Paracompactness of spaces which have covering properties weaker than paracompactness

Süleyman Önal

Middle East Technical University, Department of Mathematics, 06531 Ankara, Turkey

Received 3 August 1999; received in revised form 9 December 1999

Abstract

We prove that (i) a collectionwise normal, orthocompact, $\theta_{m}$-refinable, $[m, \aleph_{0}]$-submetacompact space is paracompact, (ii) a collectionwise normal, $[\infty, m]$-paracompact $[m, \aleph_{0}]$-submetacompact space is paracompact. This gives a sufficient condition for the paracompactness of para-Lindelöf, collectionwise normal spaces. © 2001 Elsevier Science B.V. All rights reserved.

AMS classification: 54B10; 54D20

Keywords: Orthocompact; Collectionwise normal; Paracompactness degree; $[m, n]$-paracompact

Introduction

It is well known that a subparacompact, collectionwise normal space is paracompact. The method of proof of this fact cannot be applied directly to a meta-Lindelöf or even para-Lindelöf collectionwise normal space for obtaining paracompactness of such a space, since there is no way of obtaining a discrete partial refinement of a given open cover when the cardinality of elements of the cover which contains a point is not finite. In fact, this case (para-Lindelöf + collectionwise normal $\Rightarrow$ paracompact) appears to be an open problem in [1]. But there is an example of existence of a nonparacompact, meta-Lindelöf, collectionwise normal space [2], under the assumption $V = L$. Here we give a sufficient condition for paracompactness of such a space as a corollary of our main results. For getting these results, we investigate the order type of the sets, $\{p \mid p \in O_{i}, O_{i} \in O\}$ and $\{i \mid A \cap O_{i} \neq \emptyset, O_{i} \in O\}$ instead of the cardinalities of these sets where $O$ is an open cover of a topological space which is indexed by some ordinal, $p$ is a point in the space and $A$ is a subset of the space. This enables us to obtain discrete collections whose elements are...
covered by a subfamily of $\mathcal{O}$, which has cardinality less than the cardinality of $\mathcal{O}$. After that, induction and collectionwise normality will lead us to the desired conclusion.

Let $X$ be a topological space, $m, n$ be infinite cardinals. We recall that $X$ is $[m, n]$-paracompact if each open cover of $X$ with cardinality at most $m$ has a locally $< n$ open refinement. $[m, n]$-metacompactness and $[m, n]$-subparacompactness are defined analogously. If $X$ is $[m, n]$-paracompact for each infinite cardinal $m$ then $X$ is called $[\omega, \omega]$-paracompact, as usual. If each open cover of $X$ has an open refinement which is the an union of $m$ discrete subfamilies then we write $\text{pa}(X) \leq m$. We say that $X$ is $\theta_m$-refinable if each open cover $\mathcal{O}$ of $X$ has $m$-refinements $\{V_i \mid i \in m\}$ such that for each $x \in X$ there is $i \in m$ with $|\{V \mid x \in V, \ V \in V_i\}| < m$.

1. Order type and local order type of a point

Let $X$ be a set and $\mathcal{O} = \{O_i \mid i \in \alpha\}$ be a collection of subsets of $X$ where $\alpha$ is an ordinal. For a subset $A$ of $X$ and $x \in X$ we define order type of $A$ in $\mathcal{O}$, $\text{ot}(A, \mathcal{O})$, as the unique ordinal which is isomorphic to the well ordered set $\mathcal{O}$ if each open cover $\mathcal{U}$ of $X$ has $\mathcal{U}$-refinements $\{\mathcal{V}_i \mid i \in \omega\}$ such that when $x \in X$, $\mathcal{V}$ is not isomorphic to an ordinal less than $\mathcal{O}$, $\mathcal{O}$ is an increasing family $\mathcal{O}$, $\mathcal{O}_1 \in \mathcal{O}$ and $|j > i \mid p \in O_j$, $O_j \in \mathcal{O}|$ is finite}, $Y_\beta(\mathcal{O}) = \{p \in Y_\beta \mid i = \{j < i \mid U \in O_j \neq \emptyset, O_j \in \mathcal{O}\}$ for each neighborhood $U$ of $p$ and $|j > i \mid W \cap O_j \neq \emptyset, O_j \in \mathcal{O}|$ is finite for some neighborhood $W$ of $p$). If there is no confusion we do not refer the $\mathcal{O}$ in $X_\beta(\mathcal{O})$ and the others.

Now we have two lemmas concerning $\text{ot}(x, \mathcal{O})$ and $\text{lot}(x, \mathcal{O})$, respectively, for which some of the results for the finite case are already known and used for proving several theorems, such as “collectionwise normal submetacompact spaces are paracompact”.

**Lemma 1.** Let $X$ be a topological space, $\lambda$ be an ordinal, $\mathcal{O} = \{O_i \mid i \in \beta\}$ be an open cover such that $\bigcap\{O_i \mid i \in L\}$ is a neighborhood of $x$ when $\emptyset \neq L \subset \{i \mid x \in O_i\}$ and $L$ is isomorphic to an ordinal less than $\lambda$. Then:

(i) $\bigcup_{\alpha \in \lambda} X_\alpha$ is a closed subset of $X$ for each $\alpha \in \lambda$.
(ii) The family $\{X'_\alpha \mid i \in A_\alpha\}$ is a pairwise disjoint open cover of the subspace $X_\alpha$.
(iii) $\{X'_\alpha \mid W \mid i \in A_\alpha\}$ is a discrete closed collection in $X$ when $W$ is an open subset of $X$ such that $\bigcup_{\alpha \in \lambda} X_\alpha \subset W$.
(iv) $X = \bigcup_{\alpha \in \lambda} X_\alpha$ when $\text{ot}(x, \mathcal{O}) < \lambda$ for each $x \in X$.
(v) $1 \leq \text{ot}(x, \{O_{\gamma} \mid i \in \gamma\}) < \omega$ for each $x \in X_{\alpha}$, when $\alpha$ is a successor ordinal.
(vi) $\{O_{\gamma} \mid j \in \gamma < i\}$ is an open cover for each $X_{\alpha}$, for each $j < i$ when $\alpha$ is an infinite ordinal.
Proof. To show (i), let \( p \notin \bigcup_{i \in A} X_i \). Then \( \text{ot}(p, \mathcal{O}) \geq \alpha \). Find \( L \subset \{ j \mid p \in O_j, \ O_j \in \mathcal{O} \} \) such that \( L \) is isomorphic to \( \alpha \). Hence \( \bigcap \{ O_i \mid i \in L \} \) is a neighborhood of \( p \) which is disjoint with \( \bigcup_{i \in A} X_i \), so \( \bigcup_{i \in A} X_i \) is closed. To show (ii), let \( p \in X'_{\alpha} \), and \( \alpha \in \beta \) be an infinite ordinal, find \( L \subset \{ j \mid p \in O_j, \ O_j \in \mathcal{O} \} \) such that \( L \) is isomorphic to \( \gamma \) where \( \gamma \) is limit such that \( \alpha = \gamma + n \) with \( n \in \omega \) and \( \bigcup L = i \). Let \( N = \bigcap \{ O_i \mid i \in L \} \). Then \( N \) is a neighborhood of \( p \) and \( N \cap X'_{\alpha} = \emptyset \) when \( j \neq i, \ j \in A_{\alpha} \). So \( N \cap X_{\alpha} \subseteq X'_{\alpha} \) which shows that \( X'_{\alpha} \) is open subset of \( X_{\alpha} \) and \( X'_{\alpha} \cap X'_j = \emptyset \) when \( i \neq j \). From the definition of \( X'_{\alpha} \) we have also \( X_{\alpha} = \bigcup_{i \in A_{\alpha}} X'_{\alpha} \). Now (iii) follows from (i) and (ii) and also (iv)–(vi) are obvious. \( \square \)

Lemma 2. Let \( X \) be a topological space, \( \mathcal{O} = \{ O_i \mid i \in \beta \} \) be collection of subsets of \( X \). Then we have:

(i) \( \bigcup_{i \in \beta} Y_i \) is an open subset of \( X \) for each ordinal \( \alpha \).

(ii) The family \( \{ Y'_{\alpha} \mid \alpha \in A_{\beta} \} \) is a pairwise disjoint open cover of the subspace \( Y_{\alpha} \) when \( \alpha \) is an infinite ordinal.

(iii) \( Y = \bigcup_{\alpha \in \beta} Y_{\alpha} \) when \( \text{lot}(x, \mathcal{O}) < \lambda \) for each \( x \in X \).

(iv) \( 1 \leq \text{ot}(x, \{ O_{\gamma} \mid i \leq \gamma < \beta \}) \leq \text{lot}(x, \{ O_{\gamma} \mid i \leq \gamma < \beta \}) < \omega \) for each \( x \in Y^i_{\alpha} \) when \( \alpha \) is a successor ordinal.

Proof. Since the statements (iii) and (iv) are obvious, we only give proofs of (i) and (ii). For \( p \in \bigcup_{i \in \alpha} Y_i \), let \( W \) be an open neighborhood of \( p \) such that \( \{ i \mid O_i \wedge W \neq \emptyset, \ O_i \in \mathcal{O} \} \) is isomorphic to some ordinal \( \beta \) which is less than \( \alpha \). So we have \( W \subset \bigcup_{i \in \alpha} Y_i \) and hence the set \( \bigcup_{i \in \alpha} Y_i \) is open. To prove (ii), it is easy to see that \( \{ Y'_{\alpha} \mid \alpha \in A_{\beta} \} \) is a cover for \( Y_{\alpha} \). We recall that \( \alpha = \gamma + n \) with \( \gamma \) limit ordinal and \( n \in \omega \). Let \( x \in Y'_{\alpha} \) and \( W \) be a neighborhood of \( x \) such that the set \( \{ \lambda \mid W \cap O_{\lambda} \neq \emptyset, \ O_{\lambda} \in \mathcal{O} \} \) is isomorphic to \( \alpha \). From the definition of \( Y'_{\alpha} \), the set \( \{ \lambda \mid W \cap O_{\lambda} \neq \emptyset, \ O_{\lambda} \in \mathcal{O} \} \) is finite. This led us to \( W \cap Y'_{\alpha} = \emptyset \) when \( j > i \) and \( j \in A_{\alpha} \). Similarly, we claim that \( W \cap Y'_j = \emptyset \) when \( i < j \) and \( j \in A_{\alpha} \). To show that, let \( y \in W \cap Y'_j \). Since \( y \in Y'_{\alpha} \), there is a neighborhood \( N \) of \( y \) such that the set \( \{ \lambda \mid j \in N \cap O_{\lambda} \neq \emptyset, \ O_{\lambda} \in \mathcal{O} \} \) is finite and we can choose \( N \subset W \). On the other hand \( \{ \lambda \mid j \in N \cap O_{\lambda} \neq \emptyset, \ O_{\lambda} \in \mathcal{O} \} \) contains a subset which is isomorphic to \( \gamma \) and \( \{ \lambda \mid j \in N \cap O_{\lambda} \neq \emptyset, \ O_{\lambda} \in \mathcal{O} \} \) contains a subset which is isomorphic to \( \omega \). This yields to the fact that \( \{ \lambda \mid W \cap O_{\lambda} \neq \emptyset, \ O_{\lambda} \in \mathcal{O} \} \) is isomorphic to an ordinal which is greater than or equal to \( \gamma + n \), which is a contradiction. So \( W \cap Y'_{\alpha} = \emptyset \) when \( j \neq i, \ j \in A_{\alpha} \). Hence \( Y'_{\alpha} \) is an open subset of \( Y_{\alpha} \) and \( Y'_{\alpha} \cap Y'_j = \emptyset \) when \( i \neq j, \ i, j \in A_{\alpha} \).

We note that \( \text{lot}(x, \mathcal{F}) \leq \text{ot}(x, \mathcal{O}) \) holds where \( \mathcal{F} \) is a precise cushioned refinement of \( \mathcal{O} \) and furthermore we have \( \text{lot}(x, \mathcal{F}) = \text{ot}(x, \mathcal{O}) \) if \( \mathcal{F} \) is a closure preserving collection of closed subsets. \( \square \)

2. Main results

In the next lemma, which can be proved easily using induction, \( P \) denotes an unary predicate, which is definable in ZFC, first part follows the work of Micheal–Nagami whereas second part is a “backwards Michael–Nagami” as the referee pointed out.
Lemma 3. Let \( \{ S_i \mid i \in \beta \} \) be a family of subsets of a topological space \( X \) with \( S_0 = \emptyset \) where \( \beta \) is an ordinal.

(i) Suppose for each \( \alpha \in \beta \) and closed subset \( F \) of \( \bigcup_{i \leq \alpha} S_i \) with \( F \subset S_\alpha \), there is an open collection \( O \) in \( X \) such that \( O \) satisfies \( P \) and \( F \subset \bigcup O \). Then there is an open collection \( O \) which covers \( \bigcup_{i \in \beta} S_i \) which can be written as a union of subcollections \( \{ O_i \mid i \in \text{card} \beta \} \) where each \( O_i \) satisfies \( P \).

(ii) Suppose for each \( \alpha \in \beta \) and for each closed subset \( F \) of \( X \) with \( F \subset \bigcup_{i \leq \alpha} S_i \) there is an open collection \( O \) in \( X \) which satisfies \( P \) and \( S_\alpha \cap F \subset \bigcup O \). Furthermore, suppose for each closed subset \( F \) of \( X \) with \( F \subset \bigcup_{i \in \alpha} S_i \) there is a collection of closed subsets \( \mathcal{F} \) of \( X \) which is a partial refinement of \( \{ \bigcup_{i \in \alpha} S_i \mid i \in \alpha \} \) and covers \( F \) with \( \text{card}(\mathcal{F}) \leq \beta \) for each limit ordinal \( \alpha \leq \beta \). Then for each \( F \subset \bigcup_{i \in \beta} S_i \) which is closed in \( X \), there is an open collection \( O \) in \( X \) which covers \( F \) and \( O = \{ O_i \mid i \in \text{card}(\beta) \} \) where each \( O_i \) satisfies \( P \).

Now we are ready to state and prove main theorems in this paper. In the following theorems we consider the unary \( P \) as the predicate “\( m \)-discrete open partial refinement of”.

Theorem 4. Let \( X \) be a collectionwise normal space and \( \beta \) be an infinite ordinal and \( m = \text{card} \lambda \). Suppose \( X \) has the following shrinking property: Each monotone open cover \( \mathcal{W} \) with \( \text{card}(\mathcal{W}) < \lambda \) has a refinement \( \mathcal{F} \) with \( \text{card}(\mathcal{F}) \leq m \) and which consists of closed subsets of \( X \). Then for each open cover \( \mathcal{O} = \{ O_\gamma \mid \gamma \in \beta \} \) satisfying \( \bigcap \{ O_\gamma \mid \gamma \in L \} \) is a neighborhood of \( x \) when \( \emptyset \neq L \subset \{ O_\gamma \mid x \in O_\gamma \} \) and \( L \) is isomorphic to an ordinal less than \( \lambda \), there exists an \( m \)-discrete open partial refinement which covers \( \bigcup_{\gamma \in \lambda} X_\gamma (\mathcal{O}) \).

Proof. Lemmas 1(i), 1(ii), 3(i) and collectionwise normality of \( X \) leads us to sufficiency of showing that for each closed subset \( F \) of \( X \) with \( F \subset X_\alpha (\mathcal{O}) \), there exists an \( m \)-discrete open partial refinement of \( \mathcal{O} \) which covers \( F \) for each \( \alpha \in \lambda \) and \( i \in A_\omega \). We prove this by induction on the ordinal \( \beta \) which we can assume \( m < \text{card}(\beta) \). Suppose the conclusion of the theorem holds for each open cover satisfying the hypothesis of the theorem which is indexed by an ordinal less than \( \beta \). Let \( F \) be a closed subset of \( X \) with \( F \subset X_\alpha (\mathcal{O}) \). If \( \alpha \) is a successor ordinal the result follows from Lemma 1(iv) and the collectionwise normality of \( X \). If \( \alpha \) is a limit ordinal and \( i < \beta \) consider the open cover \( \mathcal{W} = \{ O_j \cup F^c \mid j < i \} \) (see Lemma 1(vi)). We have \( F \subset X_\gamma (\mathcal{W}) \) and the result follows from the induction hypothesis. If \( \alpha \) is a limit ordinal and \( i = \beta \), by using the shrinking property we can find a closed cover \( \mathcal{H} \) with the cardinality at most \( m \) such that \( H \subset \bigcup \{ O_\gamma \mid \gamma < i_H \} \) some ordinal \( i_H \) less than \( \beta \) for each \( H \in \mathcal{H} \). We have \( H \cap F \subset \bigcup_{\alpha \in \lambda} X_\alpha (\mathcal{W}_H) \) for each \( H \in \mathcal{H} \) where \( \mathcal{W}_H = \{ O_\gamma \cup H^c \mid \gamma < i_H \} \). Now the result follows by the induction hypothesis and Lemma 3(i). \( \square \)

Note that the refinement condition in the hypothesis holds when \( X \) is an \( m \)-shrinking space or each open subset of \( X \) is \( F_m \) (i.e., a union of closed subsets of \( X \) with cardinality at most \( m \)). Let us state an analogous result for the points which are locally \( < m \).
**Theorem 5.** Let $X$ be a collectionwise normal space which has the same shrinking property as in Theorem 4 for the monotone open cover $W$ with $\text{card}(W) \leq \lambda$. Then for each open cover $\mathcal{O} = \{ O_\gamma \mid \gamma \in \beta \}$ and each closed subset $F$ of $X$ satisfying $F \subset \bigcup_{\alpha \in \lambda} Y_\alpha$, there is an $\text{card}(\lambda)$-discrete open partial refinement of $\mathcal{O}$ which covers $F$.

**Proof.** By Lemmas 2(i), 2(ii), 3(ii), it is sufficient to show that the conclusion holds for each closed subset $F$ of $X$ with $F \subset \bigcup_{\alpha \in \lambda} Y_\alpha$ for each $\alpha \in \lambda$ and $i \in A_\alpha$. There exist closed subsets $K$ and $H$ of $X$ such that $F = K \cup H$ and $\{ O_\gamma \mid \gamma \in i \}$ covers $K$ and $\{ O_\gamma \mid i \leq \gamma < \beta \}$ covers $H$. We have $1 \leq \text{ot}(x, \{ O_\gamma \mid i \leq \gamma < \beta \}) < w$ for each $x \in H$. A similar argument to the proof of Theorem 4 yields the result. \( \Box \)

Note, by replacing the conditions collectionwise normality and the shrinking property of $X$ with the condition that each open subset of $X$ is $F_m$, we get $m$-discrete closed partial refinement of open covers.

Next corollaries follow from Theorems 4 and 5 and the above note.

**Corollary 6.** Let $X$ be a collectionwise normal, shrinking space and $\mathcal{O}$ be an open cover of $X$. Then:

(i) If $\mathcal{O}$ is interior preserving and $\text{ot}(x, \mathcal{O}) < \lambda$ for each $x \in X$ then $\mathcal{O}$ has a $\text{card}(\lambda)$-discrete open refinement.

(ii) If $\text{lot}(x, \mathcal{O}) < \lambda$ for each $x \in X$ then $\mathcal{O}$ has a $\text{card}(\lambda)$-discrete open refinement.

**Corollary 7.** Let $X$ be a space such that each open subset is $F_m$ and $\mathcal{O}$ a collection of open subsets of $X$. Then:

(i) If $\mathcal{O}$ is interior preserving then $\mathcal{O}$ has an $m$-discrete closed partial refinement which covers $\bigcup_{1 \leq \alpha \leq \lambda} X_\alpha$ where $\text{card}(\lambda) = m$.

(ii) $\mathcal{O}$ has an $m$-discrete closed partial refinement which covers $\bigcup_{\alpha \in \lambda} Y_\alpha \cap \bigcup \mathcal{O}$ where $\text{card}(\lambda) = m$.

(iii) If $\mathcal{H}$ is a collection of closed sets, then there is an $m$-discrete closed partial refinement $\mathcal{F}$ of $\mathcal{H}$ such that $\bigcup \mathcal{H} \cap (\bigcup_{\alpha \leq \lambda} Y_\alpha(\mathcal{H})) \subset \bigcup \mathcal{F}$ where $\text{card}(\lambda) = m$.

**3. Sufficient conditions for paracompactness**

Next we state several corollaries for sufficient conditions of paracompactness of spaces which have certain covering properties.

**Corollary 8.** Let $X$ be an orthocompact, collectionwise normal, $\theta_m$-refinable space. Suppose each open cover which has cardinality less than $m$ has a closed shrinking. Then $\text{pa}(X) \leq m$, that is, each open cover has an $m$-discrete open refinement.

Next corollary follows from the fact that a $[m, \mathbb{R}_0]$-paracompact space is paracompact when $\text{pa}(X) \leq m$. 


Corollary 9. If $X$ is orthocompact, collectionwise normal, $\theta_m$-refinable and $[m, \aleph_0]$-submetacompact then $X$ is paracompact.

In the next corollary we remove the condition of orthocompactness in Corollary 3, by replacing $\theta_m$-refinability with a stronger condition, i.e., $[\infty, m]$-paracompactness.

Corollary 10. Let $X$ be a collectionwise normal, $[\infty, m]$-paracompact and $m$-shrinking space. Then $\text{pa}(X) \leq m$.

Corollary 11. Every $[\infty, m]$-paracompact, $[m, \aleph_0]$-submetacompact, collectionwise normal spaces are paracompact.

Next corollary is a special case of the previous corollary which gives a partial answer to the question whether para-Lindelöf collectionwise normal spaces are paracompact or not. Before stating this corollary, let us define boundedly para-Lindelöfness. We call a topological space $X$ boundedly para-Lindelöf if every open cover has an open refinement $\mathcal{O} = \{O_i \mid i \in \kappa\}$ such that $\text{lot}(x, \mathcal{O}) \leq \alpha$ for each $x \in X$ for some fixed $\alpha \in w_1$.

Corollary 12. Every collectionwise normal para-Lindelöf $[\aleph_1, \aleph_0]$-submetacompact space is paracompact. Every collectionwise normal boundedly para-Lindelöf, countably submetacompact space is paracompact.

We note that the above theorems can be applied to generalized ordered (GO) spaces to obtain the equivalence of some covering properties for these spaces, since they are orthocompact, collectionwise normal and shrinkable. Since such an equivalence of those covering properties for GO is widely known we will not state them here.

Acknowledgement

The author acknowledges Mr. Hasan Gül for the first draft of manuscript and the referee for the fruitful suggestions.

References