+)$, and $\widehat{c}_0 := \widehat{K}_1(0) \neq 0$ (see the remarks after Eqs. (3.2.25) and (3.2.27) in [49]). Noting that $e'(0,0) = c_0' s'(0,0) = c_0' \neq 0$, and taking into account (2.7), we arrive at the desired relation

$$m(\lambda) = -\frac{e(0,\lambda)}{e'(0,\lambda)} = -\frac{\mathrm{i}\sqrt{\lambda}\widehat{K}_1(0)}{c_0'}\bigl(1 + o(1)\bigr), \quad (\mathbb{C}_+ \ni)\lambda \to 0, \tag{4.23}$$

which proves (ii) with $k_+ = -\frac{\widehat{K}_1(0)}{c_0'}$. The inequality $k_+ > 0$ follows from the inclusion $m_+ \in (R)$. $\square$

Remark 4.2. Note that, if $q \in L^1(\mathbb{R}_-, (1 + |x|)\,\mathrm{d}x)$, then the analogous statements are valid for m_- (with certain constants $a_-, k_- > 0$, and $b_- \in \mathbb{R}$ instead of a_+, k_+, and b_+, respectively).

Proposition 4.3. *Let* (1.8) *be fulfilled. Then the operator* (4.1) *has no real eigenvalues, i.e.,* $\sigma_\mathrm{p}(A) \cap \mathbb{R} = \emptyset$.

Proof. By (4.18), $\ker L = \{0\}$. But $\ker A = \ker(JL) = \ker L = \{0\}$.

Further, let $\lambda > 0$ and $f \in \ker(A - \lambda)$ (the case $\lambda < 0$ is analogous). Then $f \in L^2(\mathbb{R})$ and solves (4.2) for $x > 0$ with $\lambda > 0$. Under assumption (1.8), Eq. (4.2) has two linearly independent solutions $e_\pm(x,\lambda)$ satisfying $e_\pm(x,\lambda) \approx \mathrm{e}^{\pm\mathrm{i}\sqrt{\lambda}x}$ as $x \to +\infty$ ([49, Lemma 3.1.1], see also (4.20)–(4.21)). So $f(x) = c_+ e_+(x,\lambda) + c_- e_-(x,\lambda)$ for $x > 0$ with certain $c_\pm \in \mathbb{C}$ and $f(x) \approx c_+\mathrm{e}^{\mathrm{i}\sqrt{\lambda}x} + c_-\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}x}$ as $x \to +\infty$. The latter yields $f(x)\chi_+(x) = 0$ since $f \in L^2(\mathbb{R})$. Since f is a solution of (1.1), the Cauchy uniqueness theorem yields $f = 0$ for $x < 0$. $\square$

Remark 4.4. Assume that q satisfies (1.8) on $\mathbb{R}_+$ and that the minimal symmetric operator $L_{\min}^+$ associated with the spectral problem

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \geqslant 0, \qquad y(0) = y'(0) = 0,$$

is nonnegative in $L^2(\mathbb{R}_+)$. The Friedrichs (hard) extension $L_+^D = (L_+^D)^*$ of $L_{\min}^+$ is determined by the Dirichlet boundary condition at zero (for definitions and basic facts on M.G. Krein's extension theory of nonnegative operators see [2, Section 109]). The corresponding m-coefficient is $\widetilde{m}_+(\cdot)(= -1/m_+(\cdot))$. Lemma 4.1 shows that $s(\cdot,0) \in L^\infty(\mathbb{R}_+)$ exactly when $\widetilde{m}_+(-0) = +\infty$. It follows from [45] (see also [16, Proposition 4]) that $\widetilde{m}_+(-0) = +\infty$ holds if and only if L_+^D is the Krein–von Neumann (soft) extension of $L_{\min}^+$. The latter means that the operator $L_{\min}^+$ has a unique nonnegative self-adjoint extension. Thus, Lemma 4.1 leads to the following criterion: *L_+^D is a unique nonnegative self-adjoint extension of the nonnegative operator $L_{\min}^+$ if and only if $s(\cdot,0) \in L^\infty(\mathbb{R}_+)$.*

4.2. The case of the nonnegative operator L

The proof of Theorem 1.1 is contained in this and the next subsections. The most substantial part, the implication (ii) ⇒ (i), is given by the following result:

Theorem 4.5. *Let $A = (\operatorname{sgn} x)(-\mathrm{d}^2/\mathrm{d}x^2 + q(x))$ and let $q(\cdot)$ satisfy* (1.8). *If the operator A is J-nonnegative, then it is similar to a self-adjoint operator.*

Proof. Assume that the operator A is J-nonnegative. By Proposition 2.6, $\sigma(A) \subset \mathbb{R}$. Proposition 2.7 implies that ∞ is a regular critical point of A. Moreover, (1.8) implies $\ker A = \{0\}$ (see Proposition 4.3). Hence the similarity of A is equivalent to the nonsingularity of the critical point zero of the operator A (see Proposition 2.5).

By Lemma 4.1 and (2.8), one of the asymptotic formulas (4.4), (4.5) holds for the function $m_+(\lambda) = M_+(\lambda)$. And the same is true for $m_-(\lambda) = -M_-(-\lambda)$. Consider the following four cases.

(a) *Let the solution $s(\cdot,0)$ of* (4.2) *be bounded on $\mathbb{R}$, $s(\cdot,0) \in L^\infty(\mathbb{R})$.* By Lemma 4.1(iii), for $(\mathbb{C}_+ \ni)\lambda \to 0$ we get

$$M_+(\lambda) = \mathrm{i}k_+\sqrt{\lambda}\bigl(1+o(1)\bigr), \qquad M_-(\lambda) = k_-\sqrt{\lambda}\bigl(1+o(1)\bigr), \quad k_\pm > 0.$$

Therefore, we obtain as $\lambda \to 0$

$$\frac{M_+(\lambda)+M_-(\lambda)}{M_+(\lambda)-M_-(\lambda)} = \frac{\mathrm{i}k_+\sqrt{\lambda}+k_-\sqrt{\lambda}}{\mathrm{i}k_+\sqrt{\lambda}-k_-\sqrt{\lambda}}\bigl(1+o(1)\bigr) = \frac{\mathrm{i}k_+ + k_-}{\mathrm{i}k_+ - k_-}\bigl(1+o(1)\bigr).$$

(b) *Let $s(\cdot,0) \notin L^\infty(\mathbb{R}_+)$, but $s(\cdot,0) \in L^\infty(\mathbb{R}_-)$.* Then, by Lemma 4.1,

$$M_+(\lambda) = \frac{a_+}{b_+ - \mathrm{i}\sqrt{\lambda}}\bigl(1+o(1)\bigr), \qquad M_-(\lambda) = k_-\sqrt{\lambda}\bigl(1+o(1)\bigr), \quad \lambda \to 0,$$

where $a_+ > 0$, $b_+ \in \mathbb{R}$, and $k_- > 0$. Hence we get

$$\frac{M_+(\lambda)+M_-(\lambda)}{M_+(\lambda)-M_-(\lambda)} = \frac{a_+ + k_-\sqrt{\lambda}(b_+ - \mathrm{i}\sqrt{\lambda})}{a_+ - k_-\sqrt{\lambda}(b_+ - \mathrm{i}\sqrt{\lambda})}\bigl(1+o(1)\bigr) = 1 + O\bigl(\sqrt{|\lambda|}\bigr). \tag{4.24}$$

(c) The case when $s(\cdot,0) \in L^\infty(\mathbb{R}_+)$ and $s(\cdot,0) \notin L^\infty(\mathbb{R}_-)$ is similar to (b).

(d) *Let* $s(\cdot,0)\notin L^\infty(\mathbb{R}_+)$ *and* $s(\cdot,0)\notin L^\infty(\mathbb{R}_-)$. Then, by Lemma 4.1(ii), one gets as $\lambda\to 0$

$$M_+(\lambda)=a_+(b_+-\mathrm{i}\sqrt{\lambda})^{-1}\big(1+o(1)\big),\qquad M_-(\lambda)=-a_-(b_-+\sqrt{\lambda})^{-1}\big(1+o(1)\big),$$

where $a_\pm>0$ and $b_\pm\in\mathbb{R}$. Hence,

$$\frac{M_+(\lambda)+M_-(\lambda)-c}{M_+(\lambda)-M_-(\lambda)}\approx\frac{a_+(b_-+\sqrt{\lambda})-a_-(b_+-\mathrm{i}\sqrt{\lambda})-c(b_+-\mathrm{i}\sqrt{\lambda})(b_-+\sqrt{\lambda})}{a_+(b_-+\sqrt{\lambda})+a_-(b_+-\mathrm{i}\sqrt{\lambda})},\tag{4.25}$$

as $\lambda\to 0$. If $b_+\cdot b_-=0$, then the left part of (4.25) with $c=0$ has the asymptotic behavior similar to one of the cases (a), (b), or (c). Otherwise, we put $c:=a_+b_+^{-1}-a_-b_-^{-1}$ and get

$$\frac{M_+(\lambda)+M_-(\lambda)-c}{M_+(\lambda)-M_-(\lambda)}\approx\frac{(a_+-cb_+)\sqrt{\lambda}+\mathrm{i}(a_-+cb_-)\sqrt{\lambda}}{a_+(b_-+\sqrt{\lambda})+a_-(b_+-\mathrm{i}\sqrt{\lambda})}=O(1),\quad \lambda\to 0.$$

From the above considerations, we conclude that there exists $c\in\mathbb{R}$ such that ratio (3.5) is bounded in a neighborhood of $\lambda=0$. By Theorem 3.5, 0 is not a singular critical point of A. Combining this fact with Propositions 2.5 and 2.6, we complete the proof of the similarity of A to a self-adjoint operator. □

In passing, we have proved the following fact for any (not necessarily J-nonnegative) operator $(\operatorname{sgn} x)(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+q(x))$ with q satisfying (1.8).

Proposition 4.6. *Let* $A=(\operatorname{sgn} x)(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+q(x))$ *with* q *satisfying* (1.8). *Then there exists* $c\in\mathbb{R}$ *such that ratio* (3.5) *is bounded in a neighborhood of zero.*

Remark 4.7. It should be pointed out that if L is nonnegative, then only the case (d) in the proof of Theorem 4.5 can be realized. Indeed, if $s(\cdot,0)\in L^\infty(\mathbb{R}_+)$, then (4.5) yields $m_+(x)\uparrow -0$ as $x\uparrow -0$. Hence $(-m_+)^{-1}+(-m_-)^{-1}$ takes positive values on $\mathbb{R}_-$. But $L\geqslant 0$ and Proposition 2.2 implies $(-m_+)^{-1}+(-m_-)^{-1}\in(S^{-1})$. This contradiction shows that $s(\cdot,0)\notin L^\infty(\mathbb{R}_\pm)$.

4.3. The operator L with negative eigenvalues

It is known that under condition (1.8), the negative spectrum of the operator $L=JA=-\mathrm{d}^2/\mathrm{d}x^2+q(x)$ consists of at most finite number $\kappa_-(L)$ of simple eigenvalues and (see [8, Theorem 5.3])

$$\kappa_-(L)\leqslant 1+\int_{\mathbb{R}}|x|q_-(x)\,\mathrm{d}x,\qquad q_-(x):=\big(|q(x)|-q(x)\big)/2.$$

So Propositions 1.1 and 2.5 of [14] imply that A is a *definitizable* operator (for the definitions and basic facts see [14,30,46] and [22, Appendix B]). The latter means that $\rho(A)\neq\emptyset$ and there exists a real polynomial $\mathfrak{p}$ such that $[\mathfrak{p}(A)f,f]\geqslant 0$ for all $f\in\operatorname{dom}(A^k)$, where $k=\deg\mathfrak{p}$; the polynomial $\mathfrak{p}$ is called *definitizing*. Since $\kappa_-(JA)$ is finite, there is a definitizing polynomial $\mathfrak{p}$ of minimal degree and of the form (see [14, Eq. (1.2)])

$$\mathfrak{p}(z) = z\mathfrak{q}(z)\overline{\mathfrak{q}(\bar{z})}, \qquad \deg\mathfrak{q} \leqslant \kappa_-(L). \tag{4.26}$$

The polynomial $\mathfrak{q}(z)$ is uniquely determined under the assumption that it is monic polynomial and all its zeros belongs to $\mathbb{C}_+ \cup \mathbb{R}$. A definitizable operator admits a spectral function $E(\Delta)$ with, possibly, some critical points (which belong to the set $\infty \cup \{\lambda \in \mathbb{R}\colon\ \mathfrak{p}(\lambda) = 0\}$). The properties of $E(\Delta)$ similar to that of $E(\Delta)$ from Theorem 2.4.

B. Ćurgus and H. Langer [14] investigated nonreal spectrum of indefinite $\mathcal{J}$-self-adjoint ordinary differential operators $\mathcal{A}$ assuming that $\kappa_-(\mathcal{J}\mathcal{A}) < \infty$. The following result follows from [14, Section 1.3].

Proposition 4.8. *Let* $A = (\operatorname{sgn} x)(-\mathrm{d}^2/\mathrm{d}x^2 + q(x))$ *and* $q \in L^1(\mathbb{R}, (1+|x|)\,\mathrm{d}x)$. *Let* $\mathfrak{q}$ *be defined by* (4.26). *Then*:

- (i) $\lambda \in \mathbb{R} \setminus \{0\}$ *is a zero of* $\mathfrak{q}(\cdot)$ *if and only if it is a critical point of* A; *in this case,* λ *is also an eigenvalue of* A.
- (ii) $\lambda \in \mathbb{C}_+$ $(\mathbb{C}_-)$ *is a zero of* $\mathfrak{q}(\cdot)$ (*resp.,* $\mathfrak{q}(\bar{\cdot})$) *if and only if it is a nonreal eigenvalue of* A; *in this case, the algebraic multiplicity of* λ *is finite.*

Taking Propositions 4.3 and 4.6 into account, we obtain the following description for essential and discrete parts of the operator A.

Theorem 4.9. *Let* $A = (\operatorname{sgn} x)(-\mathrm{d}^2/\mathrm{d}x^2 + q(x))$ *and* $q(\cdot)$ *satisfy* (1.8). *Then*:

- (i) *The nonreal spectrum* $\sigma(A) \setminus \mathbb{R}$ *is finite and consists of eigenvalues of finite algebraic multiplicity. If* $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$ *is an eigenvalue of* A*, then its algebraic multiplicity is equal to the multiplicity of* λ_0 *as a zero of the holomorphic function* $M_+(\lambda) - M_-(\lambda)$. *Its geometric multiplicity equals* 1.
- (ii) $\sigma_{\mathrm{p}}(A) = \sigma_{\mathrm{disc}}(A) = \sigma(A) \setminus \mathbb{R}$, $\sigma_{\mathrm{ess}}(A) = \mathbb{R}$, *and there exist a skew direct decomposition* $L^2(\mathbb{R}) = \mathfrak{H}_{\mathrm{ess}} \dot{+} \mathfrak{H}_{\mathrm{disc}}$ *such that*
$$A = A_{\mathrm{ess}} \dot{+} A_{\mathrm{disc}},$$
$$A_{\mathrm{ess}} = A \upharpoonright \big(\operatorname{dom}(A) \cap \mathfrak{H}_{\mathrm{ess}}\big), \qquad A_{\mathrm{disc}} = A \upharpoonright \big(\operatorname{dom}(A) \cap \mathfrak{H}_{\mathrm{disc}}\big),$$
$$\sigma_{\mathrm{disc}}(A) = \sigma(A_{\mathrm{disc}})\big(= \sigma(A) \setminus \mathbb{R}\big), \qquad \sigma_{\mathrm{ess}}(A) = \sigma(A_{\mathrm{ess}})(= \mathbb{R});$$
the subspace $\mathfrak{H}_{\mathrm{disc}}$ *is finite-dimensional.*
- (iii) A_{ess} *is similar to a self-adjoint operator.*

Proof. (i) follows from Proposition 4.8(i) and [42, Proposition 4.3(5)].

(ii) follows from (i) and Proposition 4.3 (see e.g. [42, Section 6]). Note only that $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(A_0) = \mathbb{R}$ (see e.g. [42, Proposition 4.3(1)]).

(iii) The operator A is a definitizable and admits a spectral function $E_A(\Delta)$. By Proposition 4.3, $\sigma_{\mathrm{p}}(A) \cap \mathbb{R} = \emptyset$. So Proposition 4.8(i) implies that $\mathfrak{q}$ has no real zeros and that the only possible critical points of A are zero and infinity (actually, 0 and ∞ *are* critical points). Further,

∞ is a regular critical point due to [14, Theorem 3.6]. Using Proposition 4.6 and arguing as in the proof of Theorem 3.5, one can prove that zero is not a singular critical point of A. Hence A_{ess}, the part of A corresponding to the real spectrum, is similar to a self-adjoint operator T. $\square$

Corollary 4.10. *Let* $A = (\operatorname{sgn} x)(-\mathrm{d}^2/\mathrm{d}x^2 + q(x))$ *and* $q(\cdot)$ *satisfy* (1.8). *Then*:

(i) λ_0 *is an eigenvalue of* A *if and only if it is a zero of* $\mathfrak{q}(z)\overline{\mathfrak{q}(\bar{z})}$; *moreover, its algebraic multiplicity coincides with the multiplicity as a zero of* $\mathfrak{q}(z)\overline{\mathfrak{q}(\bar{z})}$.
(ii) $\sigma(A) \subset \mathbb{R}$ *if and only if* A *is* J*-nonnegative.*

Proof. (i) Since $\sigma_{\mathrm{p}}(A) \cap \mathbb{R} = \emptyset$, Proposition 4.8(i) implies $\mathfrak{q}(0) \neq 0$. It follows from these facts that equality holds in [14, formula (1.3)]. Combining this and [46, Proposition II.2.1], we see that the degree $\deg \mathfrak{p}$ of polynomial $\mathfrak{p}(z) = z\mathfrak{q}(z)\overline{\mathfrak{q}(\bar{z})}$ is greater or equal than $2\kappa_-(JA)$. From this and (4.26), we obtain $\deg \mathfrak{p} = 2\kappa_-(JA) + 1$ and $\deg \mathfrak{q} = \kappa_-(JA)$. Applying the equality in [14, formula (1.3)] and [46, Proposition II.2.1] again, one gets statement (i).

(ii) For the case $JA \geqslant 0$, see Proposition 2.6. If A is not J-nonnegative, then $\kappa_-(JA) \geqslant 1$ and hence $\mathfrak{q}(\cdot) \not\equiv 1$. So $\mathfrak{q}(\cdot)$ has at least one zero λ_1, which is an eigenvalue of A due to statement (i) and is nonreal due to Proposition 4.3. $\square$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Note that the implication (ii) $\Rightarrow$ (i) follows from Theorem 4.5. The implication (i) $\Rightarrow$ (iii) is obvious. To complete the proof it suffices to mention that the equivalence (ii) $\Leftrightarrow$ (iii) was established in Corollary 4.10(ii). $\square$

Recall that the function $M_+(\cdot) - M_-(\cdot)$ is holomorphic in $\mathbb{C} \setminus \mathbb{R}$. The next result follows easily from Theorem 4.9(i) and Corollary 4.10(i).

Corollary 4.11. *Let* $A = (\operatorname{sgn} x)(-\mathrm{d}^2/\mathrm{d}x^2 + q(x))$ *and* $q(\cdot)$ *satisfy* (1.8). *Assume also that* A *is not* J*-nonnegative.*

(i) *Let* $\{z_j\}_1^n$ *be the set of nonreal zeros of the function* $M_+(\cdot) - M_-(\cdot)$, *and let* $\{k_j\}_1^n$ *be their multiplicities. Then* $\mathfrak{p} = z\prod_1^n (z - z_j)^{k_j}$ *is a definitizing polynomial of minimal degree for* A.
(ii) A *is similar to a normal operator if and only if* $k_j = 1$ *for all* $1 \leqslant j \leqslant n$.

Remark 4.12. Under the additional assumption $q \in L^1(\mathbb{R}, (1 + |x|^2)\,\mathrm{d}x)$, the equivalence (i) $\Leftrightarrow$ (iii) in Theorem 1.1 was proved in [18] by using another approach. Note also that inclusion $\sigma(A) \subset \mathbb{R}$ was established in [18, Corollary 4] under the assumption $m_\pm \in (S)$ (cf. Proposition 2.2 of the present paper).

5. Operators with decaying potentials and singular critical point 0

If $r(x) = \operatorname{sgn} x$, then the operator A defined by (1.2) is similar to a self-adjoint one whenever $L (= JA)$ is uniformly positive (see Proposition 2.7). If $0 \in \sigma_{\mathrm{ess}}(L)$, then it may occur that 0 is a critical point of A. Sturm–Liouville operators of type $-\frac{\mathrm{d}^2}{r(x)\,\mathrm{d}x^2}$ with the *singular* critical point 0 were constructed in [38]. A J-nonnegative operator of type $(\operatorname{sgn} x)(-\mathrm{d}^2/\mathrm{d}x^2 + q(x))$ with the singular critical point 0 have not been constructed explicitly, but existence of such an operator was proved in [38, Section 6.2]. The goal of this section is to construct an explicit example of such type. Our example also shows that condition (1.8) in Theorem 1.1 cannot be weaken to $q \in L^1(\mathbb{R}, (1+|x|)^\gamma\, \mathrm{d}x)$ with $\gamma < 1$.

5.1. Sharpness of condition (1.8) in Theorem 1.1

Lemma 5.1. *Let*

$$q_0(x) = -\chi_{[0,\pi/4]}(x) + 2\frac{\chi_{(\pi/4,+\infty)}(x)}{(1+x-\pi/4)^2}, \quad x \in \mathbb{R}_+. \tag{5.1}$$

Then the function

$$m_0(\lambda) = \frac{\sin(\pi\sqrt{\lambda+1}/4)/\sqrt{\lambda+1} + m_1(\lambda)\cos(\pi\sqrt{\lambda+1}/4)}{\cos(\pi\sqrt{\lambda+1}/4) - m_1(\lambda)\sqrt{\lambda+1}\sin(\pi\sqrt{\lambda+1}/4)}, \quad \lambda \in \mathbb{C}_+, \tag{5.2}$$

where

$$m_1(\lambda) = \frac{1-\mathrm{i}\sqrt{\lambda}}{1-\mathrm{i}\sqrt{\lambda}-\lambda}, \quad \lambda \in \mathbb{C}_+, \tag{5.3}$$

is the Titchmarsh–Weyl m-coefficient of the boundary value problem

$$-y''(x) + q_0(x)y(x) = \lambda y(x), \quad x \geqslant 0, \qquad y'(0) = 0. \tag{5.4}$$

Proof. Consider the Sturm–Liouville equation

$$-y''(x) + \frac{2}{(1+x)^2}y(x) = \lambda y(x), \quad x \geqslant 0. \tag{5.5}$$

It is easy to check that $f_1(x,\lambda) = \mathrm{e}^{\mathrm{i}\sqrt{\lambda}(x+1)}(\sqrt{\lambda} + \mathrm{i}/(x+1))$ solves (5.5) and $f_1(\cdot,\lambda) \in L^2(\mathbb{R})$ for $\lambda \in \mathbb{C}_+$. Further, $f_1(0,\lambda) = \mathrm{e}^{\mathrm{i}\sqrt{\lambda}}(\sqrt{\lambda}+\mathrm{i})$ and $f_1'(0,\lambda) = \mathrm{e}^{\mathrm{i}\sqrt{\lambda}}(-\sqrt{\lambda} + \mathrm{i}\lambda - \mathrm{i})$. By (2.7), we get that (5.3) is the Titchmarsh–Weyl m-coefficient of (5.5) associated with the Neumann boundary condition at zero.

Using (5.1), we obtain that the function

$$\begin{aligned} f_0(x,\lambda) = {} & \left(f_1(0,\lambda)\cos\left(\left(x-\frac{\pi}{4}\right)\sqrt{\lambda+1}\right) + f_1'(0,\lambda)\frac{\sin((x-\frac{\pi}{4})\sqrt{\lambda+1})}{\sqrt{\lambda+1}}\right)\chi_{[0,\frac{\pi}{4}]}(x) \\ & + f_1\left(x-\frac{\pi}{4},\lambda\right)\chi_{(\frac{\pi}{4},+\infty)}(x), \quad x \geqslant 0, \end{aligned} \tag{5.6}$$

is the Weyl solution of (5.4) for $\lambda \in \mathbb{C}_+$. To complete the proof, it remains to substitute (5.6) in (2.7). $\square$

Let us consider the indefinite Sturm–Liouville operator

$$A = (\operatorname{sgn} x)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q_0\big(|x|\big)\right), \quad \operatorname{dom}(A) = W_2^2(\mathbb{R}), \tag{5.7}$$

with q_0 defined by (5.1).

Theorem 5.2. *Let A be the operator defined by* (5.7) *and* (5.1). *Then*:

(i) *A is J-self-adjoint, J-nonnegative, and $\sigma(A) \subset \mathbb{R}$.*
(ii) 0 *is a simple eigenvalue of A, i.e., its algebraic multiplicity is* 1.
(iii) 0 *is a singular critical point of A.*
(iv) *A is not similar to a self-adjoint operator.*

Proof. (i) Note that q_0 is bounded on $\mathbb{R}$. Hence A is J-self-adjoint. Next, we show that the operator $L = JA = -\mathrm{d}^2/\mathrm{d}x^2 + q_0(|x|)$ is nonnegative. The potential is even, hence, by Lemma 5.1, $m_+(\lambda) = m_-(\lambda) = m_0(\lambda)$ (see (5.2)). It is easy to see that m_1 is a Krein–Stieltjes function, $m_1 \in (S)$, since it is analytic and positive on $(-\infty, 0)$. It is not difficult to see that the latter implies $m_0 \in (S)$. Proposition 2.2 yields $L \geqslant 0$. Hence $A = JL$ is J-nonnegative and, by Proposition 2.6, $\sigma(A) \subset \mathbb{R}$.

(ii) It is easily seen that $\lim_{\lambda\to 0} \lambda m_0(\lambda) = -\frac{6+\pi}{4} \neq 0$. So $\lambda = 0$ is the eigenvalue of the problem (5.4). Hence $c(x,0)\chi_+(x) \in L^2(\mathbb{R}_+)$. Furthermore, $q_0(|x|)$ is even, hence $c(x,0)\chi_-(x) \in L^2(\mathbb{R}_-)$ and $c(x,0) \in \ker L$. Since $s(x,0) \notin L^2(\mathbb{R})$, we get $\ker L = \{ac(\cdot,0)\colon a \in \mathbb{C}\}$. The equality $\ker A = \ker L$ implies $0 \in \sigma_{\mathrm{p}}(A)$.

By (5.2) and (5.3), we get

$$m_0(\lambda) = \frac{2}{k\lambda} + \frac{2\mathrm{i}}{k^2\sqrt{\lambda}} + O(\lambda), \quad \lambda \to 0,\ k := -\frac{6+\pi}{4}.$$

Further, $M_+(\cdot) = -M_-(-\cdot) = m_0(\cdot)$ since $m_+(\cdot) = m_-(\cdot) = m_0(\cdot)$. Hence,

$$\frac{\operatorname{Im}(M_+(\mathrm{i}y) + M_-(\mathrm{i}y))}{M_+(\mathrm{i}y) - M_-(\mathrm{i}y)} = \frac{\operatorname{Im} m_0(\mathrm{i}y)}{\operatorname{Re} m_0(\mathrm{i}y)} \approx \frac{-2/ky + \sqrt{2}/k^2\sqrt{y} + O(y)}{\sqrt{2}/k^2\sqrt{y} + O(y)} \approx \sqrt{\frac{2}{k^2 y}}, \tag{5.8}$$

as $y \to +0$. Noting that $\|(A_0 - \mathrm{i}y)^{-1}\| \leqslant 1/y$ and combining (3.4) with (2.9) and (5.8), after simple calculations we arrive at

$$\big\|(A - \mathrm{i}y)^{-1}\big\| \leqslant O\big(y^{-3/2}\big), \quad y \to +0.$$

Therefore, $\ker A = \ker A^2$. This completes the proof of (ii).

(iii) Combining (5.8) with Theorem 3.6(i), we conclude that 0 is a singular critical point of A.

(iv) follows from Proposition 2.5 and (iii). □

5.2. *On a question of B. Ćurgus*

It is known that infinity is a critical point of the operator (1.2). Moreover, the results of [14, 22,51,53,57] shows that the regularity of the critical point ∞ of a definitizable operator of type

(1.2) depends only on behavior of the weight function r in a neighborhood of its turning point (in our case, in a neighborhood of $x = 0$). At 6th Workshop on Operator Theory in Krein Spaces (TU Berlin, 2006), B. Ćurgus posed the following problem: *does the regularity of the critical point zero of a J-nonnegative operator of type* (1.2) *depend only on behavior of the coefficients q and r at infinity*?

Below we give the negative answer to this question.

Consider the operator

$$A_1 = (\operatorname{sgn} x)\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2\frac{\chi_{(\pi/4,+\infty)}(|x|)}{(1+|x|-\pi/4)^2}\right), \quad \operatorname{dom}(A) = W_2^2(\mathbb{R}).$$

It is easy to see that A_1 is J-self-adjoint and J-nonnegative since the potential is bounded and positive on $\mathbb{R}$. Arguing as in the proof of Lemma 5.1, we obtain that the corresponding Titchmarsh–Weyl m-coefficients are

$$M_+(\lambda) = -M_-(-\lambda) = m_2(\lambda) := \frac{\sin(\pi\sqrt{\lambda}/4)/\sqrt{\lambda} + m_1(\lambda)\cos(\pi\sqrt{\lambda}/4)}{\cos(\pi\sqrt{\lambda}/4) - m_1(\lambda)\sqrt{\lambda}\sin(\pi\sqrt{\lambda}/4)},$$

where $m_1(\cdot)$ is given by (5.3). Since $m_1(\lambda) = 1 + O(\sqrt{\lambda})$ as $\lambda \to 0$, we easily get $m_2(\lambda) = (1 + \pi/4 + O(\sqrt{\lambda}))$ as $\lambda \to 0$. Hence we obtain

$$\lim_{\mathbb{C}_+ \ni \lambda \to 0} \left| \frac{M_+(\lambda) + M_-(\lambda)}{M_+(\lambda) - M_-(\lambda)} \right| = \left| \frac{(1+\pi/4) - (1+\pi/4)}{(1+\pi/4) + (1+\pi/4)} \right| = 0 < \infty,$$

and, by Theorem 3.5, 0 is not a singular critical point of A_1.

On the other hand, the operator A considered in the previous subsection is an additive perturbation of A_1 by a potential with a compact support. However, 0 is a singular critical point of A due to Theorem 5.2(iii). Thus, *the regularity of the critical point zero of operator* (1.2) *depends not only on behavior of the weight function r, but also on local behavior of the potential q.*

6. Operators with periodic and almost-periodic potentials

Throughout this section we assume $r(x) = \operatorname{sgn} x$, so the operators L and A have the forms $L = -\mathrm{d}^2/\mathrm{d}x^2 + q(x)$ and $A = (\operatorname{sgn} x)L$. All the asymptotic formulas in this section are considered in $\mathbb{C}_+$.

6.1. The case of a periodic potential q

First, we consider the case of $\mathcal{T}$-periodic potential $q \in L^1_{\mathrm{loc}}(\mathbb{R})$, i.e., $q(x+\mathcal{T}) = q(x)$ a.e. on $\mathbb{R}$, $\mathcal{T} > 0$. It is known that in this case Eq. (2.5) is limit point at both $+\infty$ and $-\infty$. Hence, the maximal operator L corresponding to the differential expression $-\mathrm{d}^2/\mathrm{d}x^2 + q(x)$ is self-adjoint in $L^2(\mathbb{R})$.

Let $c(x,\lambda)$ and $s(x,\lambda)$ be the functions defined by (2.5), (2.6). Recall that for any $x \in \mathbb{R}$, $c(x,\lambda)$, $s(x,\lambda)$, $c'(x,\lambda)$, and $s'(x,\lambda)$ are entire functions of λ, hence so are

$$\Delta_+(\lambda) := \frac{c(\mathcal{T},\lambda) + s'(\mathcal{T},\lambda)}{2} \quad \text{and} \quad \Delta_-(\lambda) := \frac{c(\mathcal{T},\lambda) - s'(\mathcal{T},\lambda)}{2}. \tag{6.1}$$

The function $2\Delta_+(\cdot)$ is the trace of the monodromy matrix and it is called Hill's discriminant (or the Lyapunov function).

As before, we denote by $\widetilde{m}_\pm(\lambda)$ ($m_\pm(\lambda)$) the Titchmarsh–Weyl m-coefficient for (2.5) on $\mathbb{R}_\pm$ corresponding to the Dirichlet (Neumann, resp.) boundary condition at 0. Then (see [55, Section 21.2] and also [59, Section 12]),

$$\widetilde{m}_\pm(\lambda) = -\frac{1}{m_\pm(\lambda)} = \frac{\mp\Delta_-(\lambda) + \sqrt{\Delta_+^2(\lambda) - 1}}{s(\mathcal{T}, \lambda)}, \tag{6.2}$$

where the branch of the multifunction $\sqrt{\Delta_+^2(\lambda) - 1}$ is chosen such that both $\widetilde{m}_\pm$ (and so $m_\pm$) belong to the class (R).

Lemma 6.1. *Let L be a Sturm–Liouville operator with a $\mathcal{T}$-periodic potential $q \in L^1_{\mathrm{loc}}(\mathbb{R})$. Let also $\lambda_0 := \inf\sigma(L)$. Then:*

(i) $(-\infty <)$ λ_0 *is a first order zero of* $\Delta_+(\lambda) - 1$ *and* $\Delta'_+(\lambda_0) < 0$;
(ii) $s(\mathcal{T}, \lambda_0) > 0$.

This statement is well known for the case of continuous q (see [55, Section 21.4]). For the case $q \in L^1[0, \mathcal{T}]$, it can be obtained, e.g., from [59, Sections 12, 13].

Proof of Theorem 1.2. Consider the operator $L = -\mathrm{d}^2/\mathrm{d}x^2 + q(x)$ with a $\mathcal{T}$-periodic potential q and assume that $\lambda_0 (= \inf\sigma(L)) \geqslant 0$. It follows from (2.8) and (6.2) that Titchmarsh–Weyl m-coefficients for the operator $A = (\operatorname{sgn} x)L$ have the form

$$M_\pm(\lambda) = \frac{s(\mathcal{T}, \pm\lambda)}{\Delta_-(\pm\lambda) \mp \sqrt{\Delta_+^2(\pm\lambda) - 1}}. \tag{6.3}$$

By Proposition 2.7, ∞ is a regular critical point of A. At the same time, by Proposition 2.7, it suffices to consider only the case $\lambda_0 = 0$.

Assuming $\lambda_0 = 0$, consider two cases.

(a) Let $\Delta_-(0) = 0$. Lemma 6.1(i) yields that $\lambda_0 = 0$ is a first order zero of the entire function $\Delta_+(\lambda) - 1$. By Lemma 6.1(ii), $s(\mathcal{T}, 0) > 0$ and, therefore, (6.3) implies

$$M_\pm(\lambda) = \frac{s(\mathcal{T}, 0)(1 + O(\lambda))}{\pm\lambda(\Delta'_-(0) + O(\lambda)) \mp \sqrt{\pm\lambda(2\Delta'_+(0) + O(\lambda))}} \approx \pm\mathrm{i}\frac{C_1}{\sqrt{\pm\lambda}}, \tag{6.4}$$

as $\lambda \to 0$, where $C_1 = s(\mathcal{T}, 0)/\sqrt{-2\Delta'_+(0)} > 0$. Substituting (6.4) for $M_\pm(\cdot)$ in (3.5) with $c = 0$, we see that Theorem 3.5 implies that 0 is not a singular critical point of A.

(b) Suppose $\Delta_-(0) \neq 0$. Note that $\Delta_-(\lambda)$ and $\Delta_+(\lambda)$ are real if $\lambda \in \mathbb{R}$. Combining (6.3) with Lemma 6.1(ii), we get

$$M_\pm(\lambda) = \frac{s(\mathcal{T}, 0)}{\Delta_-(0) \mp \mathrm{i}C_2\sqrt{\pm\lambda}}\big[1 + O(\sqrt{\lambda})\big], \quad \lambda \to 0, \tag{6.5}$$

with $C_2 = \sqrt{-2\Delta_+'(0)} > 0$. Using Theorem 3.5 with $c = 2s(\mathcal{T}, 0)/\Delta_-(0) \in \mathbb{R} \setminus \{0\}$, we see that 0 is not a singular critical point of A.

Thus the operator A is J-nonnegative and has no singular critical points. Moreover, $\ker A = \ker L$, and $\ker L = \{0\}$ since q is $\mathcal{T}$-periodic (see e.g. [59, Section 12]). Proposition 2.5 completes the proof of Theorem 1.2. □

Remark 6.2. Let $\mathcal{T}$-periodic functions p, q, and ω be such that $1/p, q, \omega \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $p, \omega > 0$ a.e. on $\mathbb{R}$. Then the operator

$$(Ly)(x) := \frac{1}{\omega(x)}\bigl(-\bigl(p(x)y'(x)\bigr)' + q(x)y(x)\bigr)$$

defined on the maximal domain in $L^2(\mathbb{R}, \omega\,\mathrm{d}x)$ is self-adjoint and semi-bounded from below. Moreover, arguing as in the proof of Theorem 1.2 one can show that 0 *is not a singular critical point of the operator* $A = (\operatorname{sgn} x)L$ *whenever* L *is nonnegative*. If additionally the critical point ∞ is regular, then A *is similar to a self-adjoint operator*. For instance (see [14]), the latter holds if $p, 1/p \in L^\infty(-\delta, \delta)$ for certain $\delta > 0$ and $r(x) = (\operatorname{sgn} x)\omega(x)$ satisfies the conditions of Proposition 2.7(i).

6.2. *Infinite-zone and finite-zone potentials*

In this subsection we consider the cases of (real) infinite- and finite-zone potentials. Following [47], we briefly recall definitions. First note that the spectrum of the operator $L = -\mathrm{d}^2/\mathrm{d}x^2 + q(x)$ with an infinite-zone potential q is absolutely continuous and has the zone structure, i.e.,

$$\sigma(L) = \sigma_{\mathrm{ac}}(L) = \bigl[\mu_0^r, \mu_1^l\bigr] \cup \bigl[\mu_1^r, \mu_2^l\bigr] \cup \cdots, \tag{6.6}$$

where $\{\mu_j^r\}_0^\infty$ and $\{\mu_j^l\}_{j=1}^\infty$ are sequences of real numbers such that

$$\mu_0^r < \mu_1^l < \mu_1^r < \cdots < \mu_{j-1}^r < \mu_j^l < \mu_j^r < \cdots, \tag{6.7}$$

and

$$\lim_{j\to\infty} \mu_j^r = \lim_{j\to\infty} \mu_j^l = +\infty.$$

In the case of a finite-zone potential, the corresponding sequences $\{\mu_j^r\}_0^N$, $\{\mu_j^l\}_{j=1}^N$ are finite, $N < \infty$, the spectrum of L is also absolutely continuous and is given by

$$\sigma(L) = \sigma_{\mathrm{ac}}(L) = \bigl[\mu_0^r, \mu_1^l\bigr] \cup \bigl[\mu_1^r, \mu_2^l\bigr] \cup \cdots \cup \bigl[\mu_N^r, +\infty\bigr). \tag{6.8}$$

Let $N \in \mathbb{Z}_+$. Consider also sets of real numbers $\{\xi_j\}_1^N$ and $\{\epsilon_j\}_1^N$ such that $\xi_j \in [\mu_j^l, \mu_j^r]$ and $\epsilon_j \in \{-1, +1\}$ for all $j \leqslant N$. Define polynomials $R(\lambda)$, $P(\lambda)$, and $Q(\lambda)$ by

$$P(\lambda) = \prod_{j=1}^N (\lambda - \xi_j), \qquad R(\lambda) = \bigl(\lambda - \mu_0^r\bigr)\prod_{j=1}^N \bigl(\lambda - \mu_j^l\bigr)\bigl(\lambda - \mu_j^r\bigr), \tag{6.9}$$

$$Q(\lambda) = P(\lambda) \sum_{j=1}^{N} \frac{\epsilon_j \sqrt{-R(\xi_j)}}{P'(\xi_j)(\lambda - \xi_j)}. \tag{6.10}$$

Then there exists (see [47, Lemma 8.1.1]) a real polynomial $S(\lambda)$ of degree $\deg S = N + 1$ such that

$$S(\lambda) = \prod_{j=0}^{N} (\lambda - \tau_j), \quad \tau_0 \in \left(-\infty, \mu_0^r\right], \ \tau_j \in \left[\mu_j^l, \mu_j^r\right], \ j \in \{1, \dots, N\}, \tag{6.11}$$

and the following identity holds

$$P(\lambda)S(\lambda) - Q^2(\lambda) = R(\lambda). \tag{6.12}$$

According to [47, formulas (8.1.9–10)] the functions

$$m_{\pm}(\lambda) := \pm \frac{P(\lambda)}{Q(\lambda) \mp \mathrm{i}\sqrt{R(\lambda)}} \tag{6.13}$$

are the Titchmarsh–Weyl m-coefficients corresponding to the Neumann boundary value problems on $\mathbb{R}_{\pm}$ for some Sturm–Liouville operator $L = -\mathrm{d}^2/\mathrm{d}x^2 + q(x)$ with a *quasi-periodic* potential $q = \bar{q}$ (see e.g. [47, Section 10.3]). Here the multifunction $\sqrt{R(\cdot)}$ is considered on $\mathbb{C}$ with cuts along the union of intervals (6.8). The branch $\sqrt{R(\cdot)}$ of the multifunction is chosen in such a way that $\sqrt{R(\lambda_0 + \mathrm{i}0)} > 0$ for some $\lambda_0 \in (\mu_N^r, +\infty)$. So both $m_{\pm}$ belong to the class (R). In this case the spectrum of L is given by (6.8).

Definition 6.3. (See [47].) A real quasi-periodic potential q is called finite-zone if the Titchmarsh–Weyl m-coefficients $m_{\pm}$ admit the representations (6.13).

Note that if the potential q is $\mathcal{T}$-periodic and the equation $\Delta_+^2(\lambda) = 1$ (see (6.1)) has a finite number of simple roots, then q is a finite-zone potential (see [47, Sections 7.4 and 8.1]). Moreover, in this case μ_j^r and μ_j^l denote simple roots of $\Delta_+^2(\lambda) = 1$ listed in the natural order. Note also that every finite-zone potential q is bounded and its nth derivative $\frac{\mathrm{d}^n}{\mathrm{d}x^n} q$ is bounded on $\mathbb{R}$ for any $n \in \mathbb{N}$ (see [47, Section 8.3]).

A criterion of the similarity to a self-adjoint operator for (not necessary J-nonnegative) operator $A = (\operatorname{sgn} x)L$ with a finite-zone potential was obtained in [42, Theorems 7.1 and 7.2]. For the case of a J-nonnegative operator A, we present a new simple proof of [42, Corollary 7.4] based on Theorem 3.5.

Theorem 6.4. *(See [42].) Let $q(x)$ be a finite-zone potential and $\mu_0^r \geqslant 0$. Then $A = (\operatorname{sgn} x) \times (-\mathrm{d}^2/\mathrm{d}x^2 + q(x))$ is similar to a self-adjoint operator.*

Proof. Consider the operator $L = -\mathrm{d}^2/\mathrm{d}x^2 + q(x)$ with a finite-zone potential q and assume that $L \geqslant 0$. This is equivalent to $\mu_0^r \geqslant 0$ due to (6.8).

Combining (2.8) with (6.13) and (6.12), we get

$$M_{\pm}(\lambda) = \frac{P(\pm\lambda)}{Q(\pm\lambda) \mp \mathrm{i}\sqrt{R(\pm\lambda)}} = \frac{Q(\pm\lambda) \pm \mathrm{i}\sqrt{R(\pm\lambda)}}{S(\pm\lambda)}. \tag{6.14}$$

It is easy to see that

$$M_{\pm}(\lambda) = \pm\frac{\mathrm{i}}{\sqrt{\pm\lambda}}\bigl[1 + O\bigl(\lambda^{-1/2}\bigr)\bigr], \quad \lambda \to \infty,\ \lambda \in \mathbb{C}_+. \tag{6.15}$$

This implies that the function $(M_+ + M_-)(M_+ - M_-)^{-1}$ is bounded in a certain neighborhood of ∞. So ∞ is a regular critical point due to Theorem 3.5.

Let us prove that 0 is not a singular critical point. As in the periodic case, we note that 0 is not a critical point if $\mu_0^r > 0$. Further, assume that $\mu_0^r = 0$ and consider the cases analogous to that of the proof of Theorem 1.2.

(a) Let $\tau_0 = 0$ $(= \mu_0^r)$, where τ_0 is defined in (6.11). Then $R(0) = S(0) = 0$, and it follows from (6.12) that $Q(0) = 0$. By definition, $P(0) = P(\mu_0^r) \neq 0$ and, therefore, (6.13) implies that (6.4) holds with $C_1 = \frac{\prod_{j=1}^N \xi_j}{(\prod_{j=1}^N \mu_j^l \mu_j^r)^{-1/2}} > 0$.

(b) Let $\tau_0 \neq 0$ (actually, this yields $\tau_0 < 0$, see (6.11)). Then $S(0) \neq 0$. Further, $R(0) = 0$, $P(0) \neq 0$ and (6.12) implies that $Q(0) \neq 0$. Using the second representation of $M_{\pm}(\lambda)$ from (6.14), one can check that

$$M_{\pm}(\lambda) = C_2 \pm \mathrm{i}C_3\sqrt{\pm\lambda} + o\bigl(|\lambda|^{1/2}\bigr), \quad \lambda \to 0, \tag{6.16}$$

where $C_2 = Q(0)/S(0) \in \mathbb{R} \setminus \{0\}$ and $C_3 = |C_1/S(0)| > 0$.

The arguments of Section 6.1 conclude the proof. □

In the proof of Theorem 6.4, we have shown that ∞ is a regular critical point of A using the asymptotic formula (6.15) for $M_{\pm}$ and the regularity condition, Theorem 3.5. On the other hand, this fact follows from Proposition 2.7.

Now consider infinite sequences $\{\mu_j^r\}_0^\infty$, $\{\mu_j^l\}_1^\infty$, $\{\xi_j\}_1^\infty$, and $\{\epsilon_j\}_1^\infty$ such that assumptions (6.7) are fulfilled,

$$\sum_{j=1}^{\infty} \mu_j^r\bigl(\mu_j^r - \mu_j^l\bigr) < \infty, \qquad \sum_{j=1}^{\infty} \frac{1}{\mu_j^l} < \infty, \tag{6.17}$$

and $\xi_j \in [\mu_j^l, \mu_j^r]$, $\epsilon_j \in \{-1, +1\}$ for all $j \geqslant 1$. For every $N \in \mathbb{N}$, put

$$g_N = \prod_{j=1}^{N} \frac{\xi_j - \lambda}{\mu_j^l}, \qquad f_N = \bigl(\lambda - \mu_0^r\bigr)\prod_{j=1}^{N} \frac{\lambda - \mu_j^l}{\mu_j^l}\frac{\lambda - \mu_j^r}{\mu_j^l}, \tag{6.18}$$

$$k_N(\lambda) = g_N(\lambda)\sum_{j=1}^{N} \frac{\epsilon_j\sqrt{-f_N(\xi_j)}}{g_N'(\xi_j)(\lambda - \xi_j)}, \qquad h_N(\lambda) = \frac{f_N(\lambda) + k_N^2(\lambda)}{g_N(\lambda)}. \tag{6.19}$$

It is easy to see from (6.17) that g_N and f_N converge uniformly on every compact subset of $\mathbb{C}$. Denote $\lim_{N\to\infty} g_N(\lambda) =: g(\lambda)$, $\lim_{N\to\infty} f_N(\lambda) =: f(\lambda)$. [47, Theorem 9.1.1] states that there exist limits $\lim_{N\to\infty} h_N(\lambda) =: h(\lambda)$, $\lim_{N\to\infty} k_N(\lambda) =: k(\lambda)$ for all $\lambda \in \mathbb{C}$. Moreover, the functions g, f, h, and k are holomorphic in $\mathbb{C}$.

It follows from [47, Section 9.1.2] that the functions

$$m_\pm(\lambda) := \pm\frac{g(\lambda)}{k(\lambda) \mp \mathrm{i}\sqrt{f(\lambda)}} \tag{6.20}$$

are the Titchmarsh–Weyl m-coefficients on $\mathbb{R}_\pm$ (corresponding to the Neumann boundary conditions) for some Sturm–Liouville operator $L = -\mathrm{d}^2/\mathrm{d}x^2 + q(x)$ with a real bounded potential $q(\cdot)$. The branch $\sqrt{f(\cdot)}$ of the multifunction is chosen such that both $m_\pm$ belong to the class (R).

Definition 6.5. (See [47].) A real potential q is called an infinite-zone potential if the Titchmarsh–Weyl m-coefficients $m_\pm$ admit representations (6.20).

Let q be an infinite-zone potential defined as above. Since q is bounded, the operator L is self-adjoint. Its spectrum is given by (6.6). B. Levitan proved that under the additional condition $\inf(\mu^l_{j+1} - \mu^l_j) > 0$, the potential q is *almost-periodical* (see [47, Chapter 11]). Note that for a $\mathcal{T}$-periodic potential q the first inequality in (6.17) implies $q \in W_2^2[0, \mathcal{T}]$, and the second inequality in (6.17) obviously follows from asymptotic formulas for the periodic (anti-periodic) eigenvalues (see [49, Section 1.5] for details).

The following theorem is the main result of this subsection.

Theorem 6.6. *Let $L = -\mathrm{d}^2/\mathrm{d}x^2 + q(x)$ be a Sturm–Liouville operator with an infinite-zone potential q. Assume also that the spectrum $\sigma(L)$ satisfies* (6.17) *and $L \geqslant 0$ (i.e., $\mu^r_0 \geqslant 0$). Then the operator $A = (\operatorname{sgn} x)L$ is similar to a self-adjoint operator.*

The asymptotic formula (6.15) does not hold true in the infinite-zone case. Therefore, we use Proposition 2.7 to prove that ∞ is a regular critical point. The rest of the proof is also close to Section 6.1.

Proof. It is sufficient to consider the case $\mu^r_0 = 0$. Recall that the functions g, f, k, and h defined above are holomorphic in $\mathbb{C}$. Moreover, g and f admit the following representations

$$g(\lambda) = \prod_{j=1}^{\infty} \frac{\xi_j - \lambda}{\mu^l_j}, \qquad f(\lambda) = \lambda \prod_{j=1}^{N} \frac{\lambda - \mu^l_j}{\mu^l_j} \frac{\lambda - \mu^r_j}{\mu^l_j},$$

where the infinite products converge uniformly on all compact subsets of $\mathbb{C}$ due to assumptions (6.17) (see [47, Section 9]). From this and $\xi_j > \mu^r_0 = 0$, $j \in \mathbb{N}$, we see that

$$f(0) = 0, \qquad g(0) \neq 0. \tag{6.21}$$

It follows from (6.19) that

$$h_N(\lambda)g_N(\lambda) - k_N^2(\lambda) = f_N(\lambda) \quad \text{and} \quad h(\lambda)g(\lambda) - k^2(\lambda) = f(\lambda). \tag{6.22}$$

As above, the latter yields

$$M_\pm(\lambda) = \frac{g(\pm\lambda)}{k(\pm\lambda) \mp \mathrm{i}\sqrt{f(\pm\lambda)}} = \frac{k(\pm\lambda) \pm \mathrm{i}\sqrt{f(\pm\lambda)}}{h(\pm\lambda)}. \tag{6.23}$$

(a) Let $k(0)=0$. Then (6.21) and the first equality in (6.23) yield that (6.4) holds with $C_1=(\prod_{j=1}^{\infty}\xi_j)(\prod_{j=1}^{\infty}\mu_j^l\mu_j^r)^{-1/2}>0$ (as above, the product converges due to (6.17)).

(b) Let $k(0)\neq 0$. Then (6.22) and (6.21) yield $h(0)\neq 0$. Using the second representation of $M_\pm(\lambda)$ from (6.23), we get (6.16) with the constants $C_2=k(0)/h(0)\in\mathbb{R}$, $C_3=|C_1/h(0)|>0$.

Theorem 3.5 and Proposition 2.7 complete the proof. $\square$

If the potential q is periodic or finite(infinite)-zone and $\inf\sigma(L)=0$, it is easy to show that 0 *is* a critical point of A. So we have proved that 0 is *a regular critical point* in these cases.

7. Operators with nontrivial weights

In this section, we consider the J-self-adjoint operator A of the type (1.2) assuming that $q\equiv 0$. In this case assumption (1.4) is fulfilled if and only if $x\notin L^2(\mathbb{R}_\pm,|r(x)|\,\mathrm{d}x)$. In the following $\omega(\cdot)$ stands for $|r(\cdot)|$. Let us denote the corresponding operator by

$$A_\omega:=-\frac{(\operatorname{sgn}x)}{\omega(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2},\quad \operatorname{dom}(L_\omega)=\mathfrak{D}. \tag{7.1}$$

Note that the operator A_ω is J-nonnegative. Hence, by Proposition 2.6, the spectrum of A_ω is real, $\sigma(A_\omega)\subset\mathbb{R}$.

The main aim of this section is to prove Theorem 1.3. But first we need two preparatory lemmas.

Consider the spectral problem

$$-y''(x)=\lambda x^\alpha y(x),\quad x\geqslant 0,\qquad y'(0)=0, \tag{7.2}$$

with $\alpha>-1$. Denote by $z^{1/(2+\alpha)}$, $z\in\mathbb{C}\setminus\mathbb{R}_+$, the branch of the multifunction with cut along $\mathbb{R}_+$ such that $(-1)^{1/(2+\alpha)}=\mathrm{e}^{\mathrm{i}\pi/(2+\alpha)}$.

Lemma 7.1. *(See [17].) Let $\alpha>-1$. Then the function*

$$m_\alpha(\lambda):=C_\nu\mathrm{e}^{\mathrm{i}\pi\nu}\lambda^{-\nu},\quad \lambda\in\mathbb{C}_+,\ \nu=\frac{1}{2+\alpha},\ C_\nu:=\frac{\Gamma(1+\nu)}{\nu^{2\nu}\Gamma(1-\nu)}, \tag{7.3}$$

is the Titchmarsh–Weyl m-coefficient of the problem (7.2). *Here $\Gamma(\cdot)$ is the classical Γ-function.*

This result was obtained in [17] using an explicit form of the Weyl solution of Eq. (7.2) (see [32, Part III, Eq. (2.162)(1a)]). A different and simpler proof of Lemma 7.1 was given in [44] (but without computing C_ν).

As a corollary of Lemma 7.1, we obtain a simple proof of [24, Theorem 2.7].

Theorem 7.2. *(See [24].) If $\omega(x)=|x|^\alpha$, $\alpha>-1$, then A_ω is similar to a self-adjoint operator in $L^2(\mathbb{R},\omega\,\mathrm{d}x)$.*

Proof. The operator $A_{|x|^\alpha}$ is J-self-adjoint since $x\notin L^2(\mathbb{R}_\pm,|x|^\alpha\,\mathrm{d}x)$. By Lemma 7.1, we have $M_+(\cdot)=-M_-(-\cdot)=m_\alpha(\cdot)$. Hence,

$$\frac{M_+(\lambda)+M_-(\lambda)}{M_+(\lambda)-M_-(\lambda)}=\frac{1+\exp\{\mathrm{i}\pi\nu\}}{1-\exp\{\mathrm{i}\pi\nu\}},\quad \lambda\in\mathbb{C}_+. \tag{7.4}$$

By Theorem 3.4, $A_{|x|^\alpha}$ is similar to a self-adjoint operator. $\square$

Lemma 7.3. *Let $\alpha > -1$ and let p be a positive function satisfying* (1.9) *on $\mathbb{R}_+$ with $\alpha_+ = \alpha$ and certain $c_+ > 0$. Let $m_+(\cdot)$ be the Titchmarsh–Weyl m-coefficient of the problem*

$$-y''(x) = \lambda p(x)|x|^{\alpha} y(x), \quad x > 0, \qquad y'(0) = 0. \tag{7.5}$$

Then

$$m_+(\lambda) = C_\nu e^{i\pi\nu}(c_+\lambda)^{-\nu}\big(1 + o(1)\big), \quad \lambda \to 0, \ \nu = 1/(2+\alpha).$$

Proof. Without loss of generality it can be assumed that $c_+ = 1$.

Let $f_p(x,\lambda)$ denote the Weyl solution of (7.5) for $\lambda \in \mathbb{C}_+$ (we will write $f_1(x,\lambda)$ if $p \equiv 1$). It is known (see [32, Part III, Eq. (2.162)(1a)]) that the general solution of (7.2) is

$$y(x,\lambda) = c_1\sqrt{x}H_\nu^{(1)}\big(2\nu\sqrt{\lambda}x^{1/2\nu}\big) + c_2\sqrt{x}H_\nu^{(2)}\big(2\nu\sqrt{\lambda}x^{1/2\nu}\big),$$

where $c_j \in \mathbb{C}$ and $H_\nu^{(j)}(\cdot)$ are the Hankel functions (see [58, Section 3.6]). Moreover (see [58, Section 7.2]), if $-\pi + \delta \leqslant \arg z \leqslant \pi - \delta$, $\delta > 0$, then

$$H_\nu^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{i(z - \nu\pi/2 - \pi/4)}\big(1 + O\big(z^{-1}\big)\big), \quad |z| \to \infty, \tag{7.6}$$

$$H_\nu^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{-i(z - \nu\pi/2 - \pi/4)}\big(1 + O\big(z^{-1}\big)\big), \quad |z| \to \infty. \tag{7.7}$$

Note that (7.6)–(7.7) implies $f_1(x,\lambda) = \sqrt{x}H_\nu^{(1)}(2\nu\sqrt{\lambda}x^{1/2\nu})$ and

$$W\big(H_\nu^{(1)}(z), H_\nu^{(2)}(z)\big) = -\frac{4i}{\pi z}, \tag{7.8}$$

where W is the Wronskian $W(f,g)(x) := f(x)g'(x) - f'(x)g(x)$.

Let us consider the Green function

$$G(x,t;\lambda) = \varphi_1(t,\lambda)f_1(x,\lambda) - \varphi_1(x,\lambda)f_1(t,\lambda).$$

Here $\varphi_1 = c_2(\lambda)\sqrt{x}H_\nu^{(2)}(2\nu\sqrt{\lambda}x^{1/2\nu})$ and c_2 is chosen such that $W(f_1,\varphi_1) \equiv 1$ (cf. (7.8)). Using (7.7)–(7.8), after straightforward calculations we obtain

$$\left|G(x,t;\lambda)\frac{f_1(t;\lambda)}{f_1(x;\lambda)}\right| \leqslant C_1\left|\frac{\lambda^{-1/2}}{(t^{\alpha/2}+1)}\big(1 - \exp\big\{i\lambda^{1/2}2\nu\big(|t|^{1/2\nu} - |x|^{1/2\nu}\big)\big\}\big)\right|.$$

Hence, for $0 < x < t$,

$$\left|G(x,t;\lambda)\frac{f_1(t;\lambda)}{f_1(x;\lambda)}\right| \leqslant 2C_2\left|\frac{\lambda^{-1/2}}{t^{\alpha/2}+1}\right|. \tag{7.9}$$

Consider the following integral equation

$$y_p(x;\lambda) = f_1(x;\lambda) + \lambda \int_x^{+\infty} |t|^\alpha \big(p(t)-1\big) G(x,t;\lambda) y_p(t;\lambda)\,\mathrm{d}t. \tag{7.10}$$

Using a standard technique (see, for example, [49, Section 3.1]), one can show that (7.9) and (1.9) imply that the solution of (7.10) exists and is the Weyl solution of (7.5). Denoting $y_p(x,\lambda) = f_1(x,\lambda)\widetilde{y_p}(x,\lambda)$ in (7.10), one gets

$$\widetilde{y_p}(x;\lambda) = 1 + \lambda \int_x^{+\infty} |t|^\alpha \big(p(t)-1\big) G(x,t;\lambda) \frac{f_1(t;\lambda)}{f_1(x;\lambda)} \widetilde{y_p}(t,\lambda)\,\mathrm{d}t. \tag{7.11}$$

Combining (7.11) with (7.9) and (1.9), we arrive at

$$f_p(0,\lambda) = f_1(0,\lambda)\big(1+o(1)\big), \quad \lambda \to 0. \tag{7.12}$$

Analogously one obtains

$$f_p'(0,\lambda) = f_1'(0,\lambda)\big(1+o(1)\big), \quad \lambda \to 0. \tag{7.13}$$

Combining (7.12)–(7.13) with (2.7) and Lemma 7.1, we complete the proof. □

Proof of Theorem 1.3(i). By (1.9) and Lemma 7.3, we obtain

$$M_+(\lambda) = m_+(\lambda) = C_{\nu_+} \mathrm{e}^{\mathrm{i}\pi\nu_+} (c_+\lambda)^{\nu_+}\big(1+o(1)\big),$$
$$M_-(\lambda) = -m_-(-\lambda) = C_{\nu_-} (c_-\lambda)^{\nu_-}\big(1+o(1)\big), \quad \lambda \to 0,\ \lambda \in \mathbb{C}_+,$$

where $\nu_\pm = 1/(2+\alpha_\pm)$ and $c_\pm > 0$. Therefore,

$$\frac{M_+(\lambda)+M_-(\lambda)}{M_+(\lambda)-M_-(\lambda)} \approx \frac{C_{\nu_+}\mathrm{e}^{\mathrm{i}\pi\nu_+}(c_+\lambda)^{\nu_+} - C_{\nu_-}(c_-\lambda)^{\nu_-}}{C_{\nu_+}\mathrm{e}^{\mathrm{i}\pi\nu_+}(c_+\lambda)^{\nu_+} + C_{\nu_-}(c_-\lambda)^{\nu_-}}, \quad \lambda \to 0.$$

Hence $(M_+(\lambda)+M_-(\lambda))(M_+(\lambda)-M_-(\lambda))^{-1}$ is bounded in a neighborhood of 0. Thus, by Theorem 3.5, 0 is not a singular critical point of A_ω.

(ii) Condition (1.9) implies that $\omega \notin L^1(\mathbb{R}_\pm)$. Hence $\ker A_\omega = \{0\}$. Combining (i) with Propositions 2.7 and 2.5, we obtain the similarity of A_ω to a self-adjoint operator. □

Remark 7.4. M.M. Faddeev and R.G. Shterenberg proved the similarity of A_ω to a self-adjoint operator under additional assumptions (see [19, Theorem 7])

(a) $\omega(x) = p(x)x^\alpha$ $(\alpha > -1)$, where $p, p' \in AC_{\mathrm{loc}}(\mathbb{R})$ and $0 < c \leqslant p(x) \leqslant C < \infty$, $x \in \mathbb{R}$;

(b)
$$\int_{\mathbb{R}} \frac{|3(u''(x))^2/u'(x) - 2u'''(x)|}{(u'(x))^2(|u(x)|^{|\alpha|/2}+1)^{\operatorname{sgn}\alpha}}\,\mathrm{d}x < \infty, \tag{7.14}$$

where u is defined by $u(x) := (\operatorname{sgn} x)(|\frac{\alpha+2}{2}\int_0^x \sqrt{p(t)}|t|^{\alpha/2}\,\mathrm{d}t|)^{2/(2+\alpha)}$.

Note that in this case it is rather difficult to apply Theorem 3.4 because under the assumptions of Proposition 2.7 asymptotic behavior of M_+ and M_- at infinity is more complicated. We avoid these difficulties by applying the spectral theory of J-nonnegative operators.

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