Toeplitz Operators and Quantum Mechanics*

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Toeplitz operators on the Segal-Bargmann spaces of Gaussian measure square-integrable entire functions on complex \( n \)-space \( \mathbb{C}^n \) are studied. The \( C^* \)-algebra generated by the Weyl form of the canonical commutation relations consists precisely of the uniform limits of almost-periodic Toeplitz operators. The question of "which Toeplitz operators admit a symbol calculus modulo the compact operators" is raised and sufficient conditions are given for such a calculus. These conditions involve a notion of "slow oscillation at infinity."

1. Introduction

An interesting connection between some particular Bergman-type spaces of analytic functions [2] and quantum mechanics was uncovered and explored by I. E. Segal and V. Bargmann in the early 1960s [1, 11]. On these spaces, with domain \( \mathbb{C}^n \), the Fock boson creation operators are represented as multiplications by linear functions of the independent complex variables, \( z_j, j = 1, 2, \ldots, n \). This connection was called to our attention by William Arveson several years ago when he gave a talk on the role of unbounded Toeplitz operators in quantum mechanics.

Since 1960, the study of bounded Toeplitz operators on a variety of domains has been systematized. In view of this, we have revisited the Segal-Bargmann spaces in order to clarify two questions:

(1) What is the structure of the algebras of bounded Toeplitz operators on these spaces?

(2) What is the relation between bounded Toeplitz operators and the Weyl operators of boson quantum mechanics?

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Because the Segal-Bargmann function spaces consist of entire functions, there are no non-constant bounded analytic Toeplitz operators and, hence, Wiener-Hopf factorization techniques are not available. Nevertheless, we have been able to make reasonable progress on (1) and to settle (2).

To describe our results more specifically, we require some notation. We consider the space $\mathbb{C}^n$ of $n$ complex variables with $z = (z_1, z_2, \ldots, z_n)$ and $|z|^2 = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$. If $dv$ is the usual Euclidean volume measure on $\mathbb{C}^n = \mathbb{R}^{2n}$, we consider the measure $d\mu = (2\pi)^{-n} e^{-|z|^2/2} dv(z)$ on $\mathbb{C}^n$ and form the usual $L^2$-space, $L^2(\mathbb{C}^n, d\mu)$. We let $P$ denote the orthogonal projection from $L^2(\mathbb{C}^n, d\mu)$ onto the subspace $H^2(\mathbb{C}^n, d\mu)$ consisting of entire functions. For $\phi$ measurable on $\mathbb{C}^n$, the Toeplitz operator $T_\phi$ is defined by

$$T_\phi f = P(\phi f)$$

for $f$ in $H^2(\mathbb{C}^n, d\mu)$ with $\phi f$ in $L^2(\mathbb{C}^n, d\mu)$. Let $L^\infty(\mathbb{C}^n)$ be the algebra of essentially bounded measurable functions on $\mathbb{C}^n$. For $A$ a subalgebra of $L^\infty(\mathbb{C}^n)$, we denote by $\tau(A)$ the $C^*$-algebra generated by $\{T_\phi : \phi \in A\}$.

Let CCR($\mathbb{C}^n$) be the unique $C^*$-algebra generated by the Weyl operators of the canonical commutation relations over $\mathbb{C}^n$ [3]. Let AP($\mathbb{C}^n$) be the closed subalgebra of $L^\infty(\mathbb{C}^n)$ consisting of almost-periodic functions on $\mathbb{C}^n = \mathbb{R}^{2n}$. It is known [1] that the operators

$$\{e^{iT_{\Re(z)}}, w \in \mathbb{C}^n\}$$

generate CCR($\mathbb{C}^n$) (here $wz = \overline{w}_1 z_1 + \overline{w}_2 z_2 + \cdots + \overline{w}_n z_n$). In Section 5, we establish the

**Basic Identities.** For $w, v$ in $\mathbb{C}^n$,

$$T_{e^{i\Re(vz)}} e^{-|\pi z|^2/4} e^{iT_{\Re(vz)}}$$

$$T_{e^{i\Re(vz)}} T_{e^{i\Re(vz)}} e^{i\Re(wz)/2} e^{i\Re(w + z)}.$$ 

From the first identity, it follows that $\tau(\{\text{AP}(\mathbb{C}^n)\}) = \text{CCR}(\mathbb{C}^n)$. This settles Question (2) above. Since CCR($\mathbb{C}^n$) is simple [3], there is no chance for the usual sort of symbol calculus on $\tau(\{\text{AP}(\mathbb{C}^n)\})$.

For functions which oscillate less rapidly at infinity than $e^{i\text{Re}(w z)}$ there is still a chance of a symbol calculus for the associated Toeplitz algebra. Considering the case $n = 1$, we prove, in Section 4, that if $\phi$ in $L^\infty(\mathbb{C})$ is "slowly varying at infinity" then $T_\phi - T_{\overline{\phi}}$ is a compact operator for all $\zeta$ in $L^\infty(\mathbb{C})$. It follows that the $C^*$-algebra generated by all $T_\phi$ with $\phi$ and $\overline{\phi}$ "slowly varying at infinity" has a symbol calculus and we can understand the structure of this subalgebra of $\tau(\{L^\infty(\mathbb{C})\})$ reasonable well.

There are still many gaps in our understanding of $\tau(\{L^\infty(\mathbb{C}^n)\})$. We list some open problems in Section 7. We believe that the algebra $\tau(\{L^\infty(\mathbb{C}^n)\})$ is sufficiently interesting to justify further research.
2. The Segal–Bargmann Space $H^2(C^n, d\mu)$

In this section, we discuss a special Hilbert space of entire functions and its relation to the Fock boson space $F_+(C^n)$ and to $L^2(R^n, dv)$. This material is largely expository and has been pieced together from [1, 3]. For technical convenience, we have slightly modified the Gaussian measure used in [1].

For $z = (z_1, z_2, \ldots, z_n)$, $w = (w_1, w_2, \ldots, w_n)$ in $C^n$, we write $\bar{z} = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)$ for $\bar{z}_j$ the usual conjugate in $C$, $|z|^2 = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$, and $zw = z_1w_1 + z_2w_2 + \cdots + z_nw_n$. For $k = (k_1, k_2, \ldots, k_n)$ an $n$-tuple of non-negative integers, we write

$$k! = k_1!k_2!\cdots k_n!$$

$$\|k\| = k_1 + k_2 + \cdots + k_n$$

$$z^k = z_1^{k_1}z_2^{k_2}\cdots z_n^{k_n}.$$ 

All Hilbert spaces are vector spaces over the complex numbers, $C$. For the coordinates $z_j$ of $C^n$ we write $z_j = x_j + iy_j$ for $x_j, y_j$ real ($R$, the real numbers).

On $C^n$ we take the Gaussian measure (suppressing $n$)

$$d\mu(z) = (2\pi)^{-n} e^{-|z|^2/2} dv(z)$$

where $dv(z)$ is ordinary Euclidean volume on $C^n = R^{2n}$. On $R$, we write $dv = dx$. As usual, $L^2(C^n, d\mu)$ is the Hilbert space with inner product

$$\langle f, g \rangle = \int \cdots \int f(z) \overline{g(z)} d\mu(z),$$

of $\mu$ square-integrable complex-valued functions. We denote by $H^2(C^n, d\mu)$ the closed subspace of $L^2(C^n, d\mu)$ consisting of entire functions. An orthonormal basis for $H^2(C^n, d\mu)$ consists of the functions

$$\{ (2^{1/2}k!)^{-1/2} z^k : \text{all } k_j \geq 0 \}.$$ 

Since point-evaluation at $z$ in $C^n$ is a bounded linear functional on $H^2(C^n, d\mu)$, we have reproducing kernels

$$k_z(z) = e^{z^2/2}$$

in $H^2(C^n, d\mu)$ so that for all $g$ in $H^2(C^n, d\mu)$

$$g(\lambda) = \langle g(z), k_z(z) \rangle = \int \cdots \int g(z) \overline{k_z(z)} d\mu(z).$$
The spaces $H^2(C^n, d\mu)$ were introduced by Segal and studied by Bargmann [1]. Clearly, $H^2(C^n, d\mu)$ contains, as a dense subspace, the set $\mathcal{P}$ of all polynomials in the $\{z_j\}$.

The multiplicativity of $d\mu(z)$ induces a natural isometry

$$H^2(C^n, d\mu) \cong H^2(C, d\mu) \otimes \cdots \otimes H^2(C, d\mu).$$

The analogous result for $L^2(R^n, dv)$ is also standard:

$$L^2(R^n, dv) \cong L^2(R, dx) \otimes \cdots \otimes L^2(R, dx).$$

Let $F_+(C^n)$ denote the Fock boson Hilbert space completion of all symmetric tensors on $C^n$ [3]. We write

$$F_+(C^n) = C \oplus C^n \oplus (C^n \otimes C^n) \oplus (C^n \otimes C^n \otimes C^n) \oplus \cdots$$

where the $+$ denotes the symmetric part. On $F_+(C^n)$ we have the usual (boson) annihilation and creation operators $a_+(\lambda), a_+^*(\lambda)$ for $\lambda$ in $C^n$ [3]. If $e_j = (0, 0, \ldots, 1, \ldots, 0)$ (1 in the $j$th slot), then “uniqueness of the CCRs” [3] yields the existence of an isometry $S$ from $F_+(C^n)$ onto $L^2(R^n, dv)$ so that

$$S a_+^*(e_j) S^{-1} = \frac{1}{\sqrt{2}} \left( M_{x_j} - \frac{\partial}{\partial x_j} \right)$$

on a dense subspace of $L^2(R^n, dv)$. This is the “Schrodinger representation of the CCRs” [3]. Here, $(M_{x_j}f)(x) = x_jf(x)$ and $\partial/\partial x_j$ is the usual partial derivative operator with $x = (x_1, \ldots, x_n)$ in $R^n$.

Next, we note that there is a natural isometry $B$ from $L^2(R^n, dv)$ onto $H^2(C^n, d\mu)$. The map $B$ is induced via the tensor product from the one variable case. For $n = 1$, $B^{-1}$ is defined by

$$B^{-1}[(2^k k!)^{-1/2} z^k] = \left[ \pi 4^k(k!)^2 \right]^{-1/4} H_k(x) e^{-x^2/2}$$

where $H_k(x)$ is the $k$th Hermite polynomial given by

$$H_k(x) = (-1)^k e^{x^2} \left( \frac{d}{dx} \right)^k e^{-x^2}.$$ 

It is not hard to check (using standard relations among the $H_k(x)$) that

$$B \left( M_{x_j} - \frac{\partial}{\partial x_j} \right) B^{-1} = T_{z_j}$$

where $T_{z_j}$ is the operator $M_{z_j}$ (“multiplication by $z_j$”) restricted to $H^2(C^n, d\mu)$. A good reference for the Hermite polynomials is [12].
The foregoing is a sketch of the representation of the CCRs described in [1] and is summarized by

**Proposition 1.** The map \( B \circ S \) from \( F_+(C^n) \) onto \( H^2(C^n, d\mu) \) induces a unitary equivalence

\[
B \circ Sa^*_j(e_j)(B \circ S)^{-1} = \frac{1}{\sqrt{2}}T_{z_j}
\]

on the dense subspace \( \mathcal{D} \).

We now define the Toeplitz operators \( T_\phi \) on \( H^2(C^n, d\mu) \). Let \( P \) denote the orthogonal projection operator from \( L^2(C^n, d\mu) \) onto \( H^2(C^n, d\mu) \). For \( \phi \) measurable on \( C^n \) and \( \phi f \) in \( L^2(C^n, d\mu) \) for a dense set of \( f \) in \( H^2(C^n, d\mu) \), we define the (possibly unbounded) Toeplitz operator \( T_\phi \) by

\[
T_\phi f = P(\phi f)
\]

for \( f \) in \( \{ f \in H^2(C^n, d\mu) : \phi f \in L^2(C^n, d\mu) \} \). It is easy to check that \( T_\phi \) is always closed and \( T_{\bar{\phi}} \subset T^*_\phi \). Moreover, \( T_\phi \) is bounded if \( \phi \) is essentially bounded, in which case

\[
\| T_\phi \| \leq \| \phi \|_\infty.
\]

It is also easy to check for \( \alpha \) in \( C \) and \( \phi, \xi \) in \( L^\infty(C^n) \) that \( T_{\phi + \xi} = T_\phi + T_\xi \), \( T_{\alpha \phi} = \alpha T_\phi \), and \( T^*_\phi = T_{\bar{\phi}} \). Note that the domains of \( T_{z_j} \), \( T_{\bar{z}_j} \) contain \( \mathcal{D} \) as a common dense subspace. Note further, by direct calculation, that

\[
T_{\bar{z}_j} = 2 \frac{\partial}{\partial z_j}
\]

on \( \mathcal{D} \). We can now state, in somewhat simpler terms, a major identity of [1] which will be useful. We denote by \( \widetilde{k}_\lambda(z) \) the normalized kernel function at \( \lambda \)

\[
\widetilde{k}_\lambda(z) = e^{2z/2 - |\lambda|^2/4}.
\]

We also write \( t_\lambda \) for the translation operator on \( H^2(C^n, d\mu) \)

\[
(t_\lambda g)(z) = g(z - \lambda).
\]

**Proposition 2.** For \( \lambda \) in \( C^n \), we have [1]

\[
e^{iT\text{Re}(z)} = \widetilde{k}_{-i\lambda}(z) t_{-i\lambda}.
\]
Proof. Direct calculation, starting (for \( n = 1 \) and \( d/dz = D_z \)) with

\[
T_x = \frac{1}{2}(M_z + T_z^*) = \left( \frac{1}{2}M_z + D_z \right)
\]

\[
= M_{e^{-z^2/4}}D_z M_{e^{z^2/4}}
\]
on \( \mathcal{P} \), shows that

\[
T^k_x = M_{e^{-z^2/4}}D_z^k M_{e^{z^2/4}}
\]
on \( \mathcal{P} \). It follows, using finite-dimensional vector-space calculations and the natural filtration of \( \mathcal{P} \) by degree, that for \( u \) in \( \mathbb{R} \),

\[
e^{iuT_x} = M_{e^{-z^2/4}}e^{iuD_z}M_{e^{z^2/4}}
\]

\[
= \kappa_{-iu}(z) e^{-iu}
\]
on \( \mathcal{P} \). Polarization and another direct calculation now yield the general identity.

The \( \Phi(\lambda) \) are defined ([3]) for \( \lambda \) in \( C^n \) by

\[
\Phi(\lambda) = \frac{1}{\sqrt{2}} (a_+ (\lambda) + a_+^* (\lambda)).
\]

Using \( a_+ (x\lambda) = \bar{x}a_+ (\lambda) \), \( a_+^* (x\lambda) = \bar{x}a_+^* (\lambda) \) for \( x \) in \( C \), it is clear that

\[
B \circ S \Phi(e_j)(B \circ S)^{-1} = T_{x_j}
\]

\[
B \circ S \Phi(ie_j)(B \circ S)^{-1} = -T_{y_j}
\]

so that \( \Phi(\lambda) = T_{\text{Re}(\lambda)} \) and \( e^{\Phi(\lambda)} \) has the form \( e^{iT_{\text{Re}(\lambda)}} \) for \( \lambda \) in \( C^n \). It follows immediately from the discussion in [3] that

Proposition 3. The Weyl algebra \( CCR(C^n) \) is represented on \( H^2(C^n, d\mu) \) via \( B \circ S \) as the \( C^* \)-algebra generated by

\[
\{ e^{iT_{\text{Re}(\lambda)}}, \lambda \in C^n \}.
\]

As observed in [11], the Fock "number of particles" operator \( N_+ \) is given in the \( H^2(C^n, d\mu) \) representation by

\[
(B \circ S) N_+ (B \circ S)^{-1} = \frac{1}{2} \sum_{j=1}^{n} T_{z_j} T_{\bar{z}_j}
\]

\[
= \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}.
\]
This is the well-known Euler operator. We also have the identity
\[ \frac{1}{2} T_{|z|^2} = nI + \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}. \]

The operator \( SN_+ S^{-1} \) is the Hamiltonian of the quantum-mechanical harmonic oscillator (after changing units to set Planck’s constant, the mass, and the spring constant equal to one). It is given by
\[
SN_+ S^{-1} = \frac{1}{2} \sum_{j=1}^{n} \left( M_{z_j} - \frac{\partial}{\partial x_j} \right) \left( M_{\bar{z}_j} + \frac{\partial}{\partial x_j} \right)
= \frac{1}{2} ( -A + M_{|z|^2} - nI )
\]
on \( L^2(R^n, dv) \), where \( A \) is the Laplacian. The time evolution operator of \( SN_+ S^{-1} \) takes a particularly simple form on \( H^2(C^n, d\mu) \) [1]:
\[
Be^{-it \frac{SN_+ S^{-1}}{2}} B^{-1} = R_{-t}
\]
where \( (R_tg)(z) = g(e^{it}z) \) for \( t \) in \( R \).

3. TOEPLITZ OPERATORS ON \( H^2(C^n, d\mu) \)

We now begin the analysis of Toeplitz operators and \( C^* \)-algebras generated by Toeplitz operators on \( H^2(C^n, d\mu) \). The situation resembles the study of Toeplitz operators on the Bergman space of the polydisc [4, 5]. Here, significant new difficulties occur for \( n = 1 \) because of the unboundedness of the domain.

Let \( L^\infty(C^n) \) denote the algebra of essentially bounded measurable functions on \( C^n \) (\( d\mu \) and \( dv \) give the same such functions). Clearly, \( L^\infty(C^n) \) is contained in \( L^2(C^n, d\mu) \). For \( \phi \) in \( L^\infty(C^n) \), it was noted earlier that \( T_\phi \) is bounded. It is easy to check that the converse is false (for \( n = 1 \), take \( \phi(z) = 1/\sqrt{|z|} \)). For \( A \) a subalgebra of \( L^\infty(C^n) \), we will be concerned with the \( C^* \)-algebra \( \tau(A) \) generated by \( \{ T_\phi : \phi \in A \} \).

We first establish a result which, in more standard contexts, is a triviality.

**Theorem 4.** For \( \phi \) in \( L^\infty(C^n) \), \( T_\phi = 0 \) if and only if \( \phi = 0 \) a.e.

**Proof.** That \( \phi = 0 \) implies \( T_\phi = 0 \) is trivial. For the converse, we need to know that the polynomials in \( z^j, \bar{z}^k \) are dense in \( L^2(C^n, d\mu) \). This follows
from the fact that the Hermite functions \( \{ H_k(x) e^{-x^2/2} \} \) form a basis for \( L^2(R, dx) \) as discussed in Section 2. Now, for \( T_\phi = 0 \) we have

\[
0 = \langle T_\phi z^j, z^l \rangle = \langle \phi, z^j z^l \rangle
\]

for all multi-indices \( j, k \) and it follows that \( \phi = 0 \) a.e.

Next, we prove, for completeness, a result known to W. Arveson and his student, J. Spielberg (unpublished).

**Theorem 5.** If \( \phi \) in \( L^\infty(C^n) \) has compact support, then \( T_\phi \) is a compact operator.

**Proof.** Let \( e_k = (2^{k_1} k_1!)^{-1/2} z^k \) for multi-index \( k \). Since \( \phi \) is essentially bounded, with compact support, it is clear that there are positive constants \( K, a \) so that \( |\phi(z)| \leq K \) a.e. and \( \phi(z) = 0 \) for all \( z \) with \( |z| > a \). By direct calculation,

\[
\langle T_\phi e_k, e_j \rangle = (2^{k_1+k_1} k_1!)^{-1/2} \langle \phi z^k, z^j \rangle
\]

so that

\[
\begin{align*}
|\langle T_\phi e_k, e_j \rangle| &\leq (2\pi)^{-n} (2^{k_1+k_1} + \|j\| k_1!)^{-1/2} K \int_{|z| < a} \frac{1}{\pi} \int |z|^j |z|^l e^{-|z|^2/2} dv(z) \\
&\leq (2\pi)^{-n} (2^{k_1+k_1} + \|j\| k_1!)^{-1/2} K \prod_{s=1}^n \left\{ \int_{|z| < a} |z|^j |z|^l e^{-|z|^2/2} dv(z) \right\} \\
&\leq (2^{k_1+k_1} + \|j\| k_1!)^{-1/2} K \prod_{s=1}^n a^{k_1+k_1+2} \\
&\leq (2^{k_1+k_1} + \|j\| k_1!)^{-1/2} K a^{k_1+k_1+2n}.
\end{align*}
\]

It follows that

\[
\sum_{k, j} |\langle T_\phi e_k, e_j \rangle|^2 \leq K^2 a^{4n} \sum_{k, j} \frac{(a^2/2)^{k_1+k_1+\|j\|}}{k_1! j_1!} \leq K^2 a^{4n} \left\{ \sum_{k=0}^{\infty} \frac{(a^2/2)^k}{k!} \right\}^{2n} = K^2 a^{4n} e^{na^2}.
\]

Hence, by a standard argument, \( T_\phi \) is compact [10].
We denote the full algebra of compact operators by $\mathcal{K}$ on any Hilbert space. We shall see presently that $\mathcal{K}$ is contained in $\tau\{L^\infty(C^n)\}$.

A slight strengthening of Theorem 5 is easy and will be of some use. For $\phi$ in $L^\infty(C^n)$, we write $M_\phi$ for the operator “multiplication by $\phi$” on $L^2(C^n, d\mu)$.

**Corollary.** If $\phi$ is in $L^\infty(C^n)$ and has compact support, then $M_\phi P$ is in $\mathcal{K}$.

**Proof.** By a standard argument, it suffices to show that $(M_\phi P)*(M_\phi P)$ is in $\mathcal{K}$.

\[(M_\phi P)*(M_\phi P) = PM_{|\phi|^2}P\]

and, since $\phi$ has compact support, $|\phi|^2$ has compact support. It follows from Theorem 5 that $T_{|\phi|^2}$ is in $\mathcal{K}$ and so $PM_{|\phi|^2}P$ must also be in $\mathcal{K}$.

For $\mathcal{C}\{\tau(A)\}$ the closed two-sided ideal generated by all commutators $([X, Y] = XY - YX)$ of operators in $\tau(A)$, we are interested in identifying $\mathcal{K}/\mathcal{C}\{\tau(A)\}$ and in establishing a “functional calculus” for Toeplitz operators in $\tau(A)$. This seems to be a rather difficult problem and we will only be able to settle it in special cases. The bulk of the calculations from now on will be done in the case $n = 1$. We will return later to the general case.

4. **Toeplitz Operators on $H^2(C, d\mu)$**

We write $L^\infty = L^\infty(C)$ and $\bar{A} = \{\bar{\phi}: \phi \in A\}$ for $A$ any subset of $L^\infty$. To give a general context to our discussion of “functional calculus” for Toeplitz operators, we first define some “natural” subalgebras and subspaces of $L^\infty$.

**Definition 1.** $\Gamma = \{\phi \in L^\infty: (I - P) M_\phi P \in \mathcal{K}\}$.

**Definition 2.** $B = \{\phi \in L^\infty: T_\phi \in \mathcal{K}\} = \{\phi \in L^\infty: PM_\phi P \in \mathcal{K}\}$.

It is easy to check that $\Gamma$ is a uniformly closed subalgebra of $L^\infty$ while $B$ is a norm-closed conjugate closed linear subspace of $L^\infty$. Let $\Sigma$ be the smallest norm-closed and conjugate-closed subalgebra of $L^\infty$ which contains $\Gamma$. Let $Q = \Gamma \cap \bar{\Gamma}$.

**Lemma 1.** $B$ is a $\Sigma$-module ($\Sigma \cdot B \subset B$).

**Proof.** Clearly, $\Sigma$ is the closure (in the essential supremum norm) of sums $\sum_{j=1}^r \phi_j \bar{\xi}_j$, with $\phi_j, \xi_j$ in $\Gamma$. Thus, it suffices to check for $\phi, \xi$ in $\Gamma$ that...
\( \phi \zeta b \) is in \( B \) for any \( b \) in \( B \). In turn, it suffices to check that \( \zeta b \) is in \( B \) and \( \phi b \) is in \( B \) for all \( b \) in \( B \). These inclusions follow from the identities

\[
P M_{\zeta b} P = PM_{\zeta} (I - P) + P \] \[ M_b P \]
\[ = \{ PM_{\zeta} (I - P) \} M_b P + PM_{\zeta} (PM_b P) \]
\[ = \{ (I - P) M_{\zeta} P \} * M_b P + PM_{\zeta} (PM_b P), \]

\[
P M_{\phi b} P = PM_{b} \{ P + (I - P) \} M_{\phi} P \]
\[ = (PM_b P) M_{\phi} P + PM_{b} \{ (I - P) M_{\phi} P \}. \]

**Lemma 2.** For \( \phi \) in \( \Gamma \) and \( \zeta \) in \( L^\infty \), we have

\[
T_\zeta T_\phi - T_{\zeta \phi} \in \mathcal{H}
\]
\[
T_\phi T_\zeta - T_{\phi \zeta} \in \mathcal{H}.
\]

**Proof.** The second inclusion follows from the first by taking adjoints. The first inclusion follows from the identity

\[
P M_{\zeta \phi} P = PM_{\zeta} \{ P + (I - P) \} M_{\phi} P \]
\[ = (PM_{\zeta} P)(PM_{\phi} P) + PM_{\zeta} \{ (I - P) M_{\phi} P \}. \]

**Remark.** The identity

\[
\]

together with Lemma 2, shows that \( Q \) is the largest conjugate-closed subalgebra of \( L^\infty \) for which \( \tau(Q) \) has a "symbol calculus" mod \( \mathcal{H} \).

We now consider an explicit non-trivial example of a function in \( Q \). Of course, for any \( \phi \) in \( L^\infty \) with compact support, \( \phi \) is in \( Q \) by the Corollary to Theorem 5. In fact, such \( \phi \) are in \( Q \cap B \) since \( T_\phi \) is in \( \mathcal{H} \). Let \((r, \theta)\) be the polar coordinates of the point \( z = (x, y) = x + iy \) in \( C = R^2 \). As usual, \( r(z) = |z| \) and \( z = r(z) e^{i\theta(z)} \). The function \( e^{i\theta(z)} \) is continuous on \( C \setminus \{0\} \) with modulus one.

**Lemma 3.** We have \( e^{i\theta} \in Q \) and \( \mathcal{H} \subset \tau(Q) \).

**Proof.** We use the standard fact that \( X \) is in \( \mathcal{H} \) if and only if \( X^*X \) is in \( \mathcal{H} \). For \( X = (I - P) M_{e\phi} P \), we have by direct calculation with \( \{ e_k : k \geq 0 \} \) the basis for \( H^2(C, d\mu) \) discussed earlier

\[
X^*Xe_k = (1 - x_k^2) e_k, \quad k \geq 0,
\]
where
\[ x_k = \frac{1}{2^k k! \sqrt{2(2k + 1)}} \int_0^{\infty} e^{-r^2/2} r^{2k + 2} \, dr. \]

Using the standard identity
\[ \int_0^{\infty} e^{-r^2/2} r^{2k + 2} \, dr = \frac{(2k + 2)! \sqrt{\pi}}{2^{k + 1} (k + 1)! \sqrt{2}} \]
and Stirling's formula [13] (with error term), it follows that
\[ \lim_{k \to \infty} (1 - x_k^2) = 0. \]

Hence, \((1 - P) M_{\phi} P\) is in \(\mathcal{K}\) and \(e^{i\phi}\) is in \(I\).

A similar calculation for \(X = (I - P) M_{e^{i\phi}} P\) yields
\[ X^*X e_{k+1} = (1 - x_k^2) e_{k+1}, \quad k \geq 0, \]
\[ X^*X e_0 = e_0. \]

It follows, as before, that \(e^{-i\phi}\) is in \(I\) so that \(e^{i\phi}\) is in \(Q\).

Next, we consider the matrix form of \(T_{e^{i\phi}}\). Direct calculation shows, for \(x_k\) as above, that
\[ T_{e^{i\phi}} e_k = x_k e_{k+1}, \quad k \geq 0. \]

Hence, \(T_{e^{i\phi}}\) is a "weighted shift" with all weights non-zero. It follows easily that \(T_{e^{i\phi}}\) is irreducible. Since \(Q \cap B\) contains all \(\phi\) in \(L^\infty\) with compact support (by the Corollary to Theorem 5) and since \(T_{e^{i\phi}} = 0\) implies \(\phi = 0\) a.e. (by Theorem 4), we now see that \(\tau(Q)\) is an irreducible \(C^*\)-algebra containing a non-zero compact operator. It follows [6] that \(\mathcal{K} \subset \tau(Q)\).

**Remark.** For \(U_+ e_k = e_{k+1}, k \geq 0\), the "unilateral shift" on the basis \(\{e_k: k \geq 0\}\), it is immediate that \(U_+\) is in \(\tau(Q)\) and \(T_{e^{i\phi}} - U_+\) is in \(\mathcal{K}\). Moreover, it follows immediately from Lemmas 2 and 3 that \(U_+^* T_{e^{i\phi}} U_+ - T_{e^{i\phi}}\) is in \(\mathcal{K}\) for all \(\phi\) in \(L^\infty\).

We can now prove a structure theorem for \(\tau(Q)\).

**Theorem 6.** The map \(\phi \to T_{e^{i\phi}}\) induces a \(*\)-isomorphism
\[ \tau(Q)/\mathcal{K} \simeq Q/Q \cap B. \]

**Proof.** By Lemma 1, \(Q \cap B\) is a closed, conjugate-closed ideal in \(Q\). By Lemma 3, \(\mathcal{K}\) is a closed minimal two-sided ideal in \(\tau(Q)\) (minimality...
follows from standard facts about $\mathcal{N}$ [6]). Let $\pi$ be the usual quotient
homomorphism from $\tau(Q)$ onto $\tau(Q)/\mathcal{N}$. Using Lemma 2, it is easy to
check that the map $\phi \to \pi(T_\phi)$ is a $*\$-homomorphism from $Q$ onto $\tau(Q)/\mathcal{N}$.
It only remains to check the kernel of this homomorphism. Clearly, $\pi(T_\phi) = 0$ if and only if $T_\phi$ is in $\mathcal{N}$. Thus, the kernel of $\phi \to \pi(T_\phi)$ is precisely $Q \cap B$.

**Corollary.** $\mathcal{C}\{\tau(Q)\} = \mathcal{N}$.

*Proof.* By Theorem 6, $\tau(Q)/\mathcal{N}$ is abelian. It follows that all $[X, Y]$ are
in $\mathcal{N}$ and so $\mathcal{C}\{\tau(Q)\} \subset \mathcal{N}$. Minimality of $\mathcal{N}$ then implies equality.

Let $L^\infty_c = \{\phi \in L^\infty: \phi$ has compact support $\}$. So far, we know that $L^\infty_c \subset Q \cap B$ and $L^\infty_c \cup \{e^{i0}\} \subset Q$. Next, we consider another interesting subset of $Q$. We turn our attention to the radial functions

$L^\infty_r = \{z \to \phi(|z|): \phi$ in $L^\infty[0, \infty)\}$.

We write $\xi = \xi(r)$ for $\zeta$ in $L^\infty_r$.

**Remark.** $L^\infty_r$ is not a subset of $Q$. In particular, it is easy to check that
$T_{e^{ir^2}} \in \mathcal{N}$ so $e^{ir^2} \in B$. If $e^{ir^2}$ were in $Q$ (or even in $\Sigma$) then, by Lemma 1,
$e^{-ir^2}e^{ir^2} = 1 \in B$ and this is impossible.

For $\phi$ in $L^\infty_r$, it is easy to see that

$T_\phi e_k = s_k(\phi) e_k, \quad k \geq 0$,

where

$s_k(\phi) = \frac{1}{2^k k!} \int_0^\infty \phi(r) e^{-r^2/2k + 1} dr$.

It follows from Theorem 4 that the moments $\{s_k(\phi)\}$ completely determine $\phi$. Clearly, $|s_k(\phi)| \leq \|\phi\|_\infty$. In fact, more is generally true.

**Theorem 7.** For all $\phi$ in $L^\infty_r$, $\lim_{k \to \infty} |s_{k+1}(\phi) - s_k(\phi)| = 0$.

*Proof.* By Lemma 3 and the remark following it, we see that
$U^*_+ T_\phi U^- - T_\phi$ is compact for $\phi$ in $L^\infty_r$. The desired result is immediate.

We write $BC = BC(C)$ for the bounded continuous functions on $C$ and $BC_r = BC \cap L^\infty_r$.

**Definition 3.** For $\phi$ in $BC_r$, we say $\phi$ is eventually slowly varying
($\phi \in BC_r, ESV$) if for all $\epsilon > 0$, $L > 0$ there is an $N = N(\epsilon, L) > 0$ so that
$|\phi(r) - \phi(r')| < \epsilon$ whenever $|r - r'| < L$ and $r, r' > N$. 


It is easy to check that $BC_{ESV}$ is a closed, conjugate-closed subalgebra of $BC_r$. Our next objective is to prove that $BC_{ESV} \subset Q$. We need two lemmas of some independent interest. Recall that

$$\int_0^\infty e^{-r^2/2} r^{2k+1} \, dr = 2^k k!.$$ 

**Lemma 4.** For fixed $\delta$, $0 < \delta < \sqrt{2k+1}$, we have

$$1 - \frac{1}{2^k k!} \int_0^{\sqrt{2k+1} + \delta} \frac{1}{\sqrt{2k+1} - \delta} e^{-r^2/2} r^{2k+1} \, dr \leq \frac{1.1}{\delta^2}.$$ 

*Proof.* Since $(r - \sqrt{2k+1})^2 \geq \delta^2$ on $[0, \sqrt{2k+1} - \delta]$ and on $[\sqrt{2k+1} + \delta, \infty)$, it is clear that

$$\frac{1}{\delta^2} \frac{1}{2^k k!} \int_0^{\infty} (r - \sqrt{2k+1})^2 e^{-r^2/2} r^{2k+1} \, dr \geq 1 - \frac{1}{2^k k!} \int_0^{\sqrt{2k+1} + \delta} \frac{1}{\sqrt{2k+1} - \delta} e^{-r^2/2} r^{2k+1} \, dr.$$ 

The left-hand side can now be estimated, using previously mentioned integral identities and Stirling's formula with error term. The details are tedious, but routine.

**Lemma 5.** If $\phi$ is in $BC_{ESV}$ then $\lim_{k \to \infty} \{s_k(\phi) - \phi(\sqrt{2k+1})\} = 0$.

*Proof.* We write

$$|s_k(\phi) - \phi(\sqrt{2k+1})| \leq \frac{1}{2^k k!} \int_0^\infty |\phi(r) - \phi(\sqrt{2k+1})| e^{-r^2/2} r^{2k+1} \, dr$$

$$\leq \frac{1}{2^k k!} \left\{ \int_0^{\sqrt{2k+1} + \delta} \frac{1}{\sqrt{2k+1} + \delta} + \int_0^{\sqrt{2k+1} - \delta} \frac{1}{\sqrt{2k+1} - \delta} \right\} |\phi(r) - \phi(\sqrt{2k+1})| e^{-r^2/2} r^{2k+1} \, dr$$

$$+ \frac{1}{2^k k!} \int_0^{\sqrt{2k+1} + \delta} |\phi(r) - \phi(\sqrt{2k+1})| e^{-r^2/2} r^{2k+1} \, dr$$

$$\leq 2 \|\phi\| \int_0^{\sqrt{2k+1} + \delta} \left( \frac{1}{\sqrt{2k+1} + \delta} + \frac{1}{\sqrt{2k+1} - \delta} \right) |\phi(r) - \phi(\sqrt{2k+1})| e^{-r^2/2} r^{2k+1} \, dr.$$
where the last line follows from Lemma 4. Given $\varepsilon > 0$, choose $\delta$ so large that

$$\frac{2.2}{\delta^2} \| \phi \|_\infty < \frac{\varepsilon}{2}.$$ 

Since $\phi$ is in $\text{BC,ESV}$, for $\delta = \delta(\varepsilon)$ fixed as above there is an $N(\varepsilon/2, \delta)$ so that $\sqrt{2k + 1} - \delta > N(\varepsilon/2, \delta)$ implies that $|\phi(r) - \phi(\sqrt{2k + 1})| < \varepsilon/2$ for $|r - \sqrt{2k + 1}| < \delta$. It follows that if $k > \lceil N(\varepsilon/2, \delta(\varepsilon)) + \delta(\varepsilon) \rceil^2$ then

$$\frac{1}{2^k k!} \int_{\sqrt{2k + 1} - \delta}^{\sqrt{2k + 1} + \delta} |\phi(r) - \phi(\sqrt{2k + 1})| e^{-r^2/2} r^{2k+1} dr < \varepsilon/2.$$ 

It follows that $|s_k(\phi) - \phi(\sqrt{2k + 1})| < \varepsilon$ for

$$k > \lceil N(\varepsilon/2, \delta(\varepsilon)) + \delta(\varepsilon) \rceil^2$$

and we are done.

**Theorem 8.** The algebra $\text{BC,ESV}$ is contained in $Q$.

**Proof.** Since $\text{BC,ESV}$ is conjugate-closed, it suffices to prove that $\text{BC,ESV} \subset \mathcal{K}$. As before, it is enough to prove for $X = (I - P)M_\phi P$ and $\phi$ in $\text{BC,ESV}$ that $X^*X$ is compact. Direct calculation shows that

$$X^*X = PM_\phi P - (PM_\phi P)(PM_\phi P)$$

so that

$$X^*X e_k = \{s_k(|\phi|^2) - |s_k(\phi)|^2\} e_k, \quad k \geq 0.$$ 

By Lemma 5,

$$\lim_{k \to \infty} \{s_k(|\phi|^2) - |s_k(\phi)|^2\} = 0$$

and

$$\lim_{k \to \infty} \{s_k(\phi) - \phi(\sqrt{2k + 1})\} = 0.$$ 

An elementary argument using the boundedness of $\phi$ now shows that

$$\lim_{k \to \infty} \{s_k(|\phi|^2) - |s_k(\phi)|^2\} = 0$$

and so $X^*X$ is in $\mathcal{K}$.

**Remark.** It is easy to check that $e^{iar} \in \text{BC,ESV} \subset Q$ for real $a$ and
0 < \varepsilon < 1. On the other hand, $e^{ia\varepsilon}$ is not in $Q$ unless $a = 0$. This is because, by direct calculation,

$$T_{e^{ia\varepsilon}}T_{e^{ia\varepsilon}} - e^{-a^2/2}I$$

is in $\mathcal{X}$, which, if $e^{ia\varepsilon}$ were in $\Gamma$, would contradict Lemma 2 for $a \neq 0$.

The following result exhibits the relationship between $\phi(\sqrt{T_{|\varepsilon|^2}})$ and $T_\phi$ for $\phi$ in $\text{BC,ESV}$.

**Theorem 9.** For $\phi$ in $\text{BC,ESV}$, $\phi(\sqrt{T_{|\varepsilon|^2}}) - T_\phi$ is in $\mathcal{X}$.

**Proof.** This is a simple computation using Definition 3 and Lemma 5.

We have seen that $Q$ is reasonable large. We write $\text{ESV}(r, \theta)$ for the closed, conjugate-closed subalgebra of $L^\infty$ generated by $\{\text{BC}, \text{ESV}, e^{i\theta}\}$. By Lemma 3 and Theorem 8, $\text{ESV}(r, \theta) \subset Q$. We denote the closure of $L^\infty_c$ by $\text{cl}(L^\infty_c)$.

**Lemma 6.** $\text{BC,ESV} \cap B \subset \text{cl}(L^\infty_c)$.

**Proof.** For $\phi$ in $\text{BC,ESV}$ and $T_\phi$ compact, we must have

$$\lim_{k \to \infty} s_k(\phi) = 0.$$ 

By Lemma 5, this implies that

$$\lim_{k \to \infty} \phi(\sqrt{2k + 1}) = 0.$$ 

Now using Definition 3, we see that

$$\lim_{r \to \infty} \phi(r) = 0$$ 

and so $\phi$ is in $\text{cl}(L^\infty_c)$.

**Lemma 7.** For $R_re_k = e^{ik}e_k$, the map

$$m(X) = \frac{1}{2\pi} \int_0^{2\pi} R^*_\tau XR, dt$$

is a conditional expectation from $\tau\{\text{ESV}(r, \theta)\}$ onto

$$\tau\{\text{ESV}(r, \theta)\} \cap \text{diag}\{e_k\},$$

where $\text{diag}\{e_k\}$ is the set of operators which are diagonal in the basis $\{e_k\}$. 
Proof. This is easy to check using the proof of Lemma 3 and the ensuing Remark. In particular, $\mathcal{X} \subset \tau\{\ESV(r, \theta)\}$ and $U_+ \in \tau\{\ESV(r, \theta)\}$. In fact, $\tau\{\ESV(r, \theta)\}$ is generated by $\tau\{\BC, \ESV\}$ and $U_+$.

**Lemma 8.** For $\phi$ in $\ESV(r, \theta)$, and

$$a_k(\phi, r) = \frac{1}{2\pi} \int_0^{2\pi} \phi(r, \theta) e^{-ik\theta} d\theta$$

we have $a_k(\phi, r)$ in $\BC, \ESV$ for all integers $k$. Moreover, for $\varepsilon > 0$ given there is an $N = N(\varepsilon, \phi)$ so that for $r \neq 0$, $n > N$

$$\left| \phi(r, \theta) - \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) a_k(\phi, r) e^{ik\theta} \right| < \varepsilon.$$

Proof. The first statement is an easy consequence of the facts that $\BC, \ESV$ is closed and sums of the form $\sum_{k=-N}^{N} \phi_k(r) e^{ik\theta}$ are dense in $\ESV(r, \theta)$ where $\phi_k$ is in $\BC, \ESV$. Density of such sums also implies that, for $r \neq 0$ and $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ so that

$$|\phi(r, \theta) - \phi(r, \theta^\prime)| < \varepsilon$$

whenever $|\theta - \theta^\prime| < \delta$ (independent of $r \neq 0$) and $\phi$ is in $\ESV(r, \theta)$. The last statement follows from the usual proof of convergence of Cesaro means of Fourier sums of continuous functions on the circle [8].

**Theorem 10.** The map $\phi \rightarrow T_{\phi}$ induces a $\ast$-homomorphism

$$\tau\{\ESV(r, \theta)\}/\mathcal{X} \sim \ESV(r, \theta)/\ESV(r, \theta) \cap \text{cl}\{L_c^\infty\}.$$  

Proof. The arguments used in the proof of Theorem 6 give the desired result if we can see that

$$\ESV(r, \theta) \cap B \subset \text{cl}\{L_c^\infty\}.$$  

Suppose that $T_{\phi} \in \mathcal{X}$ and $\phi$ is in $\ESV(r, \theta)$. By Lemma 8, with $U_+^{-|k|} = U_+^{-k}$, $\varepsilon > 0$ and $m_0$ given, $n > N$

$$\left\| K_n - \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) T_{a_k(\phi, r)} U_+^{k-m_0} \right\| < \varepsilon$$

for some $K_n$ in $\mathcal{X}$. Hence, using Lemma 7, we see that

$$\left\| m(K_n) - \left(1 - \frac{|m_0|}{n+1}\right) T_{m_0(\phi, r)} \right\| < \varepsilon.$$
Noting that $m(K_n)$ is in $\mathcal{X}$ and letting $n \to \infty$ and then $\varepsilon \to 0$, it follows that $T_{a_{m_0}}(\phi, r)$ is in $\mathcal{X}$ for each $m_0$. Hence, by Lemma 6,

$$\lim_{r \to \infty} a_{m_0}(\phi, r) = 0$$

so that $a_{m_0}(\phi, r)$ is in $\text{cl}(L^\infty_c)$ for each $m_0$. Applying Lemma 8 again shows finally that $\phi$ is in $\text{cl}(L^\infty_c)$.

The next series of lemmas may be of some independent interest. These lemmas will be used to show that $Q \cap L^\infty_c$ is “not much bigger” than $\text{BC}, \text{ESV}$.

Let

$$g * h(a) = \int_{-\infty}^{\infty} g(a - x) h(x) \, dx$$

be the usual convolution on $\mathbb{R}$. Let $(t, b g)(x) = g(x - b)$. For $\phi$ in $L^\infty(\mathbb{R})$ and $G(x) = (1/\sqrt{\pi}) e^{-x^2}$ we define $\tilde{\phi} = G * \phi$ and then obtain

**Lemma 9.** The function $\tilde{\phi} = G * \phi$ is bounded and uniformly continuous for $\phi$ in $L^\infty$.

*Proof.* Boundedness is easy. For $b$ in $\mathbb{R}$

$$t_b \tilde{\phi} = (t_b G)^* \phi$$

so

$$t_b \tilde{\phi} - \phi = (t_b G - G)^* \phi$$

and

$$|t_b \tilde{\phi} - \phi| \leq |t_b G - G|^* |\phi|$$

$$\leq \|t_b G - G\|_1 \|\phi\|_\infty.$$ 

It is an easy consequence of the Lebesgue dominated convergence theorem that

$$\lim_{b \to 0} \|t_b G - G\|_1 = 0$$

and uniform continuity of $\tilde{\phi}$ follows immediately.

From now on, we embed $L^\infty_c$ in $L^\infty(\mathbb{R})$ by setting $\phi(r) = 0$ for $r < 0$.

**Lemma 10.** For $\phi(r)$ in $L^\infty_c$, we have

$$\lim_{k \to \infty} |s_k(\phi) - \tilde{\phi}(\sqrt{2k + 1})| = 0.$$
Proof. Let \( f_k(r) \) be given by
\[
f_k(r) = \frac{1}{2^k k!} e^{-(r + \sqrt{2k + 1})^2} (r + \sqrt{2k + 1})^{2k + 1}.
\]

Direct calculation shows that \( f_k(r) \) converges pointwise for each \( r \) and, for fixed \( M > 0 \),
\[
\left| f_k(r) - \frac{1}{\sqrt{\pi}} e^{-r^2} \right| \leq M e^{-r^2/2}.
\]

The Lebesgue dominated convergence theorem and translation-invariance of Lebesgue measure complete the proof.

Now for \( \phi \) in \( L^\infty \), we write
\[
\hat{\phi} = G^*|\phi|^2 - |G^*\phi|^2.
\]

**Lemma 11.** If \( \phi \) is in \( Q \cap L_r^\infty \), then
\[
\lim_{a \to +\infty} \hat{\phi}(a) = 0.
\]

**Proof.** It follows from Lemma 9 that \( \hat{\phi} \) is bounded and uniformly continuous. As in the proof of Theorem 8, it is easy to check that for \( \phi \) in \( Q \cap L_r^\infty \)
\[
\lim_{k \to \infty} s_k(\phi^2) - s_k(\phi^2) = 0.
\]

It follows from Lemma 10 that
\[
\lim_{k \to \infty} \hat{\phi}(\sqrt{2k + 1}) = 0.
\]

After noting that
\[
\sqrt{2k + 3} - \sqrt{2k + 1} \leq \frac{1}{\sqrt{2k + 1}},
\]
uniform continuity of \( \hat{\phi} \) gives the desired result.

**Lemma 12.** For \( \phi \) in \( BC, ESV \), we have
\[
\lim_{a \to +\infty} |\phi(a) - \hat{\phi}(a)| = 0.
\]
Proof. By Definition 3 and Lemma 9, we see that $\phi - \bar{\phi}$ is uniformly continuous for $r > 0$. Using Lemmas 5 and 10, we see that
\[
\lim_{k \to \infty} |\phi(\sqrt{2k} + 1) - \bar{\phi}(\sqrt{2k} + 1)| = 0.
\]
The argument at the end of Lemma 11 completes the proof.

**Lemma 13.** For $\phi$ in $L^\infty$, $\tilde{\phi} = G^*\phi$, $\tilde{\phi} = G^*|\phi|^2 - |G^*\phi|^2$ and $|a - b| \leq L$ we have
\[
|\tilde{\phi}(a) - \tilde{\phi}(b)| \leq K(L) \max\{\sqrt{\phi(a)}, \sqrt{\phi(b)}\}
\]
for $K(L)$ finite and independent of $\phi$.

Proof. Let $X$ be a measure space with distinct probability measures $\alpha, \beta$. At the end, $X$ will be $\mathbb{R}$ and $\text{d}\alpha = (1/\sqrt{\pi}) e^{-(a - x)^2} \text{d}x$, $\text{d}\beta = (1/\sqrt{\pi}) e^{-x^2} \text{d}x$. We define
\[
A(\alpha, \beta) = \sup \left\{ \left| \int f \text{d}\alpha - \int f \text{d}\beta \right| : f \in L^2(\alpha + \beta) \right\}
\]
\[
\|f\|_2^2 - |\langle f, 1 \rangle_\alpha|^2 \leq 1
\]
\[
\|f\|_\beta^2 - |\langle f, 1 \rangle_\beta|^2 \leq 1
\]
Here,
\[
\|f\|_2^2 = \int |f|^2 \text{d}x
\]
\[
\langle f, g \rangle_\alpha = \int f \bar{g} \text{d}x.
\]
If we define
\[
\Omega(\alpha, \beta) = \sup \left\{ \left| \int f \text{d}\alpha - \int f \text{d}\beta \right| : f \in L^2(\alpha + \beta) \right\}
\]
and $\|f\|_{\alpha + \beta}^2 - |\langle f, 1 \rangle_\alpha|^2 - |\langle f, 1 \rangle_\beta|^2 \leq 2$
then it is clear that
\[
A(\alpha, \beta) \leq \Omega(\alpha, \beta).
\]
Now write $\text{d}\alpha = g_\alpha \text{d}(\alpha + \beta)$, $\text{d}\beta = g_\beta \text{d}(\alpha + \beta)$ with $g_\alpha$, $g_\beta$ the Radon–Nikodym derivatives. Clearly, $g_\alpha$, $g_\beta$ are non-negative real-valued functions.
with \( g_\alpha + g_\beta = 1 \) a.e. \( d(\alpha + \beta) \). Thus, \( g_\alpha \) and \( g_\beta \) are in \( L^2(\alpha + \beta) \). Note that for \( \alpha \neq \beta \), \( \| g_\alpha - g_\beta \|_{\alpha + \beta} \neq 0 \). It is easy to check that
\[
\Omega(\alpha, \beta) = \sup \{ |\langle f, g_\alpha - g_\beta \rangle_{\alpha + \beta} | : f \in L^2(\alpha + \beta) \text{ and } \| f \|_{\alpha + \beta}^2 - |\langle f, g_\alpha \rangle_{\alpha + \beta}|^2 - |\langle f, g_\beta \rangle_{\alpha + \beta}|^2 \leq 2 \}.
\]

Next define
\[
e_+ = (g_\alpha \pm g_\beta)\| g_\alpha \pm g_\beta \|_{\alpha + \beta}^{-1}.
\]
Since \( g_\alpha + g_\beta = 1 \), we have \( \| g_\alpha + g_\beta \|_{\alpha + \beta} = \sqrt{2} \) and \( e_+ = 1/\sqrt{2} \). It is also easy to check that
\[
\| g_\alpha - g_\beta \|_{\alpha + \beta}^2 \leq 2
\]
with equality if and only if \( \alpha \) and \( \beta \) are mutually singular. Since \( \alpha(X) = \beta(X) = 1 \), it is clear that
\[
\langle e_+, e_- \rangle_{\alpha + \beta} = 0.
\]

We have
\[
g_\alpha = \frac{1}{2}(1 + \| g_\alpha - g_\beta \|_{\alpha + \beta} e_-)
\]
\[
g_\beta = \frac{1}{2}(1 - \| g_\alpha - g_\beta \|_{\alpha + \beta} e_-).
\]

Expanding the previous expression for \( \Omega(\alpha, \beta) \) in terms of \( e_+, e_- \) yields
\[
\Omega(\alpha, \beta) = \sup \{ \| g_\alpha - g_\beta \|_{\alpha + \beta} |\langle f, e_- \rangle_{\alpha + \beta} | : f \in L^2(\alpha + \beta) \text{ and } \| f \|_{\alpha + \beta}^2 - |\langle f, e_- \rangle_{\alpha + \beta} |^2 - \frac{1}{2} \| g_\alpha - g_\beta \|_{\alpha + \beta}^2 |\langle f, e_- \rangle_{\alpha + \beta} |^2 \leq 2 \}.
\]

Writing \( f = u + ae_+ + be_- \) with \( u \perp e_+, e_- \) yields
\[
\Omega(\alpha, \beta) = \sup \{ |b| \| g_\alpha - g_\beta \|_{\alpha + \beta} : u \in L^2(\alpha + \beta), u \perp e_+, e_- \text{ and } \| u \|_{\alpha + \beta}^2 + |b|^2 \{ 1 - \frac{1}{2} \| g_\alpha - g_\beta \|_{\alpha + \beta}^2 \} \leq 2 \} \]
\[
= \sup \{ c \| g_\alpha - g_\beta \|_{\alpha + \beta} : c \geq 0, c^2 \{ 1 - \frac{1}{2} \| g_\alpha - g_\beta \|_{\alpha + \beta}^2 \} \leq 2 \}
\]
\[
= \sqrt{2} \| g_\alpha - g_\beta \|_{\alpha + \beta} \{ 1 - \frac{1}{2} \| g_\alpha - g_\beta \|_{\alpha + \beta}^2 \}^{-1/2}.
\]

On \( X = \mathbb{R} \), with \( d\alpha = (1/\sqrt{\pi}) e^{-(a-x)^2} \, dx \) and \( d\beta = (1/\sqrt{\pi}) e^{-x^2} \, dx \) it is clear that \( \alpha \) and \( \beta \) are not mutually singular. Direct calculation now shows for this \( \alpha \) and \( \beta \) that
\[
\| g_\alpha - g_\beta \|_{\alpha + \beta}^2 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} g(a, x)(e^{-(a-x)^2} + e^{-x^2}) \, dx \equiv k(a)
\]
where

\[ g(a, x) = \frac{e^{-(a-x)^2} - e^{-x^2}}{e^{-(a-x)^2} + e^{-x^2}}^2 \]

and \( g \) is continuous on \( \mathbb{R} \times \mathbb{R} \) with \( 0 \leq g(a, x) \leq 1 \). It is easy to check that

\[
k(a) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left[ g(a, x + a) + g(a, x) \right] e^{-x^2} dx
\]

\[
= \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} g(a, x) e^{-x^2} dx.
\]

Continuity of \( k(a) \) follows by a standard "2\( \varepsilon \)" argument. Note that

\[ 0 < k(a) < 2. \]

Since

\[
Q \equiv \int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi},
\]

it is now clear that, for any fixed \( L > 0 \),

\[
\sup_{|a| \leq L} Q \left( \frac{1}{\sqrt{\pi}} e^{-(a-x)^2} dx, \frac{1}{\sqrt{\pi}} e^{-x^2} dx \right) = K(L)
\]

is finite.

It follows from the discussion above that

\[
\tilde{A}(a, b) = \sup_{\phi \in L^\infty} \{ |\phi(a) - \phi(b)| : \phi(a) \leq 1, \phi(b) \leq 1 \}
\]

is finite and, by translation invariance,

\[
\sup_{|a - b| \leq L} \tilde{A}(a, b) = \sup_{|a| \leq L} \tilde{A}(a, 0) \leq K(L).
\]

For fixed \( \phi \) in \( L^\infty \), suppose \( \phi(a) = A \) and \( \phi(b) = B \) with \( A \geq B \). Then for \( \xi = \phi A^{-1/2} \) we see that \( \xi(a) = 1 \) and \( \xi(b) = BA^{-1} \leq 1 \). It follows that

\[
|\xi(a) - \xi(b)| \leq K(L)
\]

and so

\[
|\phi(a) - \phi(b)| \leq K(L) \max \{ \sqrt{\phi(a)}, \sqrt{\phi(b)} \}
\]

whenever \( |a - b| \leq L \). The exceptional case \( A = B = 0 \) is easily treated.
We can now prove

**Theorem 11.** For \( \phi \) in \( L^\infty _r \cap Q \), we have \( \tilde{\phi} \) in \( BC, ESV \) and \( \phi - \tilde{\phi} \) in \( Q \cap B \).

**Proof.** It follows directly from Lemmas 11, 13 and Definition 3 that \( \tilde{\phi} \) is in \( BC, ESV \). Next, using Lemma 10, we see that

\[
\lim_{k \to \infty} |s_k(\phi - \tilde{\phi}) - (\tilde{\phi} - \tilde{\phi})(\sqrt{2k+1})| = 0.
\]

By Lemma 12, since \( \tilde{\phi} \) is in \( BC, ESV \),

\[
\lim_{k \to \infty} (\tilde{\phi} - \tilde{\phi})(\sqrt{2k+1}) = 0
\]

and it follows that

\[
\lim_{k \to \infty} s_k(\phi - \tilde{\phi}) = 0.
\]

The last statement is equivalent to \( T_{\phi - \tilde{\phi}} \) belonging to \( \mathcal{X} \) and so \( \phi - \tilde{\phi} \) is in \( Q \cap B \).

5. \( \tau \{L^\infty (C^n)\} \) AND \( CCR(C^n) \)

Recall the discussion in Section 2, where the Weyl algebra \( CCR(C^n) \) was represented on \( H^2(C^n, d\mu) \) as the \( C^* \)-algebra generated by

\[
\{ e^{iT_{Re(iz)}}, w \in C^n \}.
\]

The results of this section depend heavily on the

**Basic Identities.** For \( w, v \) in \( C^n \), we have

\[
T_{\phi(Re(iz))} = e^{-|w|^2/4} e^{iT_{Re(iz)}}
\]

\[
T_{v(Re(iz))} T_{\phi(Re(iz))} = e^{iv^2/2} T_{v(Re(iz))}.
\]

**Proof.** We use the fact that, for the reproducing kernels \( k_\lambda(z) = e^{2\lambda z/2}, \)

\[
\langle Xk_\lambda, k_\mu \rangle = 0 \text{ for all } \lambda, \mu \text{ in } C^n \text{ implies } X = 0.
\]

We also use Proposition 2 of Section 2, which states that \( e^{iT_{Re(iz)}} = \overline{k_{-i\mu}}(z) t_{-i\mu} \), where \( \overline{k_\lambda(z)} = e^{2\lambda z/2 - |\lambda|^2/4} \)

and \( (t, f)(z) = f(z - \lambda) \). Thus, we have

\[
\langle e^{iT_{Re(iz)}} k_\lambda, k_\mu \rangle = \langle k_{-i\mu}(z) t_{-i\mu} k_\lambda, k_\mu \rangle
\]

\[
= \exp \left\{ \frac{1}{2} i\mu \mu - \frac{1}{4} |w|^2 + \frac{1}{2} \bar{\lambda} \mu + \frac{1}{2} i\lambda w \right\}
\]
while
\[ \langle T_{e^{Re(iw)}}k_{\lambda}, k_{\mu} \rangle = \langle e^{iwz/2}e^{iwz/2}k_{\lambda}, k_{\mu} \rangle = \langle e^{iwz/2}e^{iwz/2}, e^{-iwz/2}e^{iwz/2} \rangle - \langle k_{\lambda - iw}, k_{\mu + iw} \rangle = \exp\left\{ \frac{i\lambda}{2} - \frac{i\mu}{2} + \frac{i}{2}w + \frac{1}{2}w^2 \right\} \]
and the first identity follows at once.

The second identity, a version of the usual CCR commutation relations, can be established by using the first identity and Proposition 2 of Section 2. In particular, we have
\[ T_{e^{Re(iw)}}T_{e^{Re(iw)}} = e^{-\frac{1}{4}(|w|^2 + |v|^2)}k_{-iw}(z) k_{-iv}(z) e^{-iv} = e^{-\frac{1}{4}(|w|^2 + |v|^2)}k_{-iw}(z) k_{-iv}(z) e^{-iv} \]
As our first consequence of the Basic Identities, we have

**Theorem 12.** \( \tau(\mathcal{AP}(C^n)) = \text{CCR}(C^n) \).

**Proof.** Immediate from the Basic Identities, the representation of \( \text{CCR}(C^n) \) discussed in Section 2, and the fact that \( \mathcal{AP}(C^n) \) is exactly the closure in \( L^\infty(C^n) \) of the linear combinations of characters \( e^{Re(iwz)} \).

Recall that \( \mathcal{C}(\tau(A)) \) is the commutator ideal of \( \tau(A) \). As easy consequences of Theorem 12, we have

**Corollary 1.** \( \mathcal{C}(\tau(\mathcal{AP}(C^n))) = \tau(\mathcal{AP}(C^n)) \).

**Proof.** This is immediate from the fact that \( \text{CCR}(C^n) \) is simple [3].

**Corollary 2.** \( \mathcal{C}(\tau(L^\infty(C^n))) = \tau(L^\infty(C^n)) \).

**Proof.** Since \( I \) is in \( \mathcal{C}(\tau(\mathcal{AP}(C^n))) \), it is clear that \( I \) is in \( \mathcal{C}(\tau(L^\infty(C^n))) \).

**Remark.** The fact that \( \tau(\mathcal{AP}(C^n)) \) is simple causes a striking difference between Toeplitz operators in this setting and classical Toeplitz operators on the unit circle. In fact, for \( T \) the unit-circle and \( \tau(L^\infty(T)) \) the \( C^* \)-algebra generated by all bounded Toeplitz operators on the usual Hardy space \( H^2(T, d\theta) \), the result of Douglas [7] that
\[ \tau(L^\infty(T))/\mathcal{C}(\tau(L^\infty(T))) \simeq L^\infty(T) \]
implies that \( \tau(L^\infty(T)) \) contains no simple unital \( C^* \)-subalgebra and, moreover, \( \mathcal{C} \{ \tau(A) \} \neq \tau(A) \) for \( A \) any unital subalgebra of \( L^\infty(T) \).

Let \( E\{ \mathcal{A} \} \) be the space of states (positive linear functionals \( \omega \) with \( \| \omega \| = 1 \)) of the \( C^* \)-algebra \( \mathcal{A} \). \( E\{ \mathcal{A} \} \) is given the usual weak* topology. It follows easily from the Basic Identities that

\[
\{ T_\phi : \phi \in \text{AP}(C^n) \}
\]

is a dense subset of \( CCR(C^n) = \tau\{ \text{AP}(C^n) \} \). For \( \omega \in E\{ CCR(C^n) \} \), we define \( \hat{\omega} \) by

\[
\hat{\omega}(\phi) = \omega(T_\phi)
\]

for all \( \phi \) in \( \text{AP}(C^n) \). Let \( j(\omega) = \hat{\omega} \). The state space \( E\{ \text{AP}(C^n) \} \) consists precisely of the probability measures on the Bohr compactification of \( C^n = R^{2n} \). For \( a \) in \( C^n \),

\[
\delta_a(\phi) = \phi(a)
\]

defines an element of \( E\{ \text{AP}(C^n) \} \).

**THEOREM 13.** The map \( j(\omega) = \hat{\omega} \) is a homeomorphism from \( E\{ CCR(C^n) \} \) into \( E\{ \text{AP}(C^n) \} \setminus \{ \delta_a : a \in C^n \} \).

**Proof.** Since the map \( \phi \to T_\phi \) is positive and norm-decreasing with \( T_1 = I \), it follows that \( \hat{\omega} = j(\omega) \) is in \( E\{ \text{AP}(C^n) \} \). The fact that \( j \) is continuous follows directly from the definition of \( \hat{\omega} \). The density of \( \{ T_\phi : \phi \in \text{AP}(C^n) \} \) in \( CCR(C^n) \) implies immediately that \( j \) is 1–1 and the fact that \( E\{ CCR(C^n) \} \) is compact Hausdorff then implies that \( j \) is a homeomorphism. It remains to check the range of \( j \). It follows from the Basic Identities that

\[
\hat{\omega}(e^{\text{Re}(iz)}) = e^{-|z|^2/4} \omega(e^{iT_{\text{Re}(iz)}}).
\]

Now \( e^{iT_{\text{Re}(iz)}} \) is unitary so

\[
|\omega(e^{iT_{\text{Re}(iz)}})| \leq 1
\]

and

\[
|\hat{\omega}(e^{i\text{Re}(iz)})| \leq e^{-|z|^2/4}.
\]

On the other hand, for \( a \) in \( C^n \)

\[
|\delta_a(e^{i\text{Re}(iz)})| = |e^{i\text{Re}(iz)a}| = 1.
\]

The desired result follows.
Remarks. It would be interesting to determine the range of $j$ more precisely. The fact that
\[ \| T_{e^{i \text{Re}(\lambda z)}} \| = e^{-|\lambda|^2/4} \]
shows that the map $\phi \to T_\phi$ is not bounded from below even though it is bijective.

6. THE HARMONIC OSCILLATOR ON $H^2(C^n, d\mu)$

In this section, we return to the discussion at the end of Section 2. We wish to briefly indicate the role of the reproducing kernels $k_\lambda(z)$ in the transition from the quantum mechanics of the harmonic oscillator to the classical harmonic oscillator. Specifically, we will reinterpret a special case of the general result of Hepp [9] in the context of $H^2(C^n, d\mu)$.

We note first that the “coherent states,” which play a major role in quantum mechanics (see [9] for a discussion and other references), take a very simple form in the Segal–Bargmann representation. On $H^2(C^n, d\mu)$, these states are precisely
\[ e^{i T_{\text{Re}(\lambda z)}} 1, \quad w \in C^n \]
and, by Proposition 2 of Section 2, we have
\[ e^{i T_{\text{Re}(\lambda z)}} 1 - \mathcal{K}_{-\lambda w}(z), \quad w \in C^n, \]
where $\mathcal{K}_\lambda$ is the normalized reproducing kernel discussed earlier.

After setting Planck’s constant equal to one, as in Section 2, the quantum–mechanical position and momentum operators become, in the Segal–Bargmann representation [1]
\[ q_j = T_{x_j}, \]
\[ p_j = -T_{y_j} \]
on $H^2(C^n, d\mu)$. We recall, from Section 2, that the time–evolution operator for the harmonic oscillator is just $R_{-t}$, where
\[ (R_t f)(z) = f(e^{it} z). \]

Using the fact that
\[ R_t T_z R_{-t} = e^{it} T_z, \]
on $\mathcal{D}$, it is now easy to check that
Theorem 14. The functions
\[ \tilde{q}_j(t) = \langle T_{x_j} \mathcal{R}_{-x_j}, \mathcal{R}_{-y_j} \rangle, \quad \tilde{p}_j(t) = \langle -T_{y_j} \mathcal{R}_{-x_j}, \mathcal{R}_{-y_j} \rangle, \]
satisfy the differential equations \( \tilde{q}_j(t)' = \tilde{p}_j(t), \quad \tilde{p}_j(t)' = -\tilde{q}_j(t) \) with initial values \( \tilde{q}_j(0) = x_j(\lambda), \quad \tilde{p}_j(0) = -y_j(\lambda). \)

Proof. Recall that \( x_j = (z_j + \bar{z}_j)/2, \quad y_j = (z_j - \bar{z}_j)/2i. \) The argument is direct after remembering that
\[ \langle T_{z_j} \mathcal{R}_{-x_j}, \mathcal{R}_{-y_j} \rangle = \lambda_j. \]

Remark. Theorem 14 is the trivial case of the remarkable work of Hepp [9]. The main point of this example is that forming expectations with the states \( \mathcal{R}_{-x_j} \) gives all possible initial conditions for the classical equations of decoupled harmonic oscillators. In general, Planck’s constant \( h \) must be brought into the discussion since the classical equations are only recovered as \( h \to 0 \) for more general potentials.

7. Problems and Remarks

We begin by listing several questions left unsettled in Section 4.

Problem 1. How large is \( Q \cap B \)?

Problem 2. Is \( Q \) generated by ESV(\( r, \theta \)) and \( L^\infty \)?

We also have

Problem 3. Characterize those subalgebras \( A \) of \( L^\infty(\mathbb{C}) \) where \( \mathcal{C}\{\tau(A)\} \neq \tau(A) \).

As noted in Section 5, if \( \mathcal{AP}(\mathbb{C}) \) is contained in \( A \) then \( \mathcal{C}\{\tau(A)\} = \tau(A) \). Is this the determining condition?

Finally, we note that, for \( \pi: \tau(L^\infty(\mathbb{C})) \to \tau(L^\infty)/\mathcal{J} \) the quotient homomorphism, it follows from Lemma 2 in Section 4 that \( \pi\{\tau(Q)\} \) is in the center of \( \pi\{\tau(L^\infty)\} \). The localization technique of Douglas [7] applied to \( \pi(T_{\phi}) \) yields

Theorem 15. Given \( \phi \) in \( L^\infty(\mathbb{C}) \), suppose for each \( w \) in \( \mathbb{C} \) with \( |w| = 1 \) there is a \( \phi_w \) with \( T_{\phi_w} \) Fredholm and \( \phi = \phi_w \) a.e. on some sector \( \{z: |z| - |w| < \varepsilon(w)\} \) (\( \varepsilon(w) > 0 \)). Then \( T_{\phi} \) is Fredholm.
Proof. Left to reader.

Problem 4. Can localization be further exploited for $\tau\{L^\infty(C)\}$?

Several problems are suggested by Section 5 and 6. In particular, keeping in mind the equality $\tau\{\text{AP}(C^n)\} = \text{CCR}(C^n)$, we are led to

Problem 5. To what extent does $\tau\{L^\infty(C^n)\}$ serve as a natural algebra of observables for quantum mechanics?

Problem 6. Can the known function-theoretic structure of $\text{AP}(C^n)$ be used in the analysis of $\text{CCR}(C^n)$?

Problem 7. To what extent can our analysis be applied in the physically interesting case where $C^n$ is replaced by an infinite-dimensional Hilbert space?

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References