Pancyclicity in hamiltonian graphs

D. Amar
UER de Mathématiques et Informatique, 351 bld de la Libération, 33400 Talence, France

E. Flandrin
L. R. I., Bâtiment 490, Université Paris Sud, 91405 Orsay, France

I. Fournier
L. R. I., Bâtiment 490, Université Paris Sud, 91405 Orsay, France

A. Germa
L. R. I., Bâtiment 490, Université Paris Sud, 91405 Orsay, France
ENST, Département Informatique, 46 rue Barrault, 75634 Paris Cedex 13, France

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Abstract

We prove the following theorem. If G is a hamiltonian, nonbipartite graph of minimum degree at least \((2n + 1)/S\), where n represents the order of G, then G is pancyclic.

In this article, we consider only simple, undirected graphs. The notations and definitions omitted here can be found in [1-2].

A graph G on n vertices is called pancyclic if it contains cycles of every length k, for 3 \(\leq k \leq n\). In fact various sufficient conditions for a graph to be hamiltonian are also sufficient for the graph to be pancyclic [7]. Namely the two following results are true.

**Theorem 1** [3]. Let G be a graph on \(n > 3\) vertices, with q edges. If \(q > \left(\frac{n - 1}{2}\right) + 1\) then G is pancyclic.

**Theorem 2** [3]. Let G be a graph on \(n > 3\) vertices. If the minimum degree of G, \(\delta(G)\), is at least \(n/2\) then G is pancyclic or equal to \(K_{n/2,n/2}\).

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It seems interesting to see how the bounds can be lowered if one makes the additional assumption that $G$ is hamiltonian.

For Theorem 1 this was done and the answer is given by the following theorem.

**Theorem 3** [5]. Let $G$ be an hamiltonian graph on $n$ vertices. If
\[ q > \lfloor (n - 1)^2/4 + 1 \rfloor, \]
then $G$ is pancyclic or bipartite, and the bound is best possible.

For Theorem 2, the following was conjectured independently by R. Häggkvist [6] and Mitchem and Schmeichel [7].

**Conjecture.** Let $G$ be an hamiltonian graph on $n$ vertices. If the minimum degree $\delta(G)$ is at least $(2n + 1)/5$, then $G$ is pancyclic or bipartite and the bound is best possible.

Here we prove that the condition is in fact sufficient for $n \geq 102$.

For $\delta(G) \geq (2n + 1)/5 - 1$, the result is not always true and for $n = 0$ [5] there exist hamiltonian graphs with $\delta(G) = 2n/5$ which are not pancyclic nor bipartite, as proved by the wreath product of $C_5$ and $K$, with $n = 5r$ (easily modified for the case $n \neq 0$ [5-7]). Thus, the bound is best possible.

**Notations.** Let $G = (X, E)$; for $x$ in $X$, $d(x)$ is the degree of $x$, $N(x)$ the neighbourhood of $x$. For $X \supseteq Y$, $d_Y(x)$ is the degree of $x$ restricted to $Y$ and $N_Y(x) = N(x) \cap Y$.

The cycles will be oriented if necessary. For an oriented cycle $C$, if $a \in C$, $a^+$ will be the successor of $a$ on $C$ and $a^{-l}$ the $l$th point following $a$ on $C$. For $y \in G$, $N_C^+(y)$ will be the set of the successors of the neighbours of $y$ on $C$. $a^-$, $a^{-l}$ and $N_C^-(y)$ will be analogously defined.

If $a$ and $b$ are two vertices of an oriented cycle $C$, $ab$ is the directed path of $C$ from $a$ to $b$.

$C_k \setminus C_{k+1}$ denotes two cycles of length $k$ and $k+1$ having a path of length $k - 1$ in common; $\alpha$ and $\beta$ are the extremities of that common path, and $\gamma$ is the vertex of $C_{k+1}$ not on $C_k$ (see Fig. 1); if $C_{k+1}$ is oriented, $\gamma \in \alpha \beta$.

${\mathcal H}_n$ denotes the set of the graphs of order $n$, hamiltonian, nonbipartite and of minimum degree $(2n + 1)/5$.

Fig. 1.
Theorem. Let $G = (X, E)$ be a hamiltonian, nonbipartite graph such that $|X| = n \geq 102$ and $\forall x \in X, d(x) \geq (2n + 1)/5$. Then $G$ is pancyclic.

Sketch of the proof. (A) If $G \in \mathcal{H}_n$ then $G$ contains a $C_3$.
(B) If $G \in \mathcal{H}_n$ then $G$ contains $C_k \nabla C_{k+1}$ for $3 \leq k \leq (n+27)/10$.
(C) If $G \in \mathcal{H}_n$ and if the connectivity of $G$ is at most $(n-2)/10$ then $G$ contains cycles of every length $l$ for $8 \leq l \leq (9n + 22)/10$.
(D) If $G \in \mathcal{H}_n$ and if the connectivity of $G$ is a least $(n-1)/10$, then $G$ contains cycles of every length $l$ for $(n+18)/10 \leq l \leq (9n + 11)/10$.
(E) If $G \in \mathcal{H}_n$ then $G$ contains cycles of every length $l$ for $(9n + 3)/10 \leq l \leq n$.

Proof of the theorem

(A) Proposition 1. $G$ contains a $C_3$.

The proof can be found for instance in [6], but we give it here for completeness.

Proof. As $G$ is not bipartite, let $C$ be an odd cycle of minimum length $l$, $R = G\setminus C$, and assume $l \geq 5$.

For $x \in R$, there are at most two edges from $x$ to $C$. (If not, we should have an odd cycle of length $<l$).

$C$ has no chords because of the minimality of $l$.

$|R| = n - l$, so $\sum_{z \in C} d(z) \geq 2l + 2(n - l) \geq 2n$ ($z \in C$ has two neighbours on $C$, and there are at most $2|R|$ edges between $C$ and $R$).

$$[(2n + 1)/5]l \leq \sum_{z \in C} d(z) \leq 2n.$$ 

$$2n + 1)/5 \leq 2n/l:$$
a contradiction for $l \geq 5$. \[\square\]

(B) The proof of part B is by induction on $k$.

Proposition 2. $G$ contains $C_3 \nabla C_4$.

Proof. $G$ contains a $C_3$ of vertices $a$, $b$, $c$. Let $R = G\setminus C_3$. $d_R(a) + d_R(b) + d_R(c) > |R|$ implies that two vertices among $a$, $b$, $c$ have a common neighbour in $R$. Hence we have a $C_3 \nabla C_4$. \[\square\]

Proposition 3. $G$ contains $C_{k-1} \nabla C_k$ for $k \leq (n + 27)/10$.

Proof. Proposition 2 implies that it is true for $k = 4$. Assume the result is true for some $k$ with $4 \leq k \leq (n + 17)/10$. Choose a $C_{k-1} \nabla C_k$. Let $\delta$ be the neighbour of $\beta$ on $C_k$ different from $\gamma$ and $R = G\setminus C_k$.

$$d_R(\beta) + d_R(\gamma) + d_R(\delta) \geq 3[(2n + 1)/5 - (k - 1)].$$
But
\[ k < (n + 18)/10 \iff 3[(2n + 1)/5 - (k - 1)] > n - k. \]
So
\[ d_R(\beta) + d_R(\gamma) + d_R(\delta) > |R|. \]

At least two vertices among \( \beta, \gamma, \delta \) have a common neighbour in \( R \): for \( (\beta, \gamma) \) or \( (\beta, \delta) \), the result is immediate, and for \( (\gamma, \delta) \), we have a \( C_k \land C_{k+1} \) (Fig. 2). \( \square \)

**Proposition 4.** If the connectivity of \( G \) is at most \( (n - 2)/10 \) then \( G \) contains cycles of every length between 8 and \( (9n + 22)/10 \).

**Proof.** Let \( S \) be a vertex cut of minimum cardinality and \( s = |S| \). Let \( A_1, \ldots, A_p \) be the components of \( G\setminus S \) and \( q = \min\{|A_i|\}_{i=1, \ldots, p} \). We can assume that \( |A_1| = q \).

If \( x \in A_1 \), then \( (2n + 1)/5 \leq d(x) \leq q - 1 + s \), which implies \( q \geq (2n + 6)/5 - s \).

Suppose \( G\setminus S \) has at least three components. Then
\[ n \geq 3q + s \geq 3(2n + 6)/5 - 2s \geq n + 3 \]
which is impossible. Therefore \( G \) has two components \( A_1 \) and \( A_2 \), and
\[ q = |A_1| \leq |A_2| \leq n - [(2n + 6)/5 - s] - s \leq (3n - 6)/5. \]
Let \( x \in A_2 \), then
\[ d_{A_2}(x) \geq (2n + 1)/5 - s \geq (3n + 4)/10. \]
So the subgraph of \( G \) induced by \( A_2 \) has at most \( (3n - 6)/5 \) vertices and its minimum degree \( \delta \) is at least \( (3n + 4)/10 \). According to a theorem of Faudree, Rousseau and Schelp [4], this subgraph is pan-path connected (this means that it contains paths of every length between 2 and \( |A_2| - 1 \)). The same holds for \( A_1 \).
Let $u \in S$, $v \in S$, $T = S \setminus \{u, v\}$, then $G \setminus T$ is 2-connected, and, from Menger's theorems, we can find two distinct vertices $a_1$, $a'_1$ in $A_1$ and two distinct vertices $a_2$, $a'_2$ in $A_2$ such that $a_1u$, $a_2u$, $a'_1v$, $a'_2v$ belong to $E(G)$. Using paths in $A_1$ between $a_1$ and $a'_1$ and paths in $A_2$ between $a_2$ and $a'_2$ (Fig. 3) we obtain the mentioned result. □

(D) This part needs some lemmas and is partitioned into two sets of values for the lengths of the cycles.

**Lemma 1.** Let $G$ be a graph of $\mathcal{H}$, such that:

(i) $G$ contains a $C_{k-1} \setminus C_k$ with $k \geq 2n/5$, $n \geq 20$,

(ii) $G$ does not contain a $C_k \setminus C_{k+1}$ nor a $C_{k+1} \setminus C_{k+2}$.

Then, if $H$ is a component of $G \setminus C_k$ that is not a vertex:

(a) every edge of $H$ is contained in at least two triangles of $H$,

(b) the diameter of $H$ is at most 2.

**Proof of (a).** Let $A = G \setminus C_k$, and let $xy$ be an edge of $H$. Let $B = N_A(x) \cap N_A(y)$. Let us suppose, by contradiction, that $|B| \leq 1$.

$$d(x) + d(y) = |N_H(x)| + |N_H(y)| + |N_C(x)| + |N_C(y)|,$$

$$|N_H(x)| + |N_H(y)| = |N_H(x) \cup N_H(y)| + |B|.$$ 

So

$$|N_C(x)| + |N_C(y)| \geq 2(2n + 1)/5 - |H| - 1,$$

which is always strictly greater than 2, so $x$ or $y$ has at least one neighbour on $C_{k-1}$. Without loss of generality we can suppose that there exists a vertex $a$ of $C_{k-1}$, adjacent to $x$. So there exists an orientation of $C$ such that $a^+$, $a^{+2}$ and $a^{+3}$ are different from $y$ ($n \geq 20$).

(a-1) First let us suppose that $N_A(a^-) \cap N_A(y) = \emptyset$. Then $y$ is not adjacent to:

(i) $N_A(x) \setminus B$ by definition of $B$.

(ii) $N_A(a^-)$ by hypothesis (a-1).

(iii) $N_C(a^-) \setminus \{\gamma, \alpha\}$ for, if not, we would have $C_k \setminus C_{k+1}$ as on Fig. 4.

(iv) $N_C(x) \setminus \{\gamma, \alpha\}$ for, if not, we would have $C_{k+1} \setminus C_{k+2}$ as on Fig. 5.

We shall prove that these sets are mutually disjoint: we have to prove:

(i) $(N_A(x) \setminus B) \cap N_A(a^-) = \emptyset$ and this is clear for, if not, we would have $C_{k+1} \setminus C_{k+2}$ as on Fig. 6.
(ii) \( \{y, a\} \supseteq \overline{N}_2^{-}(a^{+}) \cap \overline{N}_2^{-}(x) \) and this is clear for, if not, we would have \( C_k \cup C_{k+1} \) as on Fig. 7.

We deduce from the preceding remarks that

\[
\frac{(2n+1)}{5} \leq d(y) \leq n - (d(x) - 1) - (d(a^{+}) - 2) \\
\leq n - \left[ \frac{(2n+1)}{5} - 1 \right] - \left[ \frac{(2n+1)}{5} - 2 \right]
\]

and this is impossible for \( n \geq 13 \).

(a-2) Let us suppose now that \( N_A(a^{+}) \cap N_A(y) \neq \emptyset \) and let \( z \) be a vertex of that set: \( z \) is different from \( x \) for, if not, we would have \( C_k \cup C_{k+1} \).

\( y \) is not adjacent to:

(i) \( N_A(a^{+3}) \) for, if not, we would have \( C_k \cup C_{k+1} \) as on Fig. 8.

(ii) \( N_A(x) \cup B \) by definition of \( B \).

(iii) \( \overline{N}_2^{-}(x) \backslash \{\alpha, \gamma\} \) for, if not, we would have \( C_{k+1} \cup C_{k+2} \) as on Fig. 9.

(iv) \( \overline{N}_2^{-}(a^{+3}) \backslash \{\alpha, \gamma\} \) for, if not, we would have \( C_k \cup C_{k+1} \) as on Fig. 10.

We shall prove that these sets are mutually disjoint. We have to prove:

(i) \( \{\alpha, \gamma\} \supseteq \overline{N}_2^{-}(x) \cap \overline{N}_2^{-}(a^{+3}) \) and this is clear for, if not, we would have \( C_k \cup C_{k+1} \) as on Fig. 11.

(ii) \( N_A(a^{+3}) \cap (N_A(x) \cup B) = \emptyset \). This case is a bit more complicated. Let us suppose that there exists a vertex \( t \) in this set and let us consider the new \( C_{k-1} \cup C_k \) using the edges \( ax, xt \) and \( ta^{+3} \) as on Fig. 12.
\(a^+\) and \(a^{+2}\) play the same role for \(C_{k-1} \nabla C_k\) that \(x\) and \(y\) play for \(C_{k-1} \nabla C_k\) with \(x\) playing the role of \(a^+\).

Let \(A' = G \setminus C_k\). If \(N_A(x) \cap N_A(a^{+2}) = \emptyset\) then we find a contradiction with \(|B'| \leq 1\), where \(B' = N_A(a^+) \cap N_A(a^{+2})\). But, if \(B' \neq \emptyset\), we have \(C_{k+1} \nabla C_k\), a contradiction.

This implies \(N_A(x) \cap N_A(a^{+2}) \neq \emptyset\) and we have \(C_k \nabla C_{k+1}\).

Analogous calculations, as in (a-1), lead to a contradiction.

**Proof of (b).** Let us suppose, by contradiction, that there exist in \(H\) two vertices \(x\) and \(y\) at a distance 3 in \(H\), that is \(N_A(x) \cap N_A(y) = \emptyset\) and there is a path of length 3 between \(x\) and \(y\).

As in (a), it is easy to show that \(x\) or \(y\) is adjacent to a vertex of \(C_k \setminus \{\alpha, \beta, \gamma\}\).

Without loss of generality, we can suppose that there exists a vertex \(a\) of \(C_k \setminus \{\alpha, \beta, \gamma\}\) adjacent to \(x\) (for \(n \geq 27\)).

**Proofs analogous to the preceding ones allow us to say that:**

(i) \(\{\gamma, \beta\} \ni N^+_C(x) \cap N^+_C(a^+)\).

(ii) \(N_A(a^+) \cap N_A(x) = \emptyset\).

(iii) \(N_A(x) \cap N_A(y) = \emptyset\).

Moreover, we have

(iv) \(N_C(y) = \emptyset\).

(v) \(|N_C(x)| = |N^+_C(x)|\).

But from

\[
d(x) + d(a^+) + d(y) = |N_A(x)| + |N_A(a^+)| + |N_A(y)| + |N_C(x)|
+ |N_C(a^+)| + |N_C(y)|
= |N_A(x) \cup N_A(a^+ \cup N_A(y)| + |N_A(a^+) \cap N_A(y)|
+ |N^+_C(x) \cup N^+_C(a^+)| + |N^+_C(x) \cap N^+_C(a^+)|.
\]

We deduce that

\[
3(2n + 1)/5 \leq n + 2 + |N_A(A^+) \cap N_A(y)|
\]

and so: \(|N_A(a^+) \cap N_A(y)| \geq 2\) (at least for \(n \geq 17\)).

We can prove, similarly, that \(|N_A(a^-) \cap N_A(y)| \geq 2\).

But all this implies the existence of two different vertices \(u\) and \(v\), \(u \in N_A(y) \cap N_A(a^-)\), \(v \in N_A(y) \cap N_A(a^+)\), and consequently the existence of \(C_{k+1} \nabla C_{k+2}\).

**Proof of (b-2).** Let us suppose now that \(x\) and \(y\) have neighbours on \(C_k \setminus \{\alpha, \beta, \gamma\}\).

Without loss of generality we can suppose that \(d_C(x) \leq d_C(y)\), and let \(a\) be a neighbour of \(x\) on \(C_k\) such that between \(a\) and \(a^{+4}\) there is no vertex among \(\{\alpha, \beta, \gamma\}\).

It is easy to prove that:

(i) \(a^{+4}\) is not adjacent to \(N_A(y)\). Indeed, if it was not so, we would have \(C_{k+2} \nabla C_{k+1}\).
(ii) \(a^{+4}\) is not adjacent to \(N_A(x)\). If \(t\) belongs to \(N_A(a^{+4}) \cap N_A(x)\), by part (a) of the lemma we can find, in \(H\), a path of length 3 between \(x\) and \(t\), and then there exists \(C_k \nabla C_{k+1}\).

(iii) \(a^{+4}\) is not adjacent to \(N_C^+(y)\) (if not, there would exist \(C_k \nabla C_{k+1}\)) nor to \(N_C^{+2}(y)\) (here also, to find \(C_k \nabla C_{k+1}\), we use a triangle as on Fig. 13).

As it has been done before we can prove that:

(i) \(N_A(y) \cap N_A(x) = \emptyset\),

(ii) \(N_C^+(y) \cap N_C^{+2}(y) = \emptyset\).

So we have

\[d(a^{+4}) \leq n - d_A(y) - d_A(x) - d_C(y) - d_C(x)\]

Using the hypothesis \(d_C(x) \leq d_C(y)\) we obtain: \(d(a^{+4}) \leq n - d(y) - d(x)\) but this is impossible. This completes the proof of Lemma 1. \(\square\)

**Lemma 2.** With the hypothesis of Lemma 1, \(|V(H)| \geq n/3\).

**Proof.** Before begining the proof let us remark that, between two vertices of \(H\), there exist in \(H\) a path of length two and a path of length three, this being an easy corollary of Lemma 1.

Then, let \(x\) be a vertex of \(H\), adjacent to a vertex \(a\) of \(C_k \setminus \{\alpha, \beta, \gamma\}\), and let \(xyz\) be a path of length two of \(H\). Let us consider \(x\), \(y\) and \(a^{+2}\) (we can always choose \(x\) and \(a\) such that \(a, a^{-}\) and \(a^{+2}\) do not belong to \(\{\gamma, \alpha\}\)).

We can then prove the following:

\[N_C(a^{+2}) \cap N_C^+(y) = \emptyset, \quad d_H(a^{+2}) \leq 1, \quad N_C(a^{+2}) \cap N_C^{+2}(y) = \emptyset,\]
\[N_C(a^{+2}) \cap N_C^{+3}(z) = \emptyset, \quad N_C(a^{+2}) \cap N_C^{+4}(z) = \emptyset, \quad N_C^+(y) \cap N_C^{+3}(z) = \emptyset, \quad \{a^{+2}, a^{+}\} \supseteq N_C^{+2}(y) \cap N_C^{+3}(z),\]
\[|N_C^+(y) \cap N_C^{+3}(z)| \leq 1, \quad N_C^+(y) \cap N_C^{+4}(z) = \emptyset, \quad N_C^+(y) \cap N_C^{+2}(y) = \emptyset, \quad N_C^{+3}(y) \cap N_C^{+4}(z) = \emptyset.\]

So we have, with \(A = G \setminus (C_k \cup H)\):

\[2n + 1 \leq d(a^{+2}) + 2d(y) + 2d(z)\]
\[= |N_H(a^{+2})| + |N_A(a^{+2})| + |N_C(a^{+2})| + 2 |N_H(y)| + 2 |N_H(z)|\]
\[+ |N_C(y)| + |N_C^2(y)| + |N_C^{+2}(z)| + |N_C^{+4}(z)|.\]
which implies:

$$2n + 1 \leq |V(C_k)| + 1 + 4|V(H)| + |V(A)| \leq n + 1 + 3|V(H)|,$$

whence

$$|V(H)| \geq n/3. \Box$$

**Corollary.** If $C_{k-1} \setminus V(C_k)$ exists in $G$ with $k > 2n/3$ then $G \setminus C_k$ is an independent set.

**Lemma 3.** Let us consider two disjoint chains: the chain $P_1$ with extremities $a$ and $b$ and of length $l_1$, the chain $P_2$ with extremities $c$ and $d$ and of length $l_2$; assume that every vertex of $P_2$ has two consecutive vertices of $P_1$ in its neighbourhood. Then there exist in $G$ chains of any length between $l_1$ and $l_1 + l_2 + 1$, with extremities $a$ and $b$, containing the vertices of $P_1$ plus some vertices of $P_2$.

**Proof.** Let us denote by $c$, $c^+$, $c^{+i}, \ldots$, $c^{+l_2}$ the vertices of $P_2$. We prove by induction on $i$ that there exists a chain containing the vertices of $P_1$ and $c$, $c^+$, $c^{+i}, \ldots$, $c^{+l_2}$. This is easy when $i = 0$ (since $c$ has two consecutive neighbours on $P_1$). Let us then denote by $x_i$ and $y_i$ the pair of consecutive vertices of $P_1$ adjacent to $c^{+i}$ and by $c^{+i}(j_0 \leq i)$ the first vertex of $P_2$ adjacent to $x_i$ and $y_i$. Replacing the edge $(x_i, y_i)$ of $P_1$ by the subchain of $P_2$ of extremities $c^{+i}$ and $c^{+i}$ we obtain a new chain $P_1'$ of extremities $a$ and $b$. We denote by $P'_2$ the subchain of $P_2$ of extremities $c$ and $c^{+i}$. $P'_1$ and $P'_2$ play the same role as $P_1$ and $P_2$ and we now need to add $j_0$ vertices to $P'_1$ instead of $i + 1$ vertices: this is possible by the hypothesis of induction. $\Box$

**Proposition 5.** Assume that, for $(2n + 5)/5 \leq k \leq (9n - 9)/10$, $G$ contains at least one $C_{k-1} \setminus V(C_k)$. If, for every $C_{k-1} \setminus V(C_k)$, $G \setminus C_k$ is independent, then $G$ contains $C_k \setminus V(C_{k+1})$.

**Proof.** Assume that $G$ contains no $C_k \setminus V(C_{k+1})$. (Necessarily, $k \geq (4n + 2)/5$). Choose a $C_{k-1} \setminus V(C_k)$. Let $C_k = C$, $R = G \setminus C$.

For $y \in R$, let $T(y) = \{a \in C \mid a \in N(y) \text{ and } a^{+2} \in N(y)\}$, and $p(y) = |T(y)|$.

$|N_C(y)| \geq (2n + 1)/5$ and $y$ has no consecutive neighbours on $C$, hence:

$$2p(y) + 3[(2n + 1)/5 - p(y)] \leq k \quad \text{and} \quad p(y) \geq (6n + 3)/5 - k.$$

$y \notin T(y)$ for we should have $C_k \setminus V(C_{k+1})$ (Fig. 14).
Let $y$ and $y'$ be two vertices of $R$, $A = \{a | a \in T(y) \setminus \{\alpha\}\}$, $A' = \{a' | a' \in T(y') \setminus \{\alpha\}\}$. Then $A \cup A' \cup R$ is independent, for:

(i) If $a_1 \in A$ is joined to $a_2 \in A$ then we have $C_k \setminus C_{k+1}$ and the same holds for $a'_1 \in A'$ and $a'_2 \in A'$ (Fig. 15).

(ii) If $a \in A$ is joined to $z \in R$ (necessarily different from $y$) then we have $C_{k-1} \setminus C_k$ such that $G \setminus C_k$ contains the edge $az$, which is impossible (Fig. 16). The same holds for $a' \in A'$ and $z \in R$.

(iii) If $a \in A$ is joined to $a' \in A'$, then we have $C_{k-1} \setminus C_k$ such that $G \setminus C_k$ contains the edge $aa'$, which is impossible (Fig. 17).

If not, then $G$ contains an independent set of

$$ (p(y) - 1) + (p(y') - 1) + (n - k) $$

vertices. From (1) we have:

$$ (p(y) - 1) + (p(y') - 1) + (n - k) \geq 2[(6n + 3)/5 - k - 1] + n - k $$

$$ \geq (7n + 19)/10 \quad \text{(for } k \leq (9n - 9)/10 \text{)} $$

a contradiction for $G$ is hamiltonian.

$R \cup (N_+^+(y) \setminus \{\gamma, \beta\})$ is independent:

(i) The two sets are independent.

(ii) Assume that $a_1 \in \Gamma_+^+(y) \setminus \{\gamma, \beta\}$ is joined to some $y' \in R$ ($y'$ necessarily different from $y$); let $a_2$ be a vertex of $(A \setminus \{\gamma\}) \cap (A' \setminus \{\gamma\})$ (necessarily different from $a_1$ and $a_2$), then $G$ contains $C_k \setminus C_{k+1}$ (Fig. 18) which is impossible.

$y$ is not joined simultaneously to $\alpha$ and $\gamma$, so we can find an independent set $S$ such that:

$$ |S| \geq n - k + (2n - 4)/5, $$

$$ |S| \geq (n + 9)/10 + (2n - 4)/5 \geq (5n + 1)/10 \geq n/2 + 1/10 $$

which leads to a contradiction. 

\[ \square \]
**Propositions 5'**. For \((9n - 8)/10 \leq k \leq (9n + 1)/10\), \(G\) contains a \(C_{k+1}\) under the hypothesis of the Proposition 5.

**Proof.** The proof begins as for Proposition 5, but we just suppose we have no \(C_{k+1}\) (instead of \(C_k \cup C_{k+1}\)), so \(S = R \cup N_C^-(y)\) is an independent set (we do not need to exclude \(\{y, \beta\}\)) and we have a contradiction for \(|S| \geq (5n + 1)/10\) if \(k \leq (9n + 1)/10\). □

**Proposition 6.** If the connectivity of \(G\) is at least \((n - 1)/10\), then for
\[(n + 18)/10 \leq k \leq (3n - 2)/5\]

\(G\) contains either \(C_{k-1} \cup C_k\) or \(C_k \cup C_{k+1}\).

**Proof.** The proof will be by induction on \(k\). The result is verified for \(k \leq (n + 27)/10\) using Proposition 3. Assume that for \(k\) such that \((n + 18)/10 \leq k \leq (3n - 7)/5\), \(G\) contains \(C_{k-1} \cup C_k\) but neither \(C_k \cup C_{k+1}\) nor \(C_{k+1} \cup C_{k+2}\). We shall show that this leads to a contradiction.

Choose a \(C_{k-1} \cup C_k\). Let \(a\) and \(d\) be two vertices of \(C_{k-1}\) such that the path of \(C_{k-1}\) with extremities \(a\) and \(d\), not containing \(y\), has \(m\) vertices other than \(a\) and \(d\), with \(m \geq 2\), \(m\) as small as possible under the condition that \(a\) and \(d\) have two distinct neighbours in \(R\), \(a^+\) and \(d^+\).

(I) \(m = 2\).

Let \(b = a^+, c = a^{+2}\).

If two vertices among \(\{a, b, c, d^+\}\) have a common neighbour in \(R\), \(G\) contains either \(C_k \cup C_{k+1}\) or \(C_{k+1} \cup C_{k+2}\), which is impossible, so:
\[
d_R(a') + d_R(b) + d_R(c) + d_R(d') \leq |R|. \tag{1}
\]

Assume that a vertex \(x\) different from \(\beta\) and \(\gamma\) belongs to \(N_C^+(a') \cap N_C(b)\); this implies the existence of \(C_k \cup C_{k+1}\) (Fig. 19)

But \(a\) has no consecutive neighbours on \(C\), hence:
\[
d_C(a') + d_C(b) \leq |C| + 1. \tag{2}
\]

The same holds for \(c\) and \(d^+\):
\[
d_C(c) + d_C(d') \leq |C| + 1. \tag{3}
\]

Summing (1), (2), (3) we have:
\[
4(2n + 1)/5 \leq d(a') + d(b) + d(c) + d(d') \leq n + |C| + 2
\]

which implies \(|C| \geq (3n - 6)/5\): a contradiction.

Fig. 19.
(II) \( m = 3 \).

Let \( b = a^*, e = a^+^2, c = a^+^3 \). \( d_c(b) \geq (2n - 4)/5 \) for \( \{d') \supset N_R(b) \). It results that \( b \) has several pairs of consecutive neighbours on \( da \) (the same holds for \( c \)). Hence:

\[ N_R(a') \cap N_R(d') = \emptyset \]

for the contrary would imply (see Fig. 20) the existence of \( C_k \n C_{k+1} \).

It is now easy to see that two vertices among \( \{a', b, e, d'\} \) have no common neighbour in \( R \). Whence:

\[ d_R(a') + d_R(b) + d_R(e) + d_R(d') \leq |R|. \tag{1} \]

As for \( m = 2 \):

\[ d_c(a') + d_c(b) \leq |C| + 1. \tag{2} \]

From \( (d) \supset N_C(e) \cap N_C(d') \) (if not, we would have a \( C_k \n C_{k+1} \) as in Fig. 21), we obtain:

\[ d_c(e) + d_c(d') \leq |C| + 1. \tag{3} \]

Summing (1), (2), (3) we obtain the same contradiction as for \( m = 2 \).

(III) \( m \geq 4 \).

Let \( b = a^+, c = d^- \).

\( b, b^+, c, c^- \) have at most one neighbour in \( R \), so each one has at least:

\[ 2(2n - 4)/5 - (3n - 7)/5 - (m - 1) = (n + 4)/5 - m \] pairs of consecutive neighbours on \( da \). By analogous calculations we prove that the vertices of \( b^+c^- \) have at least \( (n + 4)/5 - m \) pairs of consecutive neighbours on \( da \). Moreover, there are at least \( (n - 1)/10 \) disjoint edges between \( C \) and \( R \) because of the connectivity of \( G \). In order to determine the largest possible value of \( m \), we consider the placement of these \( t = (n - 1)/10 \) edges \( l_1, l_2, \ldots, l_t \). On \( \beta \alpha \) at most three of the \( l_i \) are on consecutive points, for, otherwise \( m = 2 \). Thus to make \( m \) as large as possible, the edges \( l_1 \cdots l_t \) would be placed as in Fig. 22. Hence we have

\[ [(\lfloor (n - 1)/10 \rfloor - 7)/3] + 1 = r \]

maximal sets of consecutive \( l_i \) (one has seven elements, there is possibly one set with one or two elements, and the other sets all have order three). It follows that

\[ m \leq \lfloor (\lfloor (3n - 2)/5 \rfloor - \lfloor (n - 1)/10 \rfloor)/r \rfloor \]
Fig. 22.

and \((n + 4)/5 - m \geq 1\) for \(n \geq 102\). Thus each vertex of \(bc\) has at least one pair of consecutive neighbours on \(da\) (in fact, for \(n \geq 102\), \((n + 4)/5 - m \geq 4\): this can be seen with the help of a little computer program to calculate \((n + 4)/5 - m\) for a few values of \(n = 102, 103, \ldots\)).

We consider now the vertices \(a'b, c, d'\).

Using Lemma 3 with the chains \(da\) and \(bc\), we obtain a chain \(P\) of length \(|P| - 2\), of extremities \(d\) and \(a\), containing only vertices of \(C\). If we carefully examine the construction of Lemma 3, we can also see that \(P\) has two vertices at distance two in \(P\) and adjacent in \(G\). If \(a'\) and \(d'\) had a common neighbour in \(R\), we would have \(C_{k+1} \cup C_{k+2}\). If \(a'\) and \(d'\) were adjacent, we would have \(C_k \cup C_{k+1}\). So:

\[
d'_{R}(a') + d'_{R}(d') \leq |R| - 2. \tag{1}
\]

As in the case \(m = 2\)

\[
d'_{C}(a') + d'_{C}(b) \leq |C| + 1, \tag{2}
\]

\[
d'_{C}(d') + d'_{C}(c) \leq |C| + 1. \tag{3}
\]

Summing (1), (2) and (3) with \(d'_{R}(b) \leq 1\) and \(d'_{R}(c) \leq 1\), we obtain:

\[
4(2n + 1)/5 \leq d(a') + d(b) + d(c) + d(d') \leq n + |C| + 2
\]

which implies: \(|C| \geq (3n - 6)/5\): a contradiction.

This ends the proof of Proposition 6. □

**Proposition 7.** If the connectivity of \(G\) is at least \((n - 1)/10\), then for \((3n - 1)/5 \leq k \leq (9n + 1)/10\), \(G\) contains either \(C_{k-1} \nabla C_k\) or \(C_k \nabla C_{k+1}\).

**Proof.** The proof will be by induction on \(k\). From Proposition 5, \(G\) contains \(C_{k-1} \nabla C_k\) with \(k \leq (3n - 6)/5\). Assume that for \(k\) such that \((3n - 6)/5 \leq k \leq (9n - 9)/10\), \(G\) contains \(C_{k-1} \nabla C_k\) but neither \(C_k \nabla C_{k+1}\) nor \(C_{k+1} \nabla C_{k+2}\). We shall show that it is impossible. Using Proposition 5, we choose a \(C_{k-1} \nabla C_k\) such that \(G \setminus C_k\) is not independent. Let \(C_k = C\) and \(R = G \setminus C\). Let \(H\) be a component or \(R\) different from one vertex. We part the proof into two cases, depending on the minimum degree in \(H\) of a vertex \(x\) of \(H\).
Case a: For any vertex $x$ in $H$, $d_H(x) > \frac{|H|}{2}$.

Then, according to a theorem of Faudree, Rousseau and Schelp [4], $H$ is pan path connected. Because of the connectivity of $G$ we know that between $C_k$ and $H$ there are, at least, $(n - 1)/10$ vertex disjoint edges. We are interested by two such edges having their extremities on $C_k$, not consecutive, without $\{\alpha, \beta, \gamma\}$ between them but as near as possible with these conditions. Let $m$ be the minimum distance on $C_k$ of two extremities of such edges. The largest value of $m$ corresponds to Fig. 23. With a calculation analogous to that of Proposition 6, it is possible to prove that, for $n \geq 102$, $m$ is at most 24. From lemma 2:

$$|V(H)| \geq n/3 \geq 102/3 \geq m + 1.$$ 

Since $H$ is pan-connected, $G$ contains a $C_k \setminus C_k+1$ (see Fig. 24).

Case b: There exists a vertex $x$ of $H$ such that $d_H(x) \leq \frac{|H|}{2}$.

We prove successively:

1. If $y$ is a vertex of $H$ adjacent to $x$, then $d_H(y) \geq |H| - 3$. We also obtain $d_H(x) \geq |H|/2 - 3$.

2. If $y$ is a vertex of $H$ non-adjacent to $x$, then $|N_H(x) \cap N_H(y)| \geq (n - 7)/5$, which implies that $d_H(x) \geq (n - 7)/5$ and $|H| \geq (2n - 14)/5$.

3. Between $x$ and a vertex $y$ of $H$, different from $x$, there exist paths of any length between 2 and $d_H(x)$. Such paths of length at least 3 can be chosen so that there exists a chord joining two vertices at distance 2 on them.

4. Using the $(n - 1)/10$ disjoint edges between $H$ and $C_k$, we conclude.

Proof of (1). Let $a$ be vertex of $C_k$ adjacent to $x$, $a \notin \{\alpha, \beta, \gamma\}$ (since $d_H(x) \leq |H|/2 \leq (n + 3)/5$, $d_C(x) \geq (n - 2)/5$)

$$(6n + 3)/5 \leq d(a^+) + d(x) + d(y) = d_C(a^+) + d_C(x) + d_C(y) + d(a^+) + d_H(x) + d_H(y).$$

We have

$$\{\alpha, \gamma\} \supseteq N_C(y) \cap N_C(x), \quad \{\beta, \gamma\} \supseteq N_C(x) \cap N_C^{2}(a^+),$$

$$\{a\} \supseteq N_C^{2}(a^+) \cap N_C(y)$$

and, in fact $|N_C(y) \cap N_C^{2}(a^+)| \leq 1$ ($\alpha$ and $\gamma$ can not belong simultaneously to $N_C^{2}(x)$).

$|N_C(x) \cap N_C^{2}(a^+)| \leq 1.$
So:
\[ d_C(a^+) + d_C(x) + d_C(y) \leq |C| + 3. \]

By choosing the orientation of \( C \), we obtain \( d_H(a^+) \leq 1 \) because \( a^+ \) and \( a^- \) can not have distinct neighbours in \( H \) (by Lemma 1). So:
\[ d_H(a^+) \leq 1 + |R| - |H|, \quad d_H(x) \leq |H|/2 \]
by hypothesis
\[ d_H(y) \leq |H| - 1. \]

Let
\[ d_H(x) = |H|/2 - \epsilon_1, \quad d_H(y) = |H| - 1 - \epsilon_2. \]

Therefore:
\[ \frac{6n + 3}{5} \leq |C| + 3 + 1 + |R| - |H| + |H|/2 + |H| - 1 - \epsilon_1 - \epsilon_2, \]
\[ \frac{n + 3}{5} \leq 3 + |H|/2 - \epsilon_1 - \epsilon_2 \quad \text{or} \quad |H| \geq (2n - 24)/5 + 2(\epsilon_1 + \epsilon_2). \]

We know yet that: \( |H| < (2n + 6)/5 \). So we have: \( \epsilon_1 + \epsilon_2 < 3 \), and \( \epsilon_1 > 0, \quad \epsilon_2 > 0 \), which implies \( \epsilon_1 \leq 3 \) and \( \epsilon_2 \leq 3 \).

**Proof of (2).** Let us consider, as in (1), \( x, \ y \) and \( a^+ \):
\[ d_H(x) + d_H(y) = |H| - 2 + |N_H(x) \cap N_H(y)|. \]

By considering \( N_C(y), \ N_C^2(y) \) and \( N_C^3(a^+) \), we obtain:
\[ \frac{6n + 3}{5} \leq |C| + 3 + |H| - 2 + |N_H(x) \cap N_H(y)| + |R| - |H| + 1, \]
\[ \frac{n + 3}{5} \leq |N_H(x) \cap N_H(y)| + 2. \]

**Proof of (3).** Let \( B = N_H(x), \ C = H\setminus(B \cup \{x\}) \). By (1) and (2), we know that \( B \) is pan path-connected. Let \( y \) and \( z \) be two different vertices of \( B \). Between \( z \) and \( y \) there exist in \( B \) paths of every length between 2 and \( d_H(x) - 1 \) and \( z \) is adjacent to \( x \). So between \( x \) and \( y \) there exist paths of every length between 1 and \( d_H(x) \) (the existence of paths of lengths 2 and 3 between two vertices of \( H \) is an easy corollary of Lemma 1). Let \( y \) be a vertex of \( C \). By (2), we know that there exist two vertices \( z_1 \) and \( z_2 \) in \( B \) adjacent to \( y \). Using the pan path-connectivity of \( B \), we find the announced result.

**Proof of (4).** We have at least \( (n - 11)/10 \) disjoint edges between \( H\setminus\{x\} \) and \( C \); we denote by \( b_i \) (\( 1 \leq i \leq (n - 11)/10 \)) the ends of these edges on \( C \), and \( c_i \) their extremities in \( H \). From (3), \( b_i \notin \{a^{+2}, \ldots, a^{+l}, a^{-2}, \ldots, a^{-l}\} \) where \( l = d_H(x) + 1 \). So we have \( \{a^-, a^+, a^{+l+1}, a^{+l+2}, \ldots, a^{-l-1}\} \supseteq \{b_i\} \). Let us choose \( a_1 \) and \( a_2 \), adjacent to \( x \), on \( C_k \), as far as possible on \( C_k \). As \( d_H(x) \leq |H|/2 \leq (n + 3)/5 \) we have \( d_C(x) > (n - 2)/5 \). On an other hand two consecutive vertices of \( C_k \) are not adjacent to \( x \). This implies that the distance of \( a_1 \) and \( a_2 \) on \( C_k \) is greater than \( n/5 \). Let us recall that \( |H| \geq (2n - 14)/5 \) implies \( |C| \leq (3n + 14)/5 \), so we have \( (3n - 6)/5 \leq |C| \leq (3n + 14)/5 \). Using the argument developed at the beginning of (4) with \( a_1 \) and \( a_2 \) (instead of \( a \)), we see that there are not enough vertices on \( C_k \) for the \( b_i \).
Proposition 7’. If the connectivity of $G$ is at most $(n - 1)/10$ then, for
\[
(9n + 2)/10 \leq k \leq (9n + 11)/10,
\]
$G$ contains a $C_k$.

Proof. It is included in the proof of Proposition 7 (if $(9n - 8)/10 \leq k \leq (9n + 1)/10$), the existence of a $C_{k-1} \cup C_k$ does not imply the existence of a $C_k \cup C_{k+1}$ or a $C_{k+1} \cup C_{k+2}$ but nevertheless implies the existence of a $C_{k+3}$.

Proposition 8. $G$ contains cycles of every length between $(9n + 3)/10$ and $n$.

Proof. The proof of that proposition is by contradiction. We prove that the two following situations are impossible for $(9n + 3)/10 \leq l \leq n$ when $C_l$ exists:

(A) There does not exist a $C_{l-1}$ and there exists a $C_{l+2}$ such that $G \setminus C_{l+2}$ is not independent.

(B) There does not exist a $C_{l-1}$ and there does not exist a $C_{l+2}$ such that $G \setminus C_{l-1}$ is not independent.

So, if $C_l$ exists, with $(9n + 13)/10 \leq l \leq n$, $C_{l-1}$ also exists. The existence of $C_n$, then, gives the complete proof.

Part (A). We want to prove that, for $(9n + 13)/10 \leq l \leq n$, if $C_l$ exists, “there does not exist a $C_{l-1}$ and there exists a $C_{l+2}$ such that $G \setminus C_{l+2}$ is not independent” is impossible. So we suppose: $(9n + 13)/10 \leq l \leq n$, and “there does not exist a $C_{l-1}$ and there exists a $C_{l+2}$ such that $G \setminus C_{l+2}$ is not independent”. Let $C = C_{l+2}$, $R = G \setminus C$.

We shall use the following lemma.

Lemma. Let $(c, d)$ be a chord of $C$, $y$ a vertex of $R$. Then $c^+$ (or $d^+$) is not adjacent to a set of cardinality $d_c(y) - 2$ of vertices of $C$, which are $a_i^+$ or $a_i^-$, with $a_i$ adjacent to $y$ on $C$.

Proof of the lemma. One of the following cases is true, always:

Case (a): There exists $a_0$ (in $N_c(y)$) in $]c, d[\cap N_c(y)$ such that $a_0^+d^+$ is an edge.

Case (b): There exists $a_0$ in $]c^+, d[\cap N_c(y)$ such that $a_0^-d^+$ is an edge.

Case (c): For every $a$ in $]c^+, d[\cap N_c(y)$, $d^+a^-$ is not an edge and for every $a$ in $]d, c[\cap N_c(y)$, $d^+a^+$ is not an edge.

Case (a): $c^+$ is not joined to $a^+$ for $a$ in $]c, d[\cap N_c(y)$ (Fig. 25).

Case (b): $c^+$ is not joined to $a^+$ for $a$ in $]d, a_0^[-]\cap N_c(y)$ (Fig. 26).

Case (c): $c^+$ is not joined to $a^-$ for $a$ in $]a_0^+, c[\cap N_c(y)$ (Fig. 27).

Case (a): $c^+$ is not joined to $a^+$ for $a$ in $]c, a_0[-]\cap N_c(y)$ (Fig. 28).

Case (b): $c^+$ is not joined to $a^+$ for $a$ in $]d, a_0[\cap N_c(y)$ (Fig. 29).

Case (c): $c^+$ is not joined to $a^-$ for $a$ in $]a_0, d[\cap N_c(y)$ (Fig. 30).

Here also the lemma is verified.
Case (c): $d^+$ is not adjacent to $a^+$ for $a$ in $]d, c[\text{ nor to } a^-$ for $a$ in $]c^+, d[$. Here $d^+$ (and not $c^+$ as in the two first cases) is not adjacent to a set as in the lemma. So the lemma is true. \qed

Proof of Part (A) We know (Proposition 1) that $G$ contains at least a $C_3$ and consider the position of that $C_3$. There are seven different cases. As in the lemma let $C = C_{i-2}$ (we suppose it exists) and $R = G \setminus C$; we examine the seven different cases.

Case a: $R \supset C_3$; let $C_3 = \{y, y', y''\}$. Let $\{a_i\}$ be the vertices of $C$ adjacent to $y$, $a_0$ one of them. Let $\{b_j\}$ be the vertices of $C$ adjacent to $y'$. $a_0^+$ is not joined to $b_j^+$ nor $b_j^{+3}$ for if it is we can construct a $C_{i-1}$ (Fig. 31). For the same reason $\{b_j^{+2}\} \cap \{b_j^{+3}\} = \emptyset$. So

\[
\begin{align*}
    d(a_0) &\leq n - \lfloor 2d_e(y') - 1 \rfloor, \\
    d_e(y') &\geq (2n + 1)/5 - (n - l + 1), \\
    (2n + 1)/5 &\leq d(a_0) \leq n - \lceil (4n + 2)/5 - 2n + 2l - 3 \rceil
\end{align*}
\]

whence $l \leq (9n + 12)/10$.

Case b: $|V(C_3) \cap V(R)| = 2$, $C_3 \cap R = \{(y, y')\}$, $V(C_3) \cap V(C) = \{a_0\}$.
Here $a_0^+$ is not adjacent to $h^+$ nor to $h^{+2}$ for $h \neq a_0$ (Fig. 32).

The calculations are then the same as in the previous case.

Case c: $|V(C_3) \cap V(R)| = 1$ and we suppose that we cannot find $C_3$ as in Case a or b. Let $\{y\} = V(C_3) \cap V(R)$. We suppose also here that $d_R(y) \geq 1$, let $y'$ in
$N_R(y)$ and let $V(C_3) \cap V(C) = \{a_1, a_2\}$ (Fig. 33). Here we use the lemma with the chord $(a_1, a_2)$. We know that $a_1^+ \lor a_2^+ \neq \text{not adjacent to } (d_c(y') - 2)$ vertices of $C$ which are $b_{r}^+$ or $b_{l}^-$ with $b_{r}$ in $N_c(y')$. Suppose that it is $a_1^+$. On the other hand we know that $a_1^+$ is not adjacent to $a_2^+$ and, as we are not in Case $b$, $a_1^+ \neq b_{r}^+$; and in fact $a_1^+ \neq b_{r}^-$ for if we have $a_1^+ - b_{r}^-$, there exists a $C_{l-1}$ (a contradiction with the hypothesis).

So

$$d(a_1^+) \leq n - (d_c(y') - 2) - d_c(y) - 1 \quad (-1 \text{ because of } y).$$

$$d_c(y) + d_c(y') = d(y) + d(y') - [d_R(y) + d_R(y')]$$

and, as we are not in Case $a$: $d_R(y) + d_R(y') \leq |R|$, so:

$$d_c(y) + d_c(y') \geq (4n + 2)/5 - (n - l + 2),$$

$$(2n + 1)/5 \leq d(a_1^+) \leq n + 1 - (4n + 2)/5 + (n - l + 2),$$

$$l \leq (4n + 12)/5$$

(we suppose here $l \geq (9n + 13)/10$).

Case $d$: $|V(C_3) \cap V(R)| = 1$, we are not in Cases $a$ or $b$, $C_3 \cap R = \{y\}$, $d_R(y) = 0$, $C_3 \cap C = \{a_1, a_2\}$. We know that $a_1^+$ is not adjacent to $a_1^+$. Suppose that $a_1^+$ is not adjacent to any $a_1^{+2} \neq a_1^+$, then $d(a_1^+) \leq n - [2d(y) - 1]$, which is impossible for $d(y) > (2n + 1)/5$ and $d(a_1^+) > (2n + 1)/5$. So $a_1^+$ is adjacent to some $a_1^{+2}$, say $a_3^{+2}$, we can find (Fig. 34) one another $C_{l-2}$, say $C'_{l-2}$, which has the same set of vertices than $C$ except $a_3$ and $y': G\backslash C'_{l-2}$ is not an independent set, for it contains the same edges than $G\backslash C_{l-2}$ and $ya_1a_2$ is a $C_3$ whose three vertices are on the $C_{l-2}$. We shall see in the following paragraphs, that it is impossible.

Case $e$: The three vertices of $C_3$ are on $C$, not consecutive on $C$, and there
does not exist a $C_3$ as in cases $a$, $b$, $c$. Let $C_3 = \{c, d, e\}$ (Fig. 35). We use the lemma and say that:

(i) $c^+$ or $d^+$ is not adjacent to $d_C(y) - 2$ vertices $a_i^+$ and $a_i^-$. Suppose it is $c^+$.

(ii) $c^+$ or $d^+$ is not adjacent to $d_C(y') - 2$ vertices $b_i^+$ and $b_i^-$. As in Case $c$ we have: $\left(\{a^+\} \cup \{a^-\}\right) \cap \left(\{b^+\} \cup \{b^-\}\right) = \emptyset$. If here also $c^+$ is not adjacent to $d_C(y') - 2$ vertices $b_i^+$, $b_i^-:

(2n + 1)/5 < d(c^+) < (d_c(y') + d_c(y') - 4)

and this gives: $l = (4n + 7)/5 + 4$ a contradiction with hypothesis. So assume that it is $d^+$ which is not adjacent to the $d_c(y') - 2$ vertices. We can make the same thing with the edges $(e, d)$ and $(e, c)$ and find a contradiction.

Case $f$: The three vertices of $C_3$ are on $C$, two of them are consecutive on $C$, say $c$ and $e$ and we are not in cases $a$, $b$, $c$ (Fig. 36).

If $d^+$ is not adjacent to:

- $a^+$ with $a$ in $[d, c[$
- $a^-$ with $a$ in $[e^+, d[$
- $b^+$ with $b$ in $[d, c[$
- $b^-$ with $b$ in $[e^+, d[$

then we have: $d(d^+) \leq n - [d_c(y) + d_c(y') - 4]$ and this is impossible. We suppose, for example, that $d^+$ is adjacent to $a^+$ with $a$ in $[d, c[$. The proof of the lemma for the chord $(e, d)$ and the vertex $y$ gives that $e^+$ is not adjacent to $d_c(y) - 2$ vertices $a_i^+$ and $a_i^-$. With a light change in the proof of the lemma we can see, considering the chord $(c, d)$ that $e^+$ is not adjacent to $d_c(y') - 3$ vertices $b_i^+$ and $b_i^-$ and find finally:

$d(e^+) \leq n - [d_c(y) + d_c(y') - 4]

which is impossible.

Case $g$: Three vertices of $C_3$ are consecutive on $C$. Let $a_0$ be a vertex of $C \setminus \{e, d, c\}$, adjacent to $y$. Then $a_0^+$ is not adjacent to $a_i^+$ nor to $b_j^+$ for $b_j \notin \{c, d\}$ (Fig. 37). So:

$d(a_0^+) \leq n - [d_c(y) + d_c(y') - 2]

a contradiction.

This completes the proof of Part (A).
Part (B). We want to prove that, for \((9n + 13)/10 \leq l \leq n\), if \(C_l\) exists, \(\text{"There does not exist a } C_{l-1} \text{ and there does not exist a } C_{l-2} \text{ such that } G \setminus C_{l-2} \text{ is not independent"}\) is impossible.

Suppose that there does not exist \(C_{l-1}\) nor \(C_{l-2}\) such that \(G \setminus C_{l-1}\) is not independent, but \(C_l\) exists. Let \(C = C_l\); \(R = G \setminus C_l\); \(r = n - l\). For \(x\) in \(C_l\), we denote by \(a\) the vertices of \(N_C(x)\). We shall prove that there exists at least \([(n - 2)/5 - r]\) vertices \(a\) such that \(a^+\) is in \(N_C(x)\). Let

\[ T = \{a \mid a \text{ and } a^+ \text{ are in } N_C(x)\}, \quad t = |T|. \]

**Case a:** \(x\) and \(x^+\) have not a common adjacent vertex in \(R\). \(x^+\) is not adjacent to \(a^+\) (there does not exist a \(C_{l-1}\)) nor to \(a^+\) (there does not exist a \(C_{l-2}\) such that \(G \setminus C_{l-2}\) is not independent) (Fig. 38). So:

\[
\begin{align*}
\delta_C(x^+) &\leq l - |\{a^+\} \cup \{a^+\}| + 1, \\
|\{a^+\} \cup \{a^+\}| &\leq |\{a^+\} \cup \{a^+\}| + |\{a^+\} \cup \{a^+\}| - |\{a^+ \cup \{a^+\}| \\
&= 2d_C(x) - t.
\end{align*}
\]

\[ d_C(x^+) \leq 2d_C(x) + t + 1, \quad d_R(x^+) \leq r - d_R(x), \]

\[ d(x^+) \leq n - d(x) - d_C(x) + t + 1, \]

\[ 3(2n + 1)/5 - r \leq d(x^+) + d(x) + d_C(x) \leq n + t + 1, \]

\[ t \geq (n - 2)/5 - r \geq 1. \]

**Case b:** \(x\) and \(x^+\) have a common adjacent vertex in \(R\). \(x^{-3}\) is not adjacent to \(a^{-1}\) nor \(a^{-2}\) (Fig. 39). So \(d_C(x^{-3}) \leq l + t + 1 - 2d_C(x)\). But \(x\) and \(x^{-3}\) have no common adjacent vertex in \(R\). \(d_R(x^{-3}) \leq r - d_R(x)\); we find also \(t \geq (n - 2)/5 - r \geq 1\). Let \((c_0, d_0)\) be the chord of \(C\) such that the number \(k\) of vertices of \([c_0, d_0]\) is minimum. There is no edge between the vertices of \([c_0, d_0]\). There exist
respectively for $c_0^+, c_0^{++}, \ldots, c_0^{+k-1}$ vertices $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$ in $[d_0, c_0]$ such that $\alpha_i$ and $\alpha_i^+$ are in $N_C(c_0^+)$. We can construct (Fig. 40) a $C_{i-1}$. □

Remark. With the same methods, we can prove in fact that:

*If $G$ is hamiltonian, contains a $C_3$ and $\delta(G) \geq (n + 8)/3$, then $G$ has a $C_{n-1}$."

References