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MINIMUM COST PARTITIONS OF A RECTANGLE*

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We consider the problem of partitioning (in a certain manner) a rectangle into n regions of equal area so that the total lengths of the boundaries is a minimum. A closed form solution to the problem is presented.

1. Introduction

The problem of partitioning a rectangle into n regions of equal area so that the total length of the boundaries is a minimum arose in the work of S. Fuller [4] on multiprocessor solutions of partial differential equations over a two dimensional field. T. C. Hu modified the problem in the following way: The regions are required to be rectangles and the partitions are obtained by dividing existing rectangles into two rectangles.

In order to state the modified problem precisely we introduce the following definitions.

Definition 1. The set $\mathcal{D}$ of *dissections* of a rectangle, R, can be defined recursively as follows:

(1) The partition consisting of R alone is in $\mathcal{D}$.

(2) If $D \in \mathcal{D}$ and D' is a partition obtained from D by dividing one of the rectangles of D into two rectangles, then $D' \in \mathcal{D}$.

Definition 2. An *equidissection* is a dissection in which all the rectangles (or parts) have equal area.

Fig. 1a is an example of an equidissection. The partition in Fig. 1b is not an equidissection.

Fig. 1.

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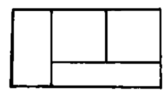
To each equidissection, D, we associate a *cost* $C(D)$, which is the sum of the lengths of all the line segments partitioning R. For example if D is



then $C(D)=\frac{5}{3}\beta+\frac{3}{4}\alpha$, where α is the length of the horizontal side of the rectangle and β is the length of the vertical side.

Hu's problem may, thus, be stated as follows: Given a rectangle R, find an equidissection of R into n parts, which is of minimum cost among all equidissections of R into n parts. We will call such an equidissection a *minimal* equidissection.

In this paper we obtain a closed form solution of Hu's problem. A class of equidissections whose members are called grids is introduced in the next section. In Section 3, by using a discrete variational technique, we show that these equidissections are extremal, that is, we show that for any rectangle a minimal equidissection can be found in the class of grids. A precise recipe for finding a grid that is minimal is obtained in Section 4.

2. Grids

Definition 3. A *vertical* (*horizontal*) *m-strip* is a column (row) of m congruent rectangles, in an equidissection, which extends vertically (horizontally) through the entire rectangle.

An example of a vertical 4-strip is given in Fig. 2. The shaded area is the 4-strip.

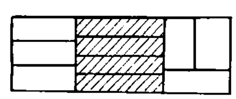


Fig. 2.

Definition 4. A *vertical* (*horizontal*) *grid* is an equidissection which consists only of vertical (horizontal) m-strips and/or vertical (horizontal) $m+1$-strips.

An example of a vertical grid is given in Fig. 3a and of a horizontal grid in Fig. 3b. Fig. 3c is a grid that is both horizontal and vertical. Fig. 2 is an equidissection which is not a grid.

We shall denote the vertical (horizontal) grid having k vertical (horizontal) m-strips and j vertical (horizontal) $m+1$-strips, by $V(m,k,j)$ $(H(m,k,j))$. It will

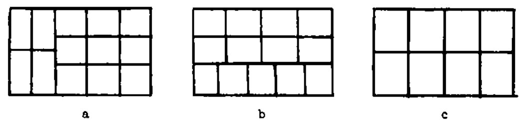


Fig. 3.

also be convenient to denote the rectangle whose horizontal side has length α and whose vertical side has length β by $R(\alpha, \beta)$.

In the following lemma we present the cost of a grid.

Lemma 1. *Let R be the rectangle $R(\alpha, \beta)$. If $D = V(m, k, j)$ on R, then*

$$C(D) = (k + j - 1)\beta + \left(m - 1 + \frac{j(m+1)}{n}\right)\alpha,$$

where $n = km + j(m+1)$ is the number of rectangles into which R is partitioned. Similarly if $D = H(m, k, j)$ on R, then

$$C(D) = \left(m - 1 + \frac{j(m+1)}{n}\right)\beta + (k + j - 1)\alpha.$$

Proof. The proof is a straightforward computation and is omitted.

3. The extremal equidissections

The main result of this section is the following theorem.

Theorem 1. *Every minimal equidissection is a grid (horizontal or vertical).*

Before we begin the proof of Theorem 1, we need to prove a rather technical lemma and state a definition.

Lemma 2. *Let R be the rectangle $R(\alpha, \beta)$. If $V(m, k, j)$ is a minimal equidissection of R, then the following hold:*

(1) *if $0 < j \leqslant m$, then $(m+1)/(j+k+1) \leqslant \beta/\alpha$,*

(2) *if $j \geqslant m$, then $m(m+1)/n \leqslant \beta/\alpha$,*

(3) *if $0 < k \leqslant m+1$, then $(k+j-1)\beta/\alpha \leqslant m$,*

(4) *if $k \geqslant m+1$, then $\beta/\alpha \leqslant m(m+1)/n$,*

where $n = km + j(m+1)$ is the number of rectangles into which R is partitioned by $V(m, k, j)$. If $H(m, k, j)$ is a minimal equidissection of R then (1)–(4) hold with α and β interchanged.

Proof. We let D be the grid $V(m, k, j)$ on R. To prove (1), we also let D' be the grid, $H(k+j, m-j, j)$ on R. We see that the number of rectangles of D' is $(m-j)(k+j)+j(k+j+1)$, which is equal to n. Hence it follows from the minimality of D that

$$C(D')-C(D)\geqslant 0.$$

Lemma 1 gives

$$C(D)=(k+j-1)\beta+\left(m-1+\frac{j(m+1)}{n}\right)\alpha,$$

and

$$C(D')=\left(k+j-1+\frac{j(k+j+1)}{n}\right)\beta+(m-1)\alpha.$$

After subtracting we have

$$\frac{j(k+j+1)}{n}\beta-\frac{j(m+1)}{n}\alpha\geqslant 0.$$

Since $j>0$, this reduces to

$$\beta/\alpha\geqslant(m+1)/(k+j+1)$$

as asserted.

We use a similar argument to establish the rest of the lemma. For parts (2), (3), and (4) we let D' be the grid, $V(m, k+m+1, j-m)$, $H(k+j-1, k, m+1-k)$, and $V(m, k-1-m, j+m)$ respectively.

We will use the notion of a *cut* in the proof of Theorem 1. A *cut* is a line in an equidissection of R which divides R into two rectangles.

Proof of Theorem 1. The proof will be carried out by induction on n, the number of rectangles in the equidissection. For $n=1$, the result is trivial. Assume the theorem holds when the equidissection has less than n rectangles.

Let D be a minimal equidissection of some rectangle R, into n parts. Assume that D is not a grid. Every equidissection must have a vertical or a horizontal cut. Therefore assume that D has a vertical cut.

We claim that D has exactly one vertical cut. If there are two vertical cuts in D, let R_1 be the rectangle on the left of the right cut and let R_2 be the rectangle on the right of the left cut (see Fig. 4).

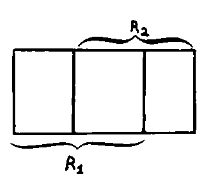


Fig. 4.

We also let D_1 be D restricted to R_1 and D_2 be D restricted to R_2. The minimality of D implies that both D_1 and D_2 are minimal. Thus by the induction hypothesis D_1 and D_2 are grids. They must be vertical grids since they both have vertical cuts. This implies that D consists only of different types of vertical strips. Thus the only way that D is not a grid is if D has an s-strip and a t-strip, where $s+1<t$. Rearrange the strips so that an s-strip and a t-strip are adjacent. (This will not alter the cost.) Let R' be the rectangle formed by the two adjacent strips. Since the rearranged equidissection is still minimal its restriction to R', which consists of the s-strip and t-strip, is also minimal; hence by the induction hypothesis, the s-strip and t-strip form a grid. But this is impossible since $s+1<t$. Thus we have reached a contradiction and therefore our claim that D has only one vertical cut holds.

We now let D_1 be the equidissection obtained by restricting D to R_1, where R_1 is the rectangle to the left of the vertical cut. We also let D_2 be the equidissection obtained by restricting D to R_2, where R_2 is the rectangle to the right of the vertical cut. Clearly D_1 and D_2 are minimal equidissections. Thus by the induction hypothesis, D_1 and D_2 are grids. They are horizontal grids since we have just shown that they have no vertical cuts. Therefore we let D_1 be $H(m, k, j)$ and D_2 be $H(l, h, i)$, where

$$m>0, \quad k>0, \quad j\geqslant 0,$$
$$l>0, \quad h>0, \quad i\geqslant 0,$$

and

$$km+j(m+1)+hl+i(l+1)=n.$$

It is convenient to let

$$n_1=km+j(m+1) \quad \text{and} \quad n_2=hl+i(l+1).$$

Observe that

$$n_1+n_2=n.$$

We can see what D looks like from Fig. 5.

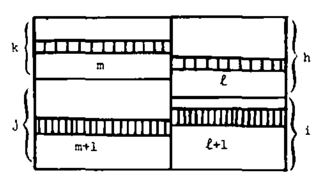


Fig. 5.

There are four cases to consider.

Case 1: $j>0, i>0$.
Case 2: $i=0, j>0$.
Case 3: $i>0, j=0$.
Case 4: $i=0, j=0$.

We shall show that each of the four cases cannot occur and thereby contradict our assumption that D is minimal but is not a grid.

Case 1. Assume $j>0$ and $i>0$. We will construct equidissections D' and D'' that consist of n rectangles. D' will have a vertical cut α/n units to the left of the vertical cut in D. Let D_1' be $H(m, k+1, j-1)$ on R_1' and D_2' be $H(l, h-1, i+1)$ on R_2' where R_1' is the rectangle to the left of the new vertical cut and R_2' is the rectangle to the right of the new vertical cut. Let D' be the dissection of R, whose restriction to R_1' is D_1' and to R_2' is D_2' (see Fig. 6).

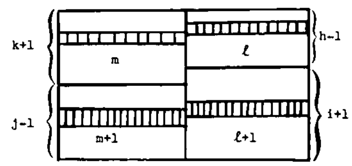


Fig. 6.

It can be shown that the area of a rectangle in D_1' is $\alpha\beta/n$ and in D_2' is also $\alpha\beta/n$. Therefore D' is an equidissection that consists of n rectangles.

The equidissection D'' is constructed similarly to the way in which D' was constructed. The vertical cut in D'' is α/n units to the right of the one in D. Let D_1'' be $H(m, k-1, j+1)$ on R_1'' and D_2'' be $K(l, h+1, i-1)$ on R_2'', where R_1'' and R_2'' are the rectangles to the left and right of the cut, respectively. Let D'' restricted to R_1'' be D_1'' and to R_2'' be D_2''. Since D' is an equidissection of R into n rectangles, by symmetry, D'' is one also.

The minimality of D gives

$$C(D')+C(D'')-2C(D)\geq 0. \tag{1}$$

Since

$$C(D)=C(D_1)+C(D_2)+\beta,$$

by Lemma 1 we have

$$C(D)=\left(m+l-1+\frac{j(m+1)}{n_1}+\frac{i(l+1)}{n_2}\right)\beta + \left((k+j-1)\frac{n_1}{n}+(h+i-1)\frac{n_2}{n}\right)\alpha. \tag{2}$$

Similarly

$$C(D')=\left(m+l-1+\frac{(j-1)(m+1)}{n_1-1}+\frac{(i+1)(l+1)}{n_2+1}\right)\beta + \left((k+j-1)\frac{n_1-1}{n}+(h+i-1)\frac{n_2+1}{n}\right)\alpha \tag{3}$$

and

$$C(D'')=\left(m+l-1+\frac{(j+1)(m+1)}{n_1+1}+\frac{(i-1)(l+1)}{n_2-1}\right)\beta$$
$$+\left((k+j-1)\frac{n_1+1}{n}+(h+i-1)\frac{n_2-1}{n}\right)\alpha \tag{4}$$

By combining (1)–(4) we obtain

$$(m+1)\left(\frac{j-1}{n_1-1}+\frac{j+1}{n_1+1}-2\frac{j}{n_1}\right)+(l+1)\left(\frac{i+1}{n_2+1}+\frac{i-1}{n_2-1}-2\frac{i}{n_2}\right)\geqslant 0. \tag{5}$$

But we have

$$\frac{j-1}{n_1-1}+\frac{j+1}{n_1+1}-\frac{2j}{n_1}=\frac{2(j-n_1)}{n_1(n_1-1)(n_1+1)}<0$$

Similarly we also have

$$\frac{i+1}{n_2+1}+\frac{i-1}{n_2-1}-\frac{2i}{n_2}<0.$$

Therefore (5) cannot be valid and we have reached a contradiction.

Case 2. Assume $j>0$ and $i=0$. Since $i=0$ there aren't any horizontal $l+1$-strips in D_2. Thus D_2 only has horizontal l strips. If $l>1$, then D_2 has a vertical cut, which means that D has two vertical cuts. Since this cannot be, $l=1$ (see Fig. 7).

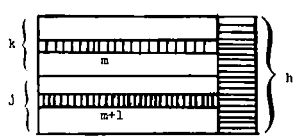


Fig. 7.

There are three subcases of Case 2.

Subcase (a). Assume $h<k+j$. Again we construct an equidissection D' having n rectangles. Let D' have a vertical cut α/n units to the left of the vertical cut in D. Let D'_1 be $H(m,k+1,j-1)$ on R'_1 and D'_2 be $H(1,h+1,0)$ on R'_2, where R'_1 and R'_2 are the rectangles to the left and right of the cut, respectively. Let D' restricted to R'_1 be D'_1 and D' restricted to R'_2 be D'_2. Just as in Case 1, each of the rectangles in D'_1 has area $\alpha\beta/n$, and each of the rectangles in D'_2 has area $\alpha\beta/n$. Thus D' is an equidissection with n rectangles.

Since

$$C(D')=C(D'_1)+C(D'_2)+\beta, \tag{6}$$

by Lemma 1 we have

$$C(D')=\left(m+\frac{(m+1)(j-1)}{n_1-1}\right)\beta+\left((k+j-1)\frac{n_1-1}{n}+\frac{h(h+1)}{n}\right)\alpha. \tag{7}$$

From (2) it follows that

$$C(D)=\left(m+\frac{j(m+1)}{n_1}\right)\beta+\left((k+j-1)\frac{n_1}{n}+\frac{(h-1)h}{n}\right)\alpha. \tag{8}$$

Thus (7), (8), and the minimality of D yield

$$(m+1)\frac{j-n_1}{n_1(n_1-1)}+\frac{2h-k-j+1}{n}\frac{\alpha}{\beta}\geqslant 0. \tag{9}$$

Because of our assumption that

$$h<k+j,$$

which is equivalent to

$$h\leqslant k+j-1,$$

we can replace h by $k+j-1$ in (9) which results in

$$(m+1)\frac{j-n_1}{n_1(n_1-1)}+\frac{k+j-1}{n}\frac{\alpha}{\beta}\geqslant 0 \tag{10}$$

We shall apply Lemma 2 to D_1 to reach a contradiction. D_1 is a minimal equidissection since D is minimal. Since D_1 is $H(m,k,j)$ on $R((n_1/n)\alpha,\beta)$, Lemma 2 gives

(i) if $k\leqslant m+1$, then

$$(k+j-1)\frac{(n_1/n)\alpha}{\beta}\leqslant m, \tag{11}$$

and

(ii) if $k\geqslant m+1$, then

$$\frac{(n_1/n)\alpha}{\beta}\leqslant\frac{m(m+1)}{n}. \tag{12}$$

Since $k+j-1>0$, (11) is the same as

$$\alpha/\beta<mn/n_1(k+j-1). \tag{13}$$

We can also simplify (12) to obtain

$$\alpha/\beta\leqslant m(m+1)/n_1. \tag{14}$$

Thus, either (13) or (14) must hold.

Combining (13) and (10) yields

$$j(m+1)-n_1-m\geqslant 0 \tag{15}$$

But since $n_1=km+j(m+1)$ (15) reduces to $-k-1\geqslant 0$, which is a contradiction.

Combining (14) and (10) yields

$$(j-n_1)n+(k+j-1)m(n_1-1)\geqslant 0$$

we replace the first n_1 with $km+j(m+1)$ and this time we obtain

$$-(k+j)n+(k+j-1)(n_1-1)\geqslant 0.$$

This is also a contradiction, since $k+j>k+j-1$ and $n>n_1-1$. Thus, we have shown that Subcase 2(a) is impossible.

Subcase (b). Assume $h>k+j$. This case is eliminated in a similar manner. We construct an equidissection D' having n rectangles and having a vertical cut α/n units to the right of the vertical cut in D. Let D' be $H(m, k-1, j+1)$ on R'_1 and D'_2 be $H(1, h-1, 0)$ on R'_2, where R'_1 and R'_2 are the rectangles to the left and the right of the cut, respectively. Let D' restricted to R'_1 be D'_1 and D' restricted to R'_2 be D'_2. Again D' is an equidissection with n rectangles.

We use Lemma 1 to compute the cost of D' and obtain

$$C(D')=\left(m+\frac{(m+1)(j+1)}{n_1+1}\right)\beta+\left((k+j-1)\frac{n_1+1}{n}+\frac{(h-2)(h-1)}{n}\right)\alpha. \quad (16)$$

The minimality of D together with (8) and (16) yield

$$(m+1)\frac{n_1-j}{(n_1+1)n_1}-\frac{2h-k-j-1}{n}\frac{\alpha}{\beta}\geqslant 0. \quad (17)$$

Recall that we are assuming $h>k+j$, or equivalently $h\geqslant k+j+1$. Replacing h by $k+j+1$ in (17) results in

$$(m+1)\frac{n_1-j}{n_1(n_1+1)}-\frac{k+j+1}{n}\frac{\alpha}{\beta}\geqslant 0. \quad (18)$$

We apply Lemma 2 to D_1 and obtain the following:

(i) if $j<m$, then

$$\frac{m+1}{j+k+1}\leqslant\frac{(n_1/n)\alpha}{\beta},$$

and

(ii) if $j\geqslant m$, then

$$\frac{(m+1)m}{n_1}\leqslant\frac{(n_1/n)\alpha}{\beta}.$$

Thus we have that either

$$n(m+1)/n_1(j+k+1)\leqslant\alpha/\beta, \quad (19)$$

or

$$n(m+1)m/n_1^2\leqslant\alpha/\beta. \quad (20)$$

Combining (18) and (19) yields $n_1-j-n_1-1\geqslant 0$, which is clearly a contradiction.

Combining (18) and (20) yields

$$n_1(n_1-j)-(k+j+1)m(n_1+1)\geqslant 0 \quad (21)$$

Since $n_1 = (k+j)m + j$, (21) becomes

$$n_1(k+j) - (k+j+1)(n_1+1) \geqslant 0$$

which is clearly impossible.

Subcase (c). Assume $h = k+j$. We let D' be $H(m+1, k, j)$ on R. The number of rectangles in D' is $k(m+1) + j(m+2)$ which is equal to n. By Lemma 1,

$$C(D') = \left(m + \frac{(m+2)j}{n}\right)\beta + (k+j-1)\alpha.$$

By (8), since $h-1 = k+j-1$, we have

$$C(D) = \left(m + \frac{(m+1)j}{n_1}\right)\beta + \left((k+j-1)\frac{n_1+h}{n}\right)\alpha$$

$$= \left(m + \frac{(m+1)j}{n_1}\right)\beta + (k+j-1)\alpha.$$

Since D is minimal it follows that

$$0 \leqslant j\left(\frac{m+2}{n} - \frac{m+1}{n_1}\right)\beta,$$

which reduces to

$$0 \leqslant n_1(m+2) - n(m+1)$$
$$= n_1(m+2) - (n_1+k+j)(m+1) = -k$$

Again, we have a contradiction. Thus, we can eliminate Case 2.

Case 3. Assume $j = 0$ and $i > 0$. This case is symmetric to Case 2. Thus Case 3 is also eliminated.

Case 4. Assume $i = 0$ and $j = 0$. We have shown that if $i = 0$, $l = 1$ and so by symmetry if $j = 0$, then $m = 1$ (see Fig. 8).

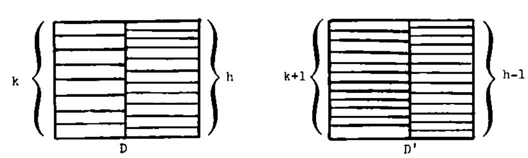


Fig. 8.

We assume $h \geqslant k$. Since D is not a grid, $h-1 > k$. We let D' be an equidissection consisting of a vertical $k+1$-strip and a vertical $h-1$-strip. D' clearly has n rectangles.

We easily compute the costs to be

$$C(D')=\frac{k(k+1)+(h-2)(h-1)}{n}\alpha+\beta,$$

$$C(D)=\frac{k(k-1)+h(h-1)}{n}\alpha+\beta.$$

After subtracting we obtain

$$0\leqslant C(D')-C(D)=\frac{2k-2(h-1)}{n}\alpha$$

Since $h-1>k$ we clearly have a contradiction and, thus, the proof of the theorem is complete.

We have thus reduced the problem of finding a minimal equidissection of a rectangle to that of finding a minimal grid.

4. The solution

Throughout this section we shall use $\lfloor x\rfloor$ to denote the greatest integer less than or equal to x and $\lceil x\rceil$ to denote the smallest integer greater than or equal to x. The closed form solution of the problem is an immediate consequence of the following theorem.

Theorem 2. *Assume we are given the positive integer, n, and the rectangle, $R=R(\alpha,\beta)$, where $\beta\leqslant\alpha$. Let s be the integer such that*

$$s(s-1)<\frac{\beta}{\alpha}n\leqslant s(s+1), \tag{22}$$

and let $r=n-\lfloor n/s\rfloor s$ and $t=\lfloor n/s\rfloor-r$.

(1) *If $(s+1)/(t+r+1)<\beta/\alpha$ then $V(s,t,r)$ is a minimal equidissection of R into n rectangles.*

(2) *If $(s-1)/(t+r)<\beta/\alpha\leqslant(s+1)/(t+r+1)$, then $H(r+t,s-r,r)$ is a minimal equidissection of R into n rectangles.*

(3) *If $\beta/\alpha\leqslant(s-1)/(t+r)$ then $V(s-1,s-r,t+2r+1-s)$ is a minimal equidissection of R into n rectangles.*

We shall apply this theorem to the following example. Let $n=47$, $\alpha=11$, and $\beta=7$. Then we have

$$n\frac{\beta}{\alpha}=\frac{47(7)}{11}=29+\frac{10}{11}.$$

Thus we find that $s=5$ since

$$20=5(4)<29+\tfrac{10}{11}<5(6)=30.$$

Now we have

$$r=47-\lfloor\tfrac{47}{5}\rfloor 5=2 \quad\text{and}\quad t=\lfloor\tfrac{47}{5}\rfloor-2=7.$$

Thus,

$$\frac{s+1}{r+t+1}=\frac{6}{10} \quad\text{and}\quad \frac{s-1}{r+t}=\frac{4}{9}.$$

We compare β/α with $(s-1)/(r+t)$ and $(s+1)/(r+t+1)$ and obtain

$$\frac{s+1}{r+t+1}=\frac{6}{10}<\frac{7}{11}=\frac{\beta}{\alpha}$$

Hence by part (1) of the theorem, $V(5,7,2)$ is a minimal equidissection.

The three lemmas that follow are essential in proving Theorem 2.

Lemma 3. *Let $R=R(\alpha,\beta)$ where $\beta\leqslant\alpha$ and let n and s be positive integers such that*

$$s(s-1)<\frac{\beta}{\alpha}n<s(s+1). \tag{23}$$

If the vertical grid, D, is a minimal equidissection of R into n rectangles, then

$$D=V(s,t,r) \quad \textit{and} \quad t>0,$$

or

$$D=V(s-1,s-r,t+2r+1-s) \quad \textit{and} \quad t+2r+1-s>0,$$

where

$$r=n-s\left\lfloor\frac{n}{s}\right\rfloor \quad \textit{and} \quad t=\left\lfloor\frac{n}{s}\right\rfloor-r.$$

Proof. Since D is a vertical grid, $D=V(m,k,j)$ for some $m>0$, $k>0$ and $j\geqslant 0$. We begin by showing that $m=s$ or $m=s-1$.

First it will be shown that $m\leqslant s$. Assume $m>s$ or equivalently

$$m\geqslant s+1. \tag{24}$$

There are three cases to consider: $j=0$, $0<j<m$, and $j\geqslant m$. If $j=0$, then

$$D=V(m,k,0)=V(m-1,0,k).$$

If we also have $k<m-1$ then part 1 of Lemma 2 gives

$$m/(k+1)\leqslant\beta/\alpha.$$

But since we assumed that $\beta\leqslant\alpha$, we have $m/(k+1)\leqslant 1$, which contradicts $k<m-1$. Thus we have $k\geqslant m-1$. Lemma 2 gives

$$m(m-1)/n\leqslant\beta/\alpha.$$

Combining this with (23) yields $m(m-1)<s(s+1)$, which contradicts (24). Thus $j\neq 0$.

If $0<j<m$, then by Lemma 2, we have

$$(m+1)/(j+k+1)\leq \beta/\alpha. \tag{25}$$

By (23) and (24) this implies

$$n(m+1)<(j+k+1)s(s+1)\leq (j+k+1)(m-1)m.$$

From this it follows that

$$n<(j+k+1)(m-1) \tag{26}$$

Since $\beta\leq\alpha$, (25) implies $m\leq j+k$. This inequality combined with (26) yields $j<-1$, which is clearly a contradiction.

We now assume $j\geq m$. Lemma 2 gives

$$m(m+1)/n\leq \beta/\alpha.$$

This inequality combined with (23) yields $m(m+1)<s(s+1)$, which contradicts (24). Thus the case $j\geq m$ is also eliminated and it follows that

$$m\leq s. \tag{27}$$

Assume now that $m<s-1$ or equivalently

$$m+1\leq s-1 \tag{28}$$

We will show that this cannot hold in each of the two cases $k<m+1$ and $k\geq m+1$.

If $k<m+1$, then by Lemma 2 we have

$$(k+j-1)\beta/\alpha\leq m.$$

Combining this inequality with (23) yields

$$s(s-1)(k+j)<nm+s(s-1) \tag{29}$$

Since $\beta\leq\alpha$, (23) gives

$$s(s-1)<n.$$

Combining this with (28) and (29) yields $s(k+j)<n$ which, implies $s<m+1$. This contradicts (28).

If $k\geq m+1$, then by Lemma 2, we have

$$\beta/\alpha\leq m(m+1)/n.$$

Since $m<s-1$ we have

$$\beta/\alpha<(s-1)s/n,$$

which contradicts (23). Therefore it holds that $m\geq s-1$, which together with (27) implies

$$m = s \quad \text{or} \quad m = s - 1.$$

We now must show that if $m = s$, then

$$j = n - \left\lfloor \frac{n}{s} \right\rfloor s = r \quad \text{and} \quad k = \left\lfloor \frac{n}{s} \right\rfloor - j = t.$$

We first show that if $m = s$, then $j < m$. If $j \geq m$, then Lemma 2 implies

$$m(m+1)/n \leq \beta/\alpha.$$

But since $m = s$, (23) is contradicted. Thus $j < m$. Since $n = m(j+k) + j$, it follows that

$$\left\lfloor \frac{n}{m} \right\rfloor = j + k,$$

and

$$j = n - \left\lfloor \frac{n}{m} \right\rfloor m = n - \left\lfloor \frac{n}{s} \right\rfloor s = r$$

Thus $k = t$ and $j = r$ and so we have

$$D = V(s, t, r).$$

It also follows that $t > 0$, since we assumed k to be positive.

We claim now that if $m = s - 1$ then it follows that

$$k = \left\lceil \frac{n}{s} \right\rceil s - n \quad \text{and} \quad j = \left\lceil \frac{n}{s} \right\rceil - k.$$

We first show that $k < m + 1$. If $k \geq m + 1$, then by Lemma 2 we have

$$\beta/\alpha \leq m(m+1)/n.$$

Since $m = s - 1$, this contradicts (23). Thus $k < m + 1$. Since $n = (m+1)(j+k) - k$, it follows that

$$\left\lceil \frac{n}{m+1} \right\rceil = j + k \quad \text{and} \quad (m+1)\left\lceil \frac{n}{m+1} \right\rceil - n = k.$$

Since $m = s - 1$, we have

$$k = s\left\lceil \frac{n}{s} \right\rceil - n \quad \text{and} \quad j = \left\lceil \frac{n}{s} \right\rceil - k. \tag{30}$$

We now assert that

$$\left\lceil \frac{n}{s} \right\rceil = \left\lfloor \frac{n}{s} \right\rfloor + 1.$$

If not, then

$$\left\lceil \frac{n}{s} \right\rceil = \left\lfloor \frac{n}{s} \right\rfloor = \frac{n}{s}.$$

Hence (30) becomes

$$k = s\frac{n}{s} - n = 0,$$

which is impossible since k is positive. Therefore

$$\left\lceil \frac{n}{s} \right\rceil = \left\lfloor \frac{n}{s} \right\rfloor + 1.$$

Combining this equation with (30) results in $k = s - r$. We also have

$$\begin{aligned} j &= \left\lceil \frac{n}{s} \right\rceil - k \\ &= \left(\left\lfloor \frac{n}{s} \right\rfloor + 1\right) - (s - r) = t + 2r - s + 1. \end{aligned}$$

Therefore we arrive at

$$D = V(s-1, s-r, t+2r-s+1).$$

We must also show that

$$t + 2r - s + 1 > 0.$$

If not, we have $t + 2r \leqslant s - 1$, which implies that $s(t+2r) \leqslant s(s-1)$. By (23), since $\beta \leqslant \alpha$, it follows that

$$s(t+2r) < n = s(t+r) + r.$$

This results in $r(s-1) < 0$, which is impossible. Thus we must have $t + 2r - s + 1 > 0$.

We therefore have our desired conclusion that D is either $V(s, t, r)$, with $t > 0$, or $V(s-1, s-r, t+2r-s+1)$ with $t+2r-s+1 > 0$.

There is a similar lemma for horizontal grids.

Lemma 4. *Let $R = R(\alpha, \beta)$ where $\beta \leqslant \alpha$ and let n and s be positive integers such that*

$$s(s-1) < \frac{\beta}{\alpha} n < s(s+1). \tag{31}$$

If the horizontal grid D is a minimal equidissection of R into n rectangles, then

$$D = H(t+r, s-r, r),$$

where

$$r = n - s\left\lfloor \frac{n}{s} \right\rfloor \quad \textit{and} \quad t = \left\lfloor \frac{n}{s} \right\rfloor - r.$$

Furthermore, we also have

$$r \leqslant t + r \quad \textit{and} \quad s - r \leqslant t + r + 1.$$

Proof. Since D is a horizontal grid $D = H(m, k, j)$ for some $m > 0$, $k > 0$ and $j \geqslant 0$. We begin by showing that $k \leqslant m + 1$ and $j \leqslant m$.

Assume that

$$k > m + 1. \tag{32}$$

By Lemma 2, we have

$$n/m(m+1) \leqslant \beta/\alpha,$$

from which it follows that $n \leqslant m(m+1)$. Therefore we have $m(j+k) \leqslant m(m+1)$, which implies $j + k \leqslant m + 1$. Since this contradicts (32), we must conclude that

$$k \leqslant m + 1. \tag{33}$$

Hence again by Lemma 2 we have

$$(k + j - 1)/m \leqslant \beta/\alpha,$$

which implies $k + j - 1 \leqslant m$. Since $k \geqslant 1$, we have

$$j \leqslant m. \tag{34}$$

We now show that the lemma holds for the case $j = 0$. If $j = 0$ then D is a vertical grid as well as a horizontal grid. Thus Lemma 3 implies that $D = V(s, t, r)$ and $t > 0$, or $D = V(s-1, s-r, t+2r+1-s)$ and $t + 2r + 1 - s > 0$. The latter cannot hold, since the inequalities, $s - r > 0$ and $t + 2r + 1 - s > 0$, imply that $V(s-1, s-r, t+2r+1-s)$ is not horizontal. Thus $D = V(s, t, r)$ and $t > 0$. Hence $r = 0$ and we have

$$\begin{aligned} D &= V(s, t, 0) \\ &= H(t, s, 0) = H(t + r, s - r, r), \end{aligned}$$

as asserted. Furthermore, since $j = 0 = r$, it follows that

$$t + r = m \quad \text{and} \quad s - r = k.$$

Thus from (33) and (34) we have

$$s - r \leqslant t + r + 1 \quad \text{and} \quad r \leqslant t + r.$$

For the case $j > 0$, we shall first show that $k + j = s$. Since $k \leqslant m + 1$, Lemma 2 gives

$$(k + j - 1)/m \leqslant \beta/\alpha.$$

Combining this inequality with (31) yields

$$n(k+j-1) < ms(s+1).$$

If $k+j \geq s+1$ then it follows that

$$n(k+j-1) < m(k+j-1)(k+j).$$

Thus we arrive at $n < m(k+j)$, which is a contradiction. Hence it must hold that

$$k+j \leq s. \tag{35}$$

Since $0 < j \leq m$, Lemma 2 gives

$$\beta/\alpha \leq (j+k+1)/(m+1).$$

This inequality combined with (31) yields

$$s(s-1)(m+1) < (j+k+1)n.$$

If $k+j \leq s-1$, then it follows that $(k+j)(m+1) < n$ which is a contradiction. Hence we have

$$k+j \geq s.$$

This inequality together with (35) yields

$$k+j = s. \tag{36}$$

Since $n = (k+j)m + j$ and $k > 0$ we have

$$\left\lfloor \frac{n}{k+j} \right\rfloor = m \quad \text{and} \quad n - \left\lfloor \frac{n}{k+j} \right\rfloor (k+j) = j.$$

By (36) we can conclude that

$$m = \left\lfloor \frac{n}{k+j} \right\rfloor = \left\lfloor \frac{n}{s} \right\rfloor = t + r,$$

and

$$j = n - \left\lfloor \frac{n}{k+j} \right\rfloor (k+j) = n - \left\lfloor \frac{n}{s} \right\rfloor s = r,$$

and

$$k = s - j = s - r,$$

Therefore we have $D = H(t+r, s-r, r)$.

Furthermore (33) and (34) yield

$$s - r = k \leq m + 1 = t + r + 1 \quad \text{and} \quad r = j \leq m = t + r.$$

This completes the proof of Lemma 4.

Lemma 5. *Let $R = R(\alpha, \beta)$ where $\beta \leq \alpha$ and let n and s be positive integers such* *that*

$$s(s-1)<\frac{\beta}{\alpha}n<s(s+1).$$

If $r=n-\lfloor n/s\rfloor s$ *and* $t=\lfloor n/s\rfloor-r$, *then the following hold:*

(1) *If* $V(s,t,r)$ *is a minimal equidissection and* $V(s,t,r)\neq H(t+r,s-r,r)$, *then*

$$(s+1)/(t+r+1)\leqslant\beta/\alpha.$$

(2) *If* $H(t+r,s-r,r)$ *is a minimal equidissection, then*

$$(s-1)/(t+r)\leqslant\beta/\alpha.$$

Furthermore, if $H(t+r,s-r,r)\neq V(s,t,r)$, *then*

$$\beta/\alpha\leqslant(s+1)/(t+r+1).$$

(3) *If* $V(s-1,s-r,t+2r+1-s)$ *is a minimal equidissection, then*

$$\beta/\alpha\leqslant(s-1)/(t+r).$$

Proof. This lemma is a consequence of Lemma 2.

(1) If $V(s,t,r)$ is minimal and $V(s,t,r)\neq H(t+r,s-r,r)$, then by Lemma 4 $V(s,t,r)$ is not a horizontal grid. Thus r must be positive. Therefore since $r<s$, Lemma 2 gives

$$(s+1)/(t+r+1)\leqslant\beta/\alpha.$$

(2) If $H(t+r,s-r,r)$ is minimal then by Lemma 4 we have

$$s-r\leqslant t+r+1.$$

Since $s-r$ is positive by Lemma 2 we have

$$(s-1)/(t+r)\leqslant\beta/\alpha.$$

If $H(t+r,s-r,r)\neq V(s,t,r)$, then r must be positive since $H(t,s,0)=V(s,t,0)$. Since $r>0$ and by Lemma 4 $r\leqslant t+r$, applying Lemma 2 to $H(t+r,s-r,r)$ yields

$$\beta/\alpha\leqslant(s+1)/(t+r+1).$$

(3) Since $0<s-r\leqslant s$, if we apply Lemma 2 to $V(s-1,s-r,t+2r+1-s)$ we get

$$\beta/\alpha\leqslant(s-1)/(t+r).$$

Proof of Theorem 2. We first prove the theorem for the case in which all the inequalities are strict. By Theorem 1 we have that a minimum equidissection of R into n rectangles is a vertical or horizontal grid. Thus by Lemmas 3 and 4, a minimal equidissection of R into n rectangles must be either $V(s, t, r)$, $H(t+r, s-r, r)$, or $V(s-1, s-r, t+2r+1-s)$.

(1) If $(s+1)/(t+r+1)<\beta/\alpha$, then Lemma 5 implies that if $V(s, t, r)\neq H(t+r, s-r, r)$, then $H(t+r, s-r, r)$ is not minimal. If it is shown that $(s-1)/(t+r)\leqslant(s+1)/(t+r+1)$, then Lemma 5 will imply that $V(s-1, s-r, t+2r+1-s)$ is not minimal. Hence the only remaining possibility will be $V(s, t, r)$ and so $V(s, t, r)$ will have to be minimal.

We assume

$$(s-1)/(t+r)>(s+1)/(t+r+1),$$

which reduces to $2(t+r)<s-1$. Thus we have

$$s(s-1)>2(t+r)s=2\left\lfloor\frac{n}{s}\right\rfloor s\geqslant n,$$

which contradicts (22) since $\beta\leqslant\alpha$. Thus we can conclude that

$$(s-1)/(t+r)\leqslant(s+1)/(t+r+1),$$

and hence that $V(s, t, r)$ is minimal.

(2) If $(s-1)/(t+r)<\beta/\alpha<(s+1)/(t+r+1)$, then Lemma 5 implies that if $V(s, t, r)\neq H(t+r, s-r, r)$ then $V(s, t, r)$ is not minimal. Lemma 5 also implies that $V(s-1, s-r, t+2r+1-s)$ is not minimal. Since $H(t+r, s-r, r)$ is the only remaining possibility, $H(t+r, s-r, r)$ is minimal.

(3) If $\beta/\alpha<(s-1)/(t+r)$, Lemma 5 gives that $H(t+r, s-r, r)$ is not minimal and also that $V(s, t, r)$ is not minimal, since $(s-1)/(t+r)\leqslant(s+1)/(t+r+1)$. Therefore $V(s-1, s-r, t+2r+1-s)$ is minimal.

The theorem for the general case in which the inequalities are not strict follows immediately from the following assertion: If the grid D on $R(\alpha, \beta)$ is minimal for $c<\beta/\alpha<d$, then D is minimal for $c<\beta/\alpha\leqslant d$. We shall use a continuity argument to prove this. Let $D_1, \ldots, D_N$ be the collection of all grids on $R(\alpha, \beta)$ having n parts and let $\gamma=\beta/\alpha$. For each $i=1,2,\ldots,N$, $f_i(\gamma)=C(D_i)/\alpha$ is a continuous function of γ over the positive reals. It follows that

$$f(\gamma)=\min_{i=1,2,\ldots,N} f_i(\gamma)$$

is also continuous. If D_j is minimal for $c<\gamma<d$, then $f(\gamma)=f_j(\gamma)$ for $c<\gamma<d$. Since $f(\gamma)$ and $f_j(\gamma)$ are two continous functions agreeing on an open interval, they must agree on the closure of the interval. Therefore

$$f(\gamma)=f_j(\gamma)\quad\text{for}\quad c<\gamma\leqslant d.$$

Note that we leave one side of the inequality strict in order to include the case $c=d$. It follows that from this that D_j is minimal for $c<\gamma\leqslant d$. This completes the proof of the theorem.

5. Related problems

Problems of a somewhat similar nature to the minimum cost partition problem are packing problems. In these problems rectangles (or boxes) of a given size or shape are packed into a rectangle (or box). This must be done in a way that minimizes the wasted space in the rectangle. Some results on packing problems can be found in [1], [2] and [3].

For a problem more closely related to the one presented here, consider partitions less restrictive than equidissections. The only requirement on the partitions is that the parts must be rectangles. Thus Fig. 1b is allowed. We conjecture that a minimum cost partition in this case is a minimum cost equidissection. The conjecture holds for $n \leq 6$.

Another question we can ask is the following: Does a result analogous to ours hold for partitions of rectangular boxes?

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