The Alternating Segment Explicit-Implicit Scheme for the Dispersive Equation

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Abstract—In this paper, we present the Alternating Segment Explicit-Implicit scheme for the dispersive equation. The scheme is unconditionally stable and is capable of parallelism. The numerical simulations show that it has better accuracy than that of some existing schemes. © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In this paper, we develop a parallel difference scheme, namely the Alternating Segment Explicit-Implicit scheme (ASEI), for the following dispersive equation:

\[ u_t + au_{xxx} = 0, \quad a \text{ is a constant.} \quad (1.1) \]

The motivation for conducting such a study is that there is no known good scheme for (1.1) that can be parallelized and is stable. Most explicit schemes are parallelizable in nature, but they are usually unstable. The implicit schemes are stable, but they are difficult to implement with parallel computation. Therefore, developing schemes for (1.1) which are stable and are capable of parallel implementation is very important. The ASEI scheme developed in this paper makes a contribution in this direction.

Some related work can be found in [1–3]. In [1], the Alternating Group Explicit scheme (AGE) was developed for (1.1) subject to the following periodic boundary condition and initial condition:

\[ u(x + L, t) = u(x, t), \quad u(x, 0) = u_0(x). \quad (1.2) \]

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In [2,3], the technique of the alternating segment was first developed for solving the diffusion equation. Here this technique is extended to the dispersive equation (1.1),(1.2), which results in the ASEI scheme. We show that this new scheme may be parallelized and also has good stability.

In Section 3, numerical simulations are performed for the ASEI scheme, the AGE scheme, and the classical fully implicit scheme (IMP). The results show that the ASEI scheme has better accuracy than the AGE scheme, and is competitive to the IMP scheme. Moreover, the ASEI scheme is superior to the IMP scheme in a parallel computing environment.

The plan of this paper is as follows. In Section 2, the ASEI scheme is developed and a brief error analysis is discussed. In Section 3, numerical simulations are performed.

2. THE ASEI SCHEME

2.1. Asymmetric Schemes

In order to construct the ASEI scheme, we first introduce four asymmetric schemes for (1.1) with the rules displayed in Figure 1. Let $h, \tau$ be the space mesh and the time mesh, where $J h = L$, and $J$ is a positive integer. We use $U_j^n$ to represent the approximate solution of $u(x_j, t^n)$, where $u(x, t)$ represents the exact solution of (1.1). Then the four asymmetric schemes are given below for $j = 1, 2, \ldots, J$, and $n = 0, 1, 2, \ldots$,

$$U_{j+1}^n - rU_{j-1}^{n+1} + rU_{j+2}^{n+1} = rU_{j-2}^n - 2rU_{j-1}^n + U_j^n + rU_{j+1}^n,$$

$$rU_{j-2}^n - rU_{j-1}^n + rU_{j+1}^n = rU_{j-3}^n - 2rU_{j-2}^n + U_{j+1}^n,$$

$$rU_{j-1}^n + U_{j+1}^n - 2rU_{j+1}^n + rU_{j+2}^n = rU_{j-2}^n - rU_{j-1}^n + U_j^n,$$

$$-rU_{j-2}^n - 2rU_{j-1}^n + U_{j+1}^n - rU_{j+2}^n = U_j^n + rU_{j+1}^n - rU_{j+2}^n,$$

where $r = (u/2)(\tau / h^3)$.

From the Taylor series expansion at $(x_j, t^n)$, we can obtain the following truncation error expressions for formulae (2.3)–(2.6):

$$T(2.3) = r h^2 \frac{\partial^2 u}{\partial x^2} + \frac{3}{2} r h^2 \frac{\partial^3 u}{\partial x^3} + \frac{1}{4} a h^2 \frac{\partial^5 u}{\partial x^5} + O(\tau + h^3),$$

$$T(2.4) = r h^2 \frac{\partial^2 u}{\partial x^2} - \frac{3}{2} r h^2 \frac{\partial^3 u}{\partial x^3} + \frac{1}{4} a h^2 \frac{\partial^5 u}{\partial x^5} + O(\tau + h^3),$$

$$T(2.5) = -r h \frac{\partial^2 u}{\partial x \partial t} + \frac{3}{2} r h^2 \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{1}{4} a h^2 \frac{\partial^5 u}{\partial x^5} + O(\tau + h^3),$$

$$T(2.6) = -r h \frac{\partial^2 u}{\partial x \partial t} - \frac{3}{2} r h^2 \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{1}{4} a h^2 \frac{\partial^5 u}{\partial x^5} + O(\tau + h^3).$$
Thus, formulae (2.3)–(2.6) are all consistent with equation (1.1). Furthermore, by the Fourier method, it can be proved that formulae (2.3) and (2.4) are unconditionally unstable, while (2.5) and (2.6) are unconditionally stable.

2.2 The ASEI Scheme

Now, we are ready to construct the ASEI scheme. Let \( J = 2K(l + l'), \ l \geq 1, \ l' \geq 5 \). We divide the mesh points \( x_1, x_2, \ldots, x_J \) into \( K \) explicit segments and \( K \) implicit segments at each time level. At the odd time levels, the explicit and implicit segment consist of \( l \) and \( l' \) mesh points, respectively, at the even time levels, they consist of \( l' - 4 \) and \( l + 4 \) points, respectively. At each time level, the \( U_j^{n+1} \) in the explicit segments are computed by the following classical explicit scheme:

\[
U_j^{n+1} = U_j^n - r \left( U_{j+2}^n - 2U_{j+1}^n + 2U_{j-1}^n - U_{j-2}^n \right). \tag{2.11}
\]

To compute the values \( U_j^{n+1} \) for the implicit segments, where \( j = i + 1, \ldots, i + k \) for some \( i \) and \( k = l' \) or \( l + 4 \), we use formulae (2.3), (2.5), (2.6), and (2.4) at the boundary points \( x_{i+1}, x_{i+2}, x_{i+k-1}, \) and \( x_{i+k} \), respectively, and the following classical implicit scheme (IMP):

\[
U_j^{n+1} + r \left( U_{j+2}^{n+1} - 2U_{j+1}^{n+1} + 2U_{j-1}^{n+1} - U_{j-2}^{n+1} \right) = U_j^n \tag{2.12}
\]

for the rest of the points. This results in the following linear system:

\[
\begin{bmatrix}
1 & -r & r & & & \\
& r & 1 & -2r & r & \\
& & -2r & 1 & -2r & r \\
& & & \ddots & \ddots & \ddots \\
& & & & -2r & 1 & -r \\
& & & & & -r & 1
\end{bmatrix}
\begin{bmatrix}
U_{i+1}^{n+1} \\
U_{i+2}^{n+1} \\
\vdots \\
U_{i+k-1}^{n+1} \\
U_{i+k}^{n+1}
\end{bmatrix}
= \begin{bmatrix}
U_{i+1}^n - 2rU_{i+2}^n + U_{i+3}^n + rU_{i+4}^n \\
\vdots \\
U_{i+k-2}^n + rU_{i+k-1}^n + U_{i+k}^n \\
-2rU_{i+k-1}^n + U_{i+k+1}^n + 2rU_{i+k+2}^n - rU_{i+k+3}^n
\end{bmatrix}
\]

Since the implicit segments and the explicit segments are independent of each other, we can solve different segments parallelly.

Generally, we arrange the computation according to the rule of “the explicit segment—the implicit segment—\ldots—-the explicit segment” for the odd time levels, and “the implicit segment—the explicit segment—\ldots—-the explicit segment” for the even ones. The flow chart of this method is displayed in Figure 2. We use \( \square \) to denote the four asymmetric schemes (2.3)–(2.6), \( \bullet \) to denote the classical implicit scheme, and \( \circ \) to denote the classical explicit scheme.

\[
(I + rG_1)U^{n+1} = (I - rG_2)U^n, \quad n = 0, 2, 4, \ldots \tag{2.13}
\]

The ASEI scheme can also be expressed as
where \( U^n = (U^n_1, \ldots, U^n_j) \),

\[
G_1 = \begin{bmatrix}
Q_l & P_l \\
& \ddots & \ddots & \ddots \\
& & Q_l & P_l \\
& & & Q_l & P_l
\end{bmatrix}
\]

and

\[
G_2 = \begin{bmatrix}
0 & -2 & 1 \\
2 & 0 & -2 & 1 \\
-1 & 2 & 0 & -2 & 1 \\
& \ddots & \ddots & \ddots \\
& -1 & 2 & 0 & -2 & 1 \\
& -1 & 2 & 0 & -1 \\
& 1 & -2 & 1 \\
1 & -1 & 2 & 0 & -1 \\
1 & 0 & & & & \\
\end{bmatrix}
\]

in which

\[
P_n = \begin{bmatrix}
0 & -1 & 1 & 0 \\
1 & 0 & -2 & 1 \\
-1 & 2 & 0 & -2 & 1 \\
& \ddots & \ddots & \ddots \\
& -1 & 2 & 0 & -2 & 1 \\
& -1 & 2 & 0 & -1 \\
& -1 & 1 & 0 \\
\end{bmatrix}_{n \times n}, \quad n = l' \text{ or } l + 4,
\]

and \( Q_n (n = l, l' - 4) \) is an \( n \times n \) zero matrix.

It is very interesting to observe that the ASEI scheme is unconditionally stable, even if it employs the unconditionally unstable schemes, such as the asymmetric schemes (2.3) and (2.4) and the classical explicit scheme (2.11). To prove the stability, we have to introduce the following Kellogg Lemmas [4].

**Lemma 2.1.** If \( \theta > 0 \) and \( (C + C^*) \) are nonnegative definite, then \( (\theta I + C)^{-1} \) exists and

\[
\| (\theta I + C)^{-1} \|_2 \leq \theta^{-1}.
\]

**Lemma 2.2.** Under the conditions of the previous lemma, the following inequality holds:

\[
\| (\theta I - C)(\theta I + C)^{-1} \|_2 \leq 1.
\]

We then have the next theorem.

**Theorem 2.1.** The ASEI scheme is unconditionally stable.

**Proof.** By eliminating \( U^{n+1} \) from (2.13), we obtain

\[
U^{n+2} = T U^n = T^2 U^{n-2} = \ldots = T^n U^0.
\]
Here
\[ T = (I + rG_2)^{-1}(I - rG_1)\left[(I + rG_1)^{-1}(I - rG_2)\right]. \]

Since \( G_1 \) and \( G_2 \) are all antisymmetric or zero definite, we can obtain the following inequalities from the Kellogg lemmas for any real number \( r \):
\[
\|T^n\|_2 \leq \|I + rG_2\|_2 \|I - rG_1\|_2 \left\|\left[(I + rG_1)^{-1}(I - rG_2)\right]\right\|_2 \\
\leq \|I - rG_2\|_2 \leq \sqrt{1 + 36r^2}. \quad (2.14)
\]

This shows that the ASEI scheme is unconditionally stable.

Finally, we briefly discuss the error analysis of the scheme. It is quite straightforward to obtain that the truncation errors for those interior points of the implicit segments and those explicit-segment points are order \( O(\tau + h^2) \). For the boundary points of the implicit segments, since the signs of (2.7) and (2.10), and the signs of (2.8) and (2.9) are opposite, the effect of the \( h \) terms in the errors can be nearly canceled. Thus, the truncation errors at these boundary points are approximately order \( O(\tau + h^2) \).

3. NUMERICAL SIMULATIONS

We perform the numerical simulations for (1.1),(1.2) using the following model problem:
\[ u_0(x) = \cos(\pi x), \quad a = 1. \quad (3.15) \]

The exact solution is \( u(x, t) = \cos(\pi x + \pi^2 t) \).

We first examine the convergence rate of the ASEI scheme. We divide the mesh points into four segments, and let \( l' = (J/4) + 2 \) and \( l = l' - 4 \). The errors \( e_h = \|U - u\|_{L^2} \) at \( T = 0.01 \) for \( \tau = 1 \) are displayed in Table 1 for four different mesh refinements. The errors appear to be order \( O(h^2) \).

Table 1. Convergence of the ASEI solution.

<table>
<thead>
<tr>
<th>( J )</th>
<th>( h )</th>
<th>( e_h \times 10^5 )</th>
<th>( \frac{e_h}{h^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.02</td>
<td>41.0</td>
<td>1.02</td>
</tr>
<tr>
<td>200</td>
<td>0.01</td>
<td>8.91</td>
<td>0.89</td>
</tr>
<tr>
<td>400</td>
<td>0.005</td>
<td>2.07</td>
<td>0.83</td>
</tr>
<tr>
<td>800</td>
<td>0.0025</td>
<td>0.50</td>
<td>0.80</td>
</tr>
</tbody>
</table>

Table 2. \( h = 2/100, \tau = 2h^3 = 0.1, 1000^{th} \) time step.

<table>
<thead>
<tr>
<th>Schemes</th>
<th>( x )</th>
<th>0.2</th>
<th>0.6</th>
<th>1.0</th>
<th>1.4</th>
<th>1.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASEI</td>
<td>ae</td>
<td>.311 \times 10^{-4}</td>
<td>.460 \times 10^{-4}</td>
<td>.297 \times 10^{-5}</td>
<td>.469 \times 10^{-4}</td>
<td>.271 \times 10^{-4}</td>
</tr>
<tr>
<td>pe</td>
<td>.399 \times 10^{-2}</td>
<td>.129 \times 10^{-1}</td>
<td>.297 \times 10^{-3}</td>
<td>.179 \times 10^{-1}</td>
<td>.324 \times 10^{-2}</td>
<td></td>
</tr>
<tr>
<td>AGE</td>
<td>ae</td>
<td>.320 \times 10^{-4}</td>
<td>.457 \times 10^{-4}</td>
<td>.382 \times 10^{-5}</td>
<td>.481 \times 10^{-4}</td>
<td>.258 \times 10^{-4}</td>
</tr>
<tr>
<td>pe</td>
<td>.411 \times 10^{-2}</td>
<td>.128 \times 10^{-1}</td>
<td>.383 \times 10^{-3}</td>
<td>.184 \times 10^{-1}</td>
<td>.309 \times 10^{-2}</td>
<td></td>
</tr>
<tr>
<td>IMP</td>
<td>ae</td>
<td>.297 \times 10^{-4}</td>
<td>.462 \times 10^{-4}</td>
<td>.120 \times 10^{-5}</td>
<td>.469 \times 10^{-4}</td>
<td>.278 \times 10^{-4}</td>
</tr>
<tr>
<td>pe</td>
<td>.382 \times 10^{-2}</td>
<td>.130 \times 10^{-1}</td>
<td>.121 \times 10^{-3}</td>
<td>.180 \times 10^{-1}</td>
<td>.332 \times 10^{-2}</td>
<td></td>
</tr>
<tr>
<td>Exact Solution</td>
<td>.779</td>
<td>-.356</td>
<td>-.999</td>
<td>-.261</td>
<td>.837</td>
<td></td>
</tr>
</tbody>
</table>
Next, we compare the ASEI solutions, the AGE solutions and the IMP solutions using the same mesh refinements. The absolute errors (ae) and the percentage errors (pe) of these three numerical solutions at the 1000th time step are displayed in Tables 2-4 for different \( r \). Our results show that the ASEI scheme is more accurate than the AGE scheme, and is competitive with the IMP scheme.

The results for small \( r \) (\( r = 0.1 \)) are less impressive than those for big \( r \) (\( r = 1 \) and \( r = 2 \)). This is because the AGE scheme is an explicit scheme, so its errors are large for big \( r \). But for small \( r \), its errors are as small as the implicit schemes like the IMP and ASEI schemes. Usually, we use big \( r \) in computation to save time.

### REFERENCES