A generalization of the Riesz–Fischer theorem and linear summability methods

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Abstract

We extend the classical Riesz–Fischer theorem to biorthogonal systems of functions in Orlicz spaces: from a given double series (not necessarily convergent but satisfying a growth condition) we construct a function (in a given Orlicz space) by a linear summation method, and recover the original double series via the coefficients of the expansion of this function with respect to the biorthogonal system. We give sufficient conditions for the regularity of some linear summation methods for double series. We are inspired by a result of Fomin who extended the Riesz–Fischer theorem to $L^p$ spaces.

Keywords: Linear summability methods; Expansions of functions; Biorthogonal systems; Orlicz spaces

1. Introduction

In this work, we extend the classical Riesz–Fischer theorem to biorthogonal systems of functions in Orlicz spaces. Our method can be used to obtain some other results for biorthogonal systems of functions in more general functional spaces. Our construction appears as a particular case of a linear summability method for (possibly) non-convergent double series, for which we...
give and prove a more general statement. This is both a classical and an active field of research; although it is not possible to give an extensive bibliography on the subject, we point out the works of Abilov and Kerimov [1], Getsadze [12,11], Kantawala [14], Andrienko and Kovalenko [5,4], Andrienko [3,2], Móricz [18], Móricz and Tandori [20], Móricz and Szalay [19], Chen [6], Rhoades [23], Szalay [25,24], Patel [21]. Summability methods are a powerful tool in Fourier analysis and have applications for example in numerical analysis and the study of mathematical physics equations, see the work of Cheong [7,8].

Let $I$ be a bounded interval in $\mathbb{R}$. If $f$ and $g$ are two Lebesgue integrable real-valued functions on $I$, we write

$$(f | g) = \int_I f(x)g(x) \, dx.$$ 

Let $(\varphi_n)_{n \in \mathbb{N}}$ be an orthonormal sequence in $L^2(I)$, that is $(\varphi_n | \varphi_m) = \delta_{n,m}$ (Kronecker symbol) for $n$ and $m$ in $\mathbb{N}$. The classical Riesz–Fischer theorem asserts that given a sequence of real numbers $(c_n)_{n \in \mathbb{N}}$ in $\ell^2(\mathbb{N})$, there exists a function $f$ in the space $L^2(I)$ such that $c_n = (f | \varphi_n)$ for all $n \in \mathbb{N}$. In other words, the numbers $c_n$ are the coefficients of the expansion of the function $f$ in the orthonormal system $(\varphi_n)_{n \in \mathbb{N}}$.

This theorem may be extended in several directions. We can ask for its validity in other spaces of functions than $L^2(I)$, or for double orthonormal systems rather than simple ones.

In [10], Fomin extended the classical one-dimensional Riesz–Fischer theorem to the $L^p$ spaces, $1 \leq p < \infty$. He observed that given a sequence of real numbers $(c_n)_{n \in \mathbb{N}}$, the condition

$$\sum_{n \in \mathbb{N}} |c_n|^2 < \infty \quad (1.1)$$

is equivalent to the condition:

there exists an increasing sequence of positive numbers $(v_n)_{n \in \mathbb{N}}$ with $v_n \to \infty$ as $n \to \infty$, such that

$$\sum_{k=0}^{\infty} \left( \frac{1}{v_k} - \frac{1}{v_{k+1}} \right) \left| \int_I \sum_{m=0}^{k} c_m v_m \varphi_m(x) \, dx \right|^2 < \infty.$$ 

This led him to the following analogue of the Riesz–Fischer theorem in $L^p(I)$ ([10, Theorem 1]):

**Theorem (Fomin).** Let $1 \leq p < \infty$, $(\varphi_n)_{n \in \mathbb{N}}$ be an orthonormal system with $\varphi_n \in L^q(I)$, where $q$ is the conjugate exponent to $p$ ($\frac{1}{p} + \frac{1}{q} = 1$) and $(c_n)_{n \in \mathbb{N}}$ be a sequence of real numbers. If for some increasing sequence of positive numbers $(v_n)_{n \in \mathbb{N}}$ with $v_n \to \infty$ as $n \to \infty$ we have

$$\sum_{k=0}^{\infty} \left( \frac{1}{v_k} - \frac{1}{v_{k+1}} \right) \left| \int_I \sum_{m=0}^{k} c_m v_m \varphi_m(x) \, dx \right|^p < \infty,$$

then there exists a function $f \in L^p(I)$ such that $c_n = (f | \varphi_n)$ for all $n \in \mathbb{N}$.

The previous theorem was extended to Orlicz classes by Mazhar [17].

Our aim is to extend the Riesz–Fischer Theorem to double orthonormal systems in Orlicz spaces.

To state our result we need to recall some definitions and introduce some notations.
In the sequel, we consider double sequences and double series of real numbers or functions. There exist several notions of convergence for them, and we will consider the following ones. Given a double sequence of real numbers \( a = (a_{m,n})_{(m,n) \in \mathbb{N}^2} \) and a real number \( \alpha \), we say that \( a \) is \( M \)-convergent to \( \alpha \) if \( a_{m,n} \to \alpha \) as \( \min(m,n) \to \infty \), and that \( a \) is \( N \)-convergent to \( \alpha \) if \( a_{m,n} \to \alpha \) as \( m + n \to \infty \). We also say that the double series
\[
\sum_{(i,j) \in \mathbb{N}^2} a_{i,j}
\] (1.2)
converges rectangularly (or in Pringsheim sense) to the real number \( S \) if
\[
\sum_{i=0}^{m} \sum_{j=0}^{n} a_{i,j} \to S \quad \text{as } \min(m,n) \to \infty.
\] (1.3)
Finite sums in (1.3) are called the rectangular partial sums of the series (1.2). Thus the series (1.2) is rectangular convergent to \( S \) if the sequence of its rectangular partial sums is \( M \)-convergent to \( S \). Other notions of convergence for multiple series are described in [15].

We will also consider Orlicz spaces [16]. Let \( \phi \) be a \( N \)-function, that is \( \phi \) is a convex continuous even function on \( \mathbb{R} \) such that \( \phi(0) = 0 \). We assume that \( \phi \) satisfies the \( \Delta_2 \)-condition, that is: there exist \( c > 0 \) and \( u_0 > 0 \) such that
\[
\phi(2u) \leq 2\phi(u)
\]
for all \( u \geq u_0 \). Define
\[
\phi^*(v) = \max_{u \geq 0} \{ u|v| - \phi(u) \}
\]
for a real number \( v \). Then \( \phi^* \) is again a \( N \)-function, called the complementary function of \( \phi \) (in the sense of Young). Let \( I \) be a bounded interval in \( \mathbb{R} \). We consider \( L_\phi(I) \) the Orlicz space associated to \( \phi \). This space is endowed by the norm
\[
\| f \|_\phi = \inf \left\{ c > 0 : \int_I \phi \left( \frac{f(x)}{c} \right) \, dx \leq 1 \right\}.
\]
The following properties are equivalent:
(i) \( \phi \) satisfies the \( \Delta_2 \)-condition,
(ii) the Orlicz space \( L_\phi(I) \) is separable,
(iii) \( \| f \|_\phi < \infty \) if and only if \( \int_I \phi(f) < \infty \).

We say that a double sequence \( (\varphi_{m,n})_{(m,n) \in \mathbb{N}^2} \) of functions in \( L^2(I) \) is a double orthonormal system if
\[
(\varphi_{k,l} | \varphi_{m,n}) = \delta_{k,m} \cdot \delta_{l,n}
\]
for all \( (k, l) \) and \( (m, n) \) in \( \mathbb{N}^2 \).

The two-dimensional Riesz–Fischer theorem asserts that given a double sequence of real numbers \( (c_{m,n})_{(m,n) \in \mathbb{N}^2} \) in \( \ell^2(\mathbb{N}^2) \), there exists a function \( f \) in \( L^2(I) \) such that \( c_{m,n} = (f | \varphi_{m,n}) \) for all \( (m, n) \) in \( \mathbb{N}^2 \).

We write \( S \) for the space of all double sequences of real numbers \( u = (u_{m,n})_{(m,n) \in \mathbb{N}^2} \). We define the partial difference operators on \( S \):
\[
\Delta_{1,0}(u)_{m,n} = u_{m,n} - u_{m+1,n}.
\]
and
\[ \Delta_{0,1}(u)_{m,n} = u_{m,n} - u_{m,n+1}, \]
for \((m, n) \in \mathbb{N}^2\). Then \(\Delta_{1,0}\) and \(\Delta_{0,1}\) commute and we define \(\Delta_{1,1} = \Delta_{1,0} \Delta_{0,1} = \Delta_{0,1} \Delta_{1,0}\). Thus given a double sequence of real numbers \(v = (v_{m,n})_{(m,n) \in \mathbb{N}^2}\), we have
\[
\Delta_{1,1}(v)(m, n) = v_{m,n} - v_{m+1,n} - v_{m,n+1} + v_{m+1,n+1}.
\]
Note that if \(V\) is the double sequence of the rectangular sums of \(v\), that is \(V_{m,n} = \sum_{i=0}^{m} \sum_{j=0}^{n} v_{i,j}\) for \((m, n) \in \mathbb{N}^2\), then a straightforward computation gives
\[
v_{m+1,n+1} = \Delta_{1,1}(V)(m, n).
\] (1.4)

We write \(\frac{1}{v}\) for the double sequence \(\left(\frac{1}{v_{m,n}}\right)_{(m,n) \in \mathbb{N}^2}\) provided that none of the \(v_{m,n}\) is zero.

Note that a simple sequence \(v\) of non-zero real numbers is increasing if and only if \(\frac{1}{v_n} - \frac{1}{v_{n+1}} \geq 0\) for all \(n \in \mathbb{N}\). So the inequality \(\Delta_{1,1}t\left(\frac{1}{v}\right) \geq 0\) is the analogue of this condition in the case of a double sequence \(v\).

We can now state our main result, which extends the classical Riesz–Fischer theorem to biorthogonal systems of functions in Orlicz Spaces.

**Theorem 1.** Assume that \((\varphi_{m,n})_{(m,n) \in \mathbb{N}^2}\) is in \(L_{\phi^*} \cap L^2(I)\), where \(\phi^*\) is the complementary function to \(\phi\).

Let \((c_{m,n})_{(m,n) \in \mathbb{N}^2}\) be a double sequence of real numbers such that there exists a double sequence \((v_{m,n})_{(m,n) \in \mathbb{N}^2}\) of positive numbers with
\[
v_{m,n} \to \infty \quad \text{as } m + n \to \infty,
\] (1.5)
\[
\Delta_{1,1}\left(\frac{1}{v}\right)(k, l) \geq 0
\] (1.6)
for all \((k, l) \in \mathbb{N}^2\), and
\[
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \Delta_{1,1}\left(\frac{1}{v}\right)(k, l) \int_I \phi\left(\sum_{i=0}^{k} \sum_{j=0}^{l} v_{i,j} c_{i,j} \varphi_{i,j}(x)\right) dx < \infty.
\] (1.7)

Then there exists a function \(f \in L_{\phi}\) such that \(c_{m,n} = (f, \varphi_{m,n})\) for all \((m, n) \in \mathbb{N}^2\).

This paper is organized as follows. In Section 2, we give the proof of Theorem 1. In Section 3, we define some linear summability methods for double series and extend to double series a summability method considered by Fomin in [10]. We give a general theorem of regularity for these methods.

### 2. Proof of Theorem 1

We first note that
\[
v_{0,0} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \Delta_{1,1}\left(\frac{1}{v}\right)(k, l) = 1.
\] (2.1)
This follows from the identity
\[
\sum_{k=0}^{m} \sum_{l=0}^{n} \Delta_{1,1} \left( \frac{1}{v} \right) (k, l) = \frac{1}{v_{i,j}} - \frac{1}{v_{m+1,j}} - \frac{1}{v_{i,n+1}} + \frac{1}{v_{m+1,n+1}} \tag{2.2}
\]
for all \( i \leq m \) and \( j \leq n \) in \( \mathbb{N} \).

For \( x \in I \), we consider the series
\[
f(x) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \Delta_{1,1} \left( \frac{1}{v} \right) (k, l) \sum_{i=0}^{k} \sum_{j=0}^{l} v_{i,j} c_{i,j} \varphi_{i,j}(x),
\]
\[
g(x) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \Delta_{1,1} \left( \frac{1}{v} \right) (k, l) \sum_{i=0}^{k} \sum_{j=0}^{l} v_{i,j} c_{i,j} \varphi_{i,j}(x)
\]
and
\[
f_{m,n}(x) = \sum_{k=0}^{m} \sum_{l=0}^{n} \Delta_{1,1} \left( \frac{1}{v} \right) (k, l) \sum_{i=0}^{k} \sum_{j=0}^{l} v_{i,j} c_{i,j} \varphi_{i,j}(x)
\]
the rectangular partial sums of \( f \) for \((m, n) \in \mathbb{N}^2\).

From (1.7) we deduce
\[
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \Delta_{1,1} \left( \frac{1}{v} \right) (k, l) \phi \left( \sum_{i=0}^{k} \sum_{j=0}^{l} v_{i,j} c_{i,j} \varphi_{i,j}(x) \right) < \infty
\]
for almost every \( x \in I \). So by (2.1) and Jensen inequality for the convex function \( \phi \), we have \( g(x) < \infty \) for almost every \( x \in I \). So \(|f(x)| \leq g(x) < \infty \) for almost every \( x \in I \). Now we will show that
(i) \( f \in L_\phi \);
(ii) \( (f|\varphi_{k,l}) = c_{k,l} \) for all \((k, l) \in \mathbb{N}^2 \).

To prove (i), it is sufficient to prove \( \int_I \phi(v_{0,0}f) < \infty \), but this follows immediately from (2.1), Jensen inequality for \( \phi \) and (1.7). Note that the same argument shows that \( g \in L_\phi \).

Then (ii) follows from the Lebesgue dominated convergence theorem. Indeed, let \((k, l) \in \mathbb{N}^2 \).

As \( g \in L_\phi \) and \( \varphi_{k,l} \in L_{\phi^*} \) the product \( g \varphi_{k,l} \) is in \( L^1 \). We have \(|f_{m,n}\varphi_{k,l}| \leq g \varphi_{k,l} \) for all \((m, n) \in \mathbb{N}^2 \) and \( f_{m,n}(x) \varphi_{k,l}(x) \rightarrow f(x) \varphi_{k,l}(x) \) as \( \min(m,n) \rightarrow \infty \). So \((f_{m,n}|\varphi_{k,l}) \rightarrow (f|\varphi_{k,l}) \) as \( \min(m,n) \rightarrow \infty \). Since \((f_{m,n}|\varphi_{k,l}) = c_{k,l} \) if \( m \) and \( n \) are large enough, assertion (ii) is proved. This completes the proof of Theorem 1.

Remark. If we consider \( \phi(t) = t^2 \) in Theorem 1, we then get the classical two-dimensional Riesz–Fischer theorem, as a consequence of the following lemma.

Lemma 1. Let \((\varphi_{m,n})(m,n)\in\mathbb{N}^2\) be a double orthonormal system in \( L^2(I) \). A double sequence of real numbers \((c_{m,n})(m,n)\in\mathbb{N}^2\) is in \( \ell^2(\mathbb{N}^2) \) if and only if there exists a double sequence \( v = (v_{m,n})(m,n)\in\mathbb{N}^2 \) of positive numbers such that \( \Delta_{1,1}(v) \geq 0 \), \( v_{m,n} \rightarrow \infty \) as \( m + n \rightarrow \infty \) and
\[
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \Delta_{1,1} \left( \frac{1}{v} \right) (k, l) \int_I \left| \sum_{i=0}^{k} \sum_{j=0}^{l} v_{i,j} c_{i,j} \varphi_{i,j}(x) \right|^2 dx < \infty. \tag{2.3}
\]
Proof of the lemma. Let \( v = (v_{m,n})_{(m,n)\in \mathbb{N}^2} \) be a double sequence of positive numbers. For \((m,n) \in \mathbb{N}^2\), we have, by orthonormality:

\[
\sum_{k=0}^{m} \sum_{l=0}^{n} \Delta_{1,1} \left( \frac{1}{v} \right) (k,l) \int_I \sum_{i=0}^{k} \sum_{j=0}^{l} v_{i,j} c_{i,j} \varphi_{i,j}(x) \left( \frac{1}{v} \right) \, dx
= \sum_{k=0}^{m} \sum_{l=0}^{n} \Delta_{1,1} \left( \frac{1}{v} \right) (k,l) \sum_{i=0}^{k} \sum_{j=0}^{l} v_{i,j}^2 |c_{i,j}|^2
= \sum_{i=0}^{m} \sum_{j=0}^{n} v_{i,j}^2 |c_{i,j}|^2 \sum_{k=i}^{m} \sum_{l=j}^{n} \Delta_{1,1} \left( \frac{1}{v} \right) (k,l).
\]

(2.4)

Now suppose that there exists a double sequence \((v_{m,n})_{(m,n)\in \mathbb{N}^2}\) of positive numbers such that \( v_{m,n} \to \infty \) as \( m + n \to \infty \) and that (2.3) holds. Then by (2.2) we have

\[
\frac{1}{v_{i,j}} \sum_{k=i}^{\infty} \sum_{l=j}^{\infty} \Delta_{1,1} \left( \frac{1}{v} \right) (k,l) = 1.
\]

Thus letting \( m \to \infty \) and \( n \to \infty \) in (2.4), we obtain

\[
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} v_{i,j} |c_{i,j}|^2 < \infty
\]

so a fortiori

\[
\sum_{(i,j)\in \mathbb{N}^2} |c_{i,j}|^2 < \infty.
\]

(2.5)

Conversely, if (2.5) holds, one can construct a double sequence of positive numbers \( v = (v_{m,n})_{(m,n)\in \mathbb{N}^2} \) such that \( v_{m,n} \to \infty \) as \( m + n \to \infty \) and that (2.3) holds. Indeed, as we have

\[
\sum_{i=0}^{m} \sum_{j=0}^{n} v_{i,j}^2 |c_{i,j}|^2 \sum_{k=i}^{m} \sum_{l=j}^{n} \Delta_{1,1} \left( \frac{1}{v} \right) (k,l) \leq \frac{1}{v_{0,0}} \sum_{i=0}^{m} \sum_{j=0}^{n} v_{i,j}^2 |c_{i,j}|^2,
\]

it is sufficient (and possible, compare with [9]) to construct \( v \) such that

\[
\sum_{(i,j)\in \mathbb{N}^2} v_{i,j}^2 |c_{i,j}|^2 < \infty. \quad \Box
\]

3. Linear summability methods

Linear summability methods for simple series are a classical and powerful tool in Fourier Analysis. A general account on linear summability methods for series can be found in [13,22] and in [26] for applications to Fourier Analysis.

Our aim is to define some linear summability methods for double series.

We first recall some definitions for simple series. Let \( \Lambda = (\lambda_{i,j})_{(i,j)\in \mathbb{N}^2} \) be a lower triangular infinite matrix of numbers. We define a linear transformation (also denoted by \( \Lambda \)) on sequences of real numbers by setting

\[
t_n = \sum_{k=0}^{n} \lambda_{n,k} s_k
\]

(3.1)
for all \( n \in \mathbb{N} \), where \( s = (s_n)_{n \in \mathbb{N}} \) is a given sequence of real numbers. Eqs. (3.1) define a sequence \( t = (t_n)_{n \in \mathbb{N}} \) of real numbers and we write \( t = \Lambda s \).

Note that general and not only lower triangular matrices may of course be considered to define linear transformations [13].

It is natural, when dealing with a linear transformation \( \Lambda \) on sequences, to ask the following questions:

- Does \( \Lambda \) transform some non-convergent sequences to convergent sequences?
- Does \( \Lambda \) transform convergent sequences to convergent sequences?
- If \( s \) converges to \( \sigma \), does \( \Lambda s \) converge to \( \sigma \)?

The last question leads to the following definition: we say that the transformation \( \Lambda \) is regular if whenever \( \lim_{n \to \infty} s_n = \sigma \) then we also have \( \lim_{n \to \infty} t_n = \sigma \).

The following proposition gives well-known conditions for the regularity of a linear transformation \( \Lambda \) [13].

**Proposition 1.** Assume that \( \lambda_{n,k} \geq 0 \) for all \( (n, k) \in \mathbb{N}^2 \) and \( \sum_{k=0}^{\infty} \lambda_{n,k} \rightarrow 1 \) as \( n \rightarrow \infty \). The transformation \( \Lambda \) is regular if and only if \( \lambda_{n,k} \rightarrow 0 \) as \( n \rightarrow \infty \), for all \( k \in \mathbb{N} \).

Linear transformations are interesting to provide a meaning for the limit of a non-convergent sequence, and specially for the sequence of the partial sums of a divergent series. More precisely, with the previous notations, suppose moreover that \( s_n = \sum_{k=0}^{n} a_k \) for a sequence \( a = (a_n)_{n \in \mathbb{N}} \) of real numbers. If \( t = \Lambda s \) converges to \( \tau \), we say that the series \( \sum a_n \) is \( \Lambda \)-summable to \( \tau \).

In [10], Fomin introduced the following definition:

**Definition 1.** Let \( v = (v_n)_{n \in \mathbb{N}} \) be an increasing sequence of positive numbers, with \( v_n \to \infty \) as \( n \to \infty \). The series of real numbers \( \sum_{n=0}^{\infty} a_n \) is \((R, v)\)-summable to the number \( S \) if

\[
\lim_{n \to \infty} \frac{1}{v_{n+1}} \sum_{k=0}^{n} (v_{n+1} - v_k) a_k = S.
\]

The \((R, v)\)-summability is in fact a linear summability method for series, as a computation shows, using Abel transformation,

\[
\frac{1}{v_{n+1}} \sum_{k=0}^{n} (v_{n+1} - v_k) a_k = \sum_{k=0}^{n} \frac{v_{k+1} - v_k}{v_{n+1}} A_k,
\]

where \( A_n = \sum_{k=0}^{n} a_k \), for all \( n \in \mathbb{N} \).

We also see by Proposition 1 that the \((R, v)\) method is regular.

We now turn to linear summability methods for double series.

First, we introduce an analogue of \((R, v)\)-summability method for double series. Note that, by Abel transformation, the Definition 1 can be reformulated as follows: given an increasing sequence of positive numbers \( (v_n)_{n \in \mathbb{N}} \) the series \( \sum_{n=0}^{\infty} a_n \) is \((R, v)\)-summable to \( S \) if

\[
\sum_{k=0}^{n} \left( \frac{1}{v_k} - \frac{1}{v_{k+1}} \right) \sum_{i=0}^{k} v_i a_i
\]

converges to \( S \) as \( n \to \infty \).

**Definition 2.** Let \( v = (v_{m,n})_{(m,n) \in \mathbb{N}^2} \) be a double sequence of positive numbers with \( \Delta_{1,1}(v) \geq 0 \) and \( v_{m,n} \to \infty \) as \( m + n \to \infty \). The series of real numbers \( \sum_{(m,n) \in \mathbb{N}^2} a_{m,n} \) is \((R, v)\)-summable...
to the number $S$ if
\[
\sum_{k=0}^{m} \sum_{l=0}^{n} \Delta_{1,1} \left( \frac{1}{v} \right) (k, l) \sum_{i=0}^{k} \sum_{j=0}^{l} v_{i,j} a_{i,j}
\]
converges to $S$ as $m \to \infty$ and $n \to \infty$.

Going back to the proof of Theorem 1, we see that the series $\sum_{(i,j)\in\mathbb{N}^2} c_{i,j} \varphi_{i,j}$ is $(R, v)$-summable to $f(x)$ for almost all $x$ in $I$.

We now introduce some linear transformations for double sequences of real numbers. Let $A = (\lambda_{(m,n),(k,l)})_{(m,n)\in\mathbb{N}^2,(k,l)\in\mathbb{N}^2}$ where $\lambda_{(m,n),(k,l)}$ are real numbers and
\[
\lambda_{(m,n),(k,l)} = 0, \quad k \geq m + 1, \ l \geq n + 1,
\]
for all $(m, n) \in \mathbb{N}^2$ and $(k, l) \in \mathbb{N}^2$.

Given a double sequence of real numbers $s = (s_{m,n})_{(m,n)\in\mathbb{N}^2}$, we define the sequence $t = (t_{m,n})_{(m,n)\in\mathbb{N}^2}$ by setting
\[
t_{m,n} = \sum_{k=0}^{m} \sum_{l=0}^{n} \lambda_{(m,n),(k,l)} s_{k,l},
\]
for all $(m, n) \in \mathbb{N}^2$ and we write $t = A s$. The transformation $A$ defines an linear operator on $S$.

Given a double sequence $v$ of positive numbers, we will now see that the $(R, v)$-summability method is related to a linear transformation
\[
\Omega = (\omega_{(m,n),(k,l)})_{(m,n)\in\mathbb{N}^2,(k,l)\in\mathbb{N}^2}.
\]

We need the following notation: if $(m, n) \in \mathbb{N}^2$ we write $D_{(m,n)}$ for the double sequence which sends $(i, j) \in \mathbb{N}^2$ to $D_{(m,n),(i,j)}$ with
\[
D_{(m,n),(i,j)} = \sum_{k=i}^{m} \sum_{l=j}^{n} \Delta_{1,1} \left( \frac{1}{v} \right) (k, l)
\]
if $i \leq m$ and $j \leq n$, and $D_{(m,n),(i,j)} = 0$ otherwise.

We also need an analogue of Abel transformation formula, which we state in the following useful but elementary proposition.

**Proposition 2.** Let $u$ and $v$ be double sequences of real numbers, and $U$ be the sequence of rectangular partial sums of $u$:
\[
U_{m,n} = \sum_{k=0}^{m} \sum_{l=0}^{n} u_{k,l}
\]
for all $(m, n) \in \mathbb{N}^2$. Then we have
\[
\sum_{i=0}^{k} \sum_{j=0}^{l} v_{i,j} u_{i,j} = v_{k,l} U_{k,l} + \sum_{i=0}^{k-1} \Delta_{1,0}(v)(i, l) U_{i,l} + \sum_{j=0}^{l-1} \Delta_{0,1}(v)(k, j) U_{k,j} + \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} \Delta_{1,1}(v)(i, j) U_{i,j}
\]
for $(k, l) \in \mathbb{N}^2$. 


Proof. Write
\[
\sum_{i=0}^{k} \sum_{j=0}^{l} v_{i,j} u_{i,j} = v_{0,0} u_{0,0} + \sum_{j=1}^{l} v_{0,j} u_{0,j} + \sum_{i=1}^{k} v_{i,0} u_{i,0} + \sum_{j=1}^{l} \sum_{i=1}^{k} v_{i,j} u_{i,j}.
\]
Replacing \(u_{i,j}\) in the last term of the right-hand side by
\[
U_{i-1,j-1} - U_{i-1,j} - U_{i,j-1} + U_{i,j},
\]
according to (1.4), and changing the indexes, we obtain
\[
\sum_{i=0}^{k} \sum_{j=0}^{l} v_{i,j} u_{i,j} = \sum_{j=0}^{l} \sum_{i=0}^{k-1} v_{i+1,j+1} U_{i,j} - \sum_{i=0}^{k-1} \sum_{j=0}^{l} v_{i+1,j} U_{i,j}
- \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} v_{i,j+1} U_{i,j} + \sum_{i=0}^{k-1} \sum_{j=0}^{l} v_{i,j} U_{i,j}
\]
from which the formula (3.3) follows. \(\square\)

We can now see how \((R, v)\)-summability is related to a linear transformation for sequences.

**Proposition 3.** Let \((a_{m,n})_{(m,n)\in N^2}\) be a double sequence of real numbers, and let \(A_{m,n} = \sum_{k=0}^{m} \sum_{l=0}^{n} a_{k,l}\) for all \((m, n)\) in \(N^2\). Let
\[
B_{m,n} = \sum_{k=0}^{m} \sum_{l=0}^{n} \Delta_{1,1} \left( \frac{1}{v} \right) (k, l) \sum_{i=0}^{k} \sum_{j=0}^{l} v_{i,j} a_{i,j}
\]
for all \((m, n)\) in \(N^2\). Then we have
\[
B_{m,n} = \sum_{k=0}^{m} \sum_{l=0}^{n} \omega_{(m,n),(k,l)} A_{k,l}
\]
(3.4)
where
\[
\omega_{(m,n),(k,l)} = \Delta_{1,1} (v D_{(m,n)}) (k, l)
\]
(3.5)
for all \((m, n)\) in \(N^2\) and \((k, l)\) in \(N^2\).

**Proof.** Let \((m, n) \in N^2\). By Abel transformation formula, we obtain
\[
B_{m,n} = \sum_{i=0}^{m} \sum_{j=0}^{n} \left( \sum_{k=i}^{m} \sum_{l=j}^{n} \Delta_{1,1} \left( \frac{1}{v} \right) (k, l) c_{(k,l),(i,j)} \right) A_{i,j}
\]
where
\[
c_{(k,l),(i,j)} = v_{k,l},
\]
c\((k, l), (i, j)\) = \(\Delta_{1,1}(v)(i, j),\) \(0 \leq i \leq k - 1,\) \(0 \leq j \leq l - 1,\)
\[
c_{(k,l),(i,l)} = \Delta_{1,0}(v)(i, l),\) \(0 \leq i \leq k - 1,\)
\[
c_{(k,l),(j, l)} = \Delta_{0,1}(v)(k, j),\) \(0 \leq j \leq l - 1 \).
Now we have
\[
\omega_{(m,n),(i,j)} = \sum_{k=i}^{m} \sum_{l=j}^{n} \Delta_{1,1} \left( \frac{1}{\nu} \right) (k,l)c(k,l),(i,j)
\]
\[
= \Delta_{1,1} \left( \frac{1}{\nu} \right) (i,j)v_{i,j} + \Delta_{0,1}(v)(i,j) \sum_{l=j+1}^{n} \Delta_{1,1} \left( \frac{1}{\nu} \right) (i,l)
\]
\[
+ \Delta_{1,0}(v)(i,j) \sum_{k=i+1}^{m} \Delta_{1,1} \left( \frac{1}{\nu} \right) (k,j)
\]
\[
+ \Delta_{1,1}(v)(i,j) \sum_{k=i+1}^{m} \sum_{l=j+1}^{n} \Delta_{1,1} \left( \frac{1}{\nu} \right) (k,l).
\]

The computations of the previous sums give
\[
\omega_{(m,n),(i,j)} = v_{i,j} \left( \frac{1}{\nu_{i,j}} - \frac{1}{\nu_{i,n+1}} - \frac{1}{\nu_{m+1,j}} + \frac{1}{\nu_{m+1,n+1}} \right)
\]
\[
- v_{i,j+1} \left( \frac{1}{\nu_{i,j+1}} - \frac{1}{\nu_{i,n+1}} - \frac{1}{\nu_{m+1,j+1}} + \frac{1}{\nu_{m+1,n+1}} \right)
\]
\[
- v_{i+1,j} \left( \frac{1}{\nu_{i+1,j}} - \frac{1}{\nu_{m+1,j}} - \frac{1}{\nu_{i+1,n+1}} + \frac{1}{\nu_{m+1,n+1}} \right)
\]
\[
+ v_{i+1,j+1} \left( \frac{1}{\nu_{i+1,j+1}} - \frac{1}{\nu_{m+1,j+1}} - \frac{1}{\nu_{i+1,n+1}} + \frac{1}{\nu_{m+1,n+1}} \right).
\]

According to (2.2), this gives (3.5). \qed

The variety of notions of convergence for double sequences leads to various notions of regularity for a linear transformation \( \Lambda \) of double sequences. We consider the following ones. For \( \sigma \in \mathbb{R} \), we say that:

- \( \Lambda \) is \( M-M \)-regular if \( t_{m,n} \to \sigma \) as \( \min(m,n) \to \infty \) whenever \( s_{m,n} \to \sigma \) as \( \min(m,n) \to \infty \),
- \( \Lambda \) is \( M-N \)-regular if \( t_{m,n} \to \sigma \) as \( \min(m,n) \to \infty \) whenever \( s_{m,n} \to \sigma \) as \( m+n \to \infty \).

In other words, \( \Lambda \) is \( M-M \)-regular if it transforms \( M \)-convergent sequences to \( M \)-convergent sequences with the same limit, and \( M-N \)-regular if it transforms \( N \)-convergent sequences to \( M \)-convergent sequences with the same limit.

The next theorem gives conditions for the regularity of \( \Lambda \).

**Theorem 2.** Let \( \Lambda \) be the linear transformation defined by (3.2). Assume that
\[
\lambda_{(m,n),(k,l)} \geq 0
\]
for all \((m, n) \in \mathbb{N}^2 \) and \((k, l) \in \mathbb{N}^2 \).

(a) Assume that:

1. \( \lambda_{(m,n),(k,l)} \to 0 \) as \( m+n \to \infty \) for all \((m, n) \in \mathbb{N}^2 \),
2. \( \sum_{k=0}^{m} \sum_{l=0}^{n} \lambda_{(m,n),(k,l)} \to 1 \) as \( \min(m,n) \to \infty \),
3. For every \( \varepsilon > 0 \), there exists \((m_0, n_0) \in \mathbb{N}^2 \) such that
\[
\sum_{k=m_0+1}^{m} \sum_{l=0}^{n_0} \lambda_{(m,n),(k,l)} < \varepsilon
\]
and
\[ \sum_{k=0}^{m_0} \sum_{l=n_0+1}^{n} \lambda_{(m,n),(k,l)} < \epsilon \]
for all \((m, n) \in \mathbb{N}^2\) such that \(m \geq m_0\) and \(n \geq n_0\).

Then \(\Lambda\) is \(M-N\)-regular.

(b) Assume that the previous conditions (1) and (2) are satisfied, and that (4) there exists \(M > 0\) such that
\[ \sum_{k=m_0+1}^{m} \sum_{l=0}^{n_0} \lambda_{(m,n),(k,l)} \leq M \]
and
\[ \sum_{k=0}^{m_0} \sum_{l=n_0+1}^{n} \lambda_{(m,n),(k,l)} \leq M \]
for all \((m, n) \in \mathbb{N}^2\) and \((m_0, n_0) \in \mathbb{N}^2\) such that \(m \geq m_0\) and \(n \geq n_0\).

Then \(\Lambda\) is \(M-N\)-regular.

**Proof.** Suppose conditions (1)–(3) are satisfied. Let \(t = \Lambda s\) and suppose \(s\) is \(M\)-convergent to \(\sigma\). Let \((m_0, n_0) \in \mathbb{N}^2\) and \((m, n) \in \mathbb{N}^2\) with \(m_0 \leq m\) and \(n_0 \leq n\). Write
\[ t_{m,n} - \sigma = (I) + (II) + (III) + (IV), \]
where

\[ (I) = \sum_{k=0}^{m_0} \sum_{l=0}^{n_0} \lambda_{(m,n),(k,l)} (s_{k,l} - \sigma), \]
\[ (II) = \sum_{k=m_0+1}^{m} \sum_{l=0}^{n_0} \lambda_{(m,n),(k,l)} (s_{k,l} - \sigma) + \sum_{k=0}^{m_0} \sum_{l=n_0+1}^{n} \lambda_{(m,n),(k,l)} (s_{k,l} - \sigma), \]
\[ (III) = \sum_{k=m_0+1}^{m} \sum_{l=n_0+1}^{n} \lambda_{(m,n),(k,l)} (s_{k,l} - \sigma), \]
\[ (IV) = \sum_{k=0}^{m} \sum_{l=0}^{n} (\lambda_{(m,n),(k,l)} - 1)\sigma. \]

The conclusion follows.

Proof of Part (b) of the theorem is analogous. \(\square\)

**Corollary.** \(\Omega\) is \(M-N\)-regular.

**Proof.** Note that we have, from the proof of Proposition 3
\[ \omega_{(m,n),(i,j)} = \frac{1}{v_{i,n+1}} (v_{i,j+1} - v_{i,j}) + \frac{1}{v_{m+1,j}} (v_{i+1,j} - v_{i,j}) \]
\[ + \frac{1}{v_{m+1,j+1}} (v_{i,j+1} - v_{i+1,j+1}) + \frac{1}{v_{i+1,n+1}} (v_{i+1,j} - v_{i+1,j+1}) \]
\[ + \frac{1}{v_{m+1,n+1}} \Delta_{1,1} (v(i, j)), \]
for all \((m, n)\) and \((i, j)\) in \(\mathbb{N}^2\). It follows that condition (1) is satisfied. Conditions (2) and (4) follow from the analogue of (2.2) for \(vD_{(m,n)}\) and (3.5). \(\square\)
Remark. In Part (a) of Theorem 2, assumptions (1)–(3) may be replaced by (1), (2) and

$$(3') \sum_{k=0}^{m} \sum_{l=0}^{n} (\lambda(m,n),(k,l) - \lambda(m',n'),(k,l)) \leq \sum_{k=m+1}^{m'} \sum_{l=n+1}^{n'} \lambda(m',n'),(k,l)$$

for all $(m,n) \in \mathbb{N}^2$ and $(m',n') \in \mathbb{N}^2$ with $m \leq m'$ and $n \leq n'$.

Indeed, write

$$A_{m,n} = \sum_{k=0}^{m} \sum_{l=0}^{n} \lambda(m,n),(k,l)$$

for $(m,n) \in \mathbb{N}^2$. Let $\varepsilon > 0$. By assumption (2), we have $|A_{m',n'} - A_{m,n}| < \varepsilon$ provided that $m \leq m'$, $n \leq n'$, when $m$ and $n$ are large enough.

Since it follows from assumption (4) that

$$\sum_{k=0}^{m} \sum_{l=n+1}^{n'} \lambda(m',n'),(k,l) + \sum_{k=m+1}^{m'} \sum_{l=0}^{n} \lambda(m',n'),(k,l) \leq A_{m',n'} - A_{m,n},$$

we then obtain assumption (3).

References

[1] V.A. Abilov, M.K. Kerimov, Sharp estimates for the convergence rate of double Fourier series in terms of orthogonal polynomials in the space $L_2((a, b) \times (c, d); p(x)q(y))$, Zh. Vychisl. Mat. Mat. Fiz. 49 (8) (2009) 1364–1368.


