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The Postulational Foundations of Linear Systems*

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I. INTRODUCTION

In this work we propose and compare six sets of postulates. The dynamic behavior of various linear input-output systems can be derived from each of them. Moreover, every postulational system that has appeared in the literature can be related to one of the sets of postulates suggested here. One of our objectives is to devise the weakest postulates one can use and still obtain there from time-domain or frequency-domain characterizations for various types of input-output systems. In this sense our postulates are a refinement of those suggested elsewhere. We accomplish this by using the kernel or convolution representations for the system to extend the system from the very restricted domains indicated in the postulates onto much larger spaces of distributions.

The second objective is to compare the six sets of postulates, which we denote by P^*, P, Q^*, Q, S^*, and S. The P^*, Q^*, and S^* sets of postulates are appropriate for systems that are active, whereas the P, Q, and S sets of postulates impose the passivity hypothesis. (By an active system we mean one for which the passivity hypothesis may or may not hold; thus, we view a passive system as being a special case of an active one.) On the other hand, the P and P^* sets are suitable for systems that are single-valued, whereas the Q^*, Q, S, and S^* sets provide greater generality by allowing multivalued systems to be taken into account. (Our use of the word "multivaluedness" allows single-valuedness as a special case.) Moreover, we will establish the system of implications indicated in Fig. 1. In this figure the symbol $P \Rightarrow Q$ means that any system that satisfies the P postulates will also satisfy the Q postulates; the other implications signs are interpreted similarly.

We also trace out how the time-domain and frequency-domain characterizations for the system are developed from the proposed postulates. It is through this very discussion that the aforementioned objectives are acheived.

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For example, the equivalence of the Q and S postulates are derived from the frequency-domain characterization of the system. Most of the proofs for this discussion already appear in the literature and are therefore omitted here; in such cases we refer the reader to the bibliography. Theorems 5 and 9 have the character of a survey; they gather together a variety of diverse results in the literature.

	Single-valued systems		Multivalued systems		
Active systems:	P^*		Q^*	$\Leftrightarrow$	S^*
	$\Uparrow$		$\Uparrow$		$\Uparrow$
Passive systems:	P	$\Rightarrow$	Q	$\Leftrightarrow$	S

FIG. 1.

Raisbeck [1] was apparently the first to propose a postulational approach to linear passive systems that are not restricted to special types of physical systems such as the lumped networks. His postulates resemble the P^* and P postulates stated below, but his analysis is only formal. The earliest rigorous and complete theories for passive systems were those due to König and Meixner [2] and Youla, Castriota, and Carlin [3]. The former authors use P-type postulates, whereas the latter authors use Q-type postulates. Subsequent theories were offered by Zemanian [4] using P-type postulates, by Wohlers and Beltrami [5] using P-type, Q-type, and S-type postulates, by Newcomb and his associates [6]-[8] using Q-type postulates, and by Guttinger [9] using S-type postulates. Finally, we note that the causality of active systems whose impulse responses are Laplace-transformable have been discussed by a number of authors. See, for example, [10] and the bibliography therein.

We assume throughout that the dependent variables are one-dimensional rather than n-dimensional ($n > 1$) vectors since almost all the important ideas can be discussed in this context. The n-dimensional discussion follows essentially the same development but requires a more complicated notation. We also require that the input and output variables be real since nothing is gained by allowing complex quantities. Furthermore, we think of the input variable and the output variable as being a voltage v and a current u respectively; as a result, we call the system function an admittance. Actually, v and u may be anything at all. However, in order for our definition of passivity to have a physical significance, v and u must be such that their product (if it exists) represents power being absorbed by the system. An example of such a system is the one-port: an electrical network wherein v and u occur at the same pair of terminals.

The symbols and terminology used in this paper follow that employed in [11], and we refer the reader to that source or to [12] for a more detailed discussion of the definitions used here. $\mathscr{D}$ and $\mathscr{D}'$ denote the conventional Schwartz spaces of testing functions of compact support and distributions, respectively. Also, $\mathscr{S}$ is the space of testing functions of rapid descent and $\mathscr{S}'$ the space of distributions of slow growth. $\mathscr{E}'$ is the space of distributions with compact supports. The strong topologies for $\mathscr{D}$ and $\mathscr{S}$ and the weak topologies for $\mathscr{D}'$, $\mathscr{S}'$, and $\mathscr{E}'$ are understood.

$\mathscr{R}$ denotes the real line and throughout this paper t, τ, x, σ, and ω are variables in $\mathscr{R}$; also, p is a complex variable with $p = \sigma + i\omega$. $[a, b]$ and (a, b) denote respectively a closed and an open interval on the real line with end-points a and b, $a < b$; the notations $[a, b)$ and $(a, b]$ are defined similarly. $\operatorname{supp} f$ denotes the support of either a conventional function or distribution f. $f^{(n)}$ denotes the nth derivative of f. $\check{f}$ is the transpose of f; i.e., $\check{f}(t) = f(-t)$. A smooth function is one having continuous derivatives of all orders at all points of its domain.

$\mathscr{D}_R$ (respectively, $\mathscr{D}'_R$) denotes the space of smooth functions (respectively, distributions) on $\mathscr{R}$ whose supports are bounded on the left. $\mathscr{D}'_R$ is not the dual of $\mathscr{D}_R$; also, $\mathscr{D}_R \subset \mathscr{D}'_R$. $\mathscr{D}'_+$ is the space of distributions on $\mathscr{R}$ whose supports are bounded on the left at the origin. Thus, $f \in \mathscr{D}'_+$ if and only if $f \in \mathscr{D}'$ and $\operatorname{supp} f \subset [0, \infty)$. A sequence $\{\phi_n\}$ converges in $\mathscr{D}_R$ if and only if there is a fixed real number $T > -\infty$ such that $\operatorname{supp} \phi_n \subset [T, \infty)$ for all n and, for each nonnegative integer k, $\{\phi_n^{(k)}\}_n$ converges uniformly on every compact subset of $\mathscr{R}$. Similarly, a sequence $\{f_n\}$ converges in $\mathscr{D}'_R$ if and only if, for some fixed real number $T > -\infty$, $\operatorname{supp} f_n \subset [T, \infty)$ for all n and $\{f_n\}$ converges in $\mathscr{D}'$. $\mathscr{D}$ is dense in $\mathscr{D}'$ and $\mathscr{D}'_R$.

The notation $y = y(t) \in \mathscr{X}|_t$ [or $y = y(t, \tau) \in \mathscr{X}|_{t,\tau}$] indicates that y is a conventional function or distribution on the real line $-\infty < t < \infty$ [or, respectively, on the (t, τ) plane] and belongs to the space $\mathscr{X}$.

A standard form in distribution theory is Schwartz' kernel representation [13], which is defined in the following way. Let $y = y(t, \tau) \in \mathscr{D}'|_{t,\tau}$ and let $v = v(\tau) \in \mathscr{D}|_\tau$. Then, $y \cdot v = y(t, \tau) \cdot v(\tau)$ denotes a distribution in $\mathscr{D}'|_t$ defined as follows: For any $\phi = \phi(t) \in \mathscr{D}|_t$,

$$\langle y \cdot v, \phi \rangle = \langle y(t, \tau), v(\tau)\phi(t) \rangle.$$

Thus, $v \mapsto y \cdot v$ is a mapping of $\mathscr{D}$ into $\mathscr{D}'$ and is in fact linear and continuous from $\mathscr{D}$ into $\mathscr{D}'$. The converse also happens to be true: every continuous linear mapping of $\mathscr{D}$ into $\mathscr{D}'$ has a kernel representation. Under suitable restrictions on y this mapping can be extended from $\mathscr{D}$ onto wider spaces of distributions (say, onto $\mathscr{A}$). But, in the latter case the right-hand side of the above definition may not possess a sense. (The extension is made via the continuity of the mapping on $\mathscr{A}$ and the denseness of $\mathscr{D}$ in $\mathscr{A}$.) We shall make use of these facts later on.

II. The Admittance Formulism

We represent the input-output system as an operator $\mathfrak{N}$ mapping the input variable v into the output variable u; thus, we write $u = \mathfrak{N}v$. It is understood that u and v are conventional functions or distributions on the real time axis $-\infty < t < \infty$. We now impose a series of postulates in order to obtain several characterizations of $\mathfrak{N}$.

P1. *$\mathfrak{N}$ is a single-valued mapping of $\mathscr{D}$ into $\mathscr{D}'$.*

As is indicated here, we at first restrict the domain of $\mathfrak{N}$ to the space $\mathscr{D}$ in order to obtain as weak a postulate as possible. Later on, $\mathfrak{N}$ will be extended in a unique way onto various spaces of distributions by means of Schwartz' kernel representation.

P2. *$\mathfrak{N}$ is linear on $\mathscr{D}$.*

P3. *$\mathfrak{N}$ is continuous from $\mathscr{D}$ into $\mathscr{D}'$.*

These first three postulates allow us to invoke Schwartz' kernel theorem [13] to show that $\mathfrak{N}$ has a kernel representation on $\mathscr{D}$.

Theorem 1. *$\mathfrak{N}$ satisfies* P1, P2, *and* P3 *if and only if there exists a unique kernel $y = y(t, \tau) \in \mathscr{D}' |_{t,\tau}$ such that $\mathfrak{N} = y \cdot$ on $\mathscr{D}$. (That is, $\mathfrak{N}v = y \cdot v$ for all $v \in \mathscr{D}$.)*

We call y the admittance of the system.

P4. *$\mathfrak{N}$ is time-invariant on $\mathscr{D}$.*

To explain this, let σ_x be the shifting operator defined on any conventional function or distribution $f(t)$ by $\sigma_x f(t) = f(t + x)$ where x is any real number. Then, the postulate means that $\mathfrak{N}$ commutes with σ_x whenever $\mathfrak{N}$ operates on a member of $\mathscr{D}$ (i.e., $\sigma_x \mathfrak{N} v = \mathfrak{N} \sigma_x v$ for all $v \in \mathscr{D}$ and all x.)

Under this additional postulate the kernel representation becomes a convolution representation [12; Vol. II, pp. 53-54]; that is, $y(t, \tau)$ becomes $y(t - \tau)$ where $y(t)$ is now a member of $\mathscr{D}' |_t$. In particular, we have

Theorem 2. *$\mathfrak{N}$ satisfies* P1 *through* P4 *if and only if there exists a unique $y = y(t) \in \mathscr{D}' |_t$ such that $\mathfrak{N} = y *$ (i.e., $\mathfrak{N}v = y * v = \langle y(\tau), v(t - \tau)\rangle$ for all $v \in \mathscr{D}$.)*

We can now extend $\mathfrak{N}$ via its convolution representation onto the space $\mathscr{E}'$. That is, for any $v \in \mathscr{E}'$, $\mathfrak{N}v$ is defined as $y * v \in \mathscr{D}'$. Because $\mathscr{D}$ is dense in $\mathscr{E}'$, this extension of $\mathfrak{N}$ is unique: There cannot be another continuous linear mapping of $\mathscr{E}'$ into $\mathscr{D}'$ that agrees with $\mathfrak{N}$ on $\mathscr{D}$ but differs from $\mathfrak{N}$ on some other member of $\mathscr{E}'$. Now, y can be identified as the unit impulse response of $\mathfrak{N}$; that is, $y = \mathfrak{N}\delta = y * \delta$ where δ denotes the delta functional. Furthermore, if the distribution y happens to be suitably restricted, we can extend $\mathfrak{N}$ onto still larger spaces of distributions via the convolution representation. For example, if $y \in \mathscr{D}'_R$, then $\mathfrak{N}$ can be extended onto $\mathscr{D}'_R$, and, if $y \in \mathscr{E}'$, then $\mathfrak{N}$ can be extended onto all of $\mathscr{D}'$. These extensions are also unique because $\mathscr{D}$ is dense in $\mathscr{D}'_R$ as well as in $\mathscr{D}'$.

Since every distribution has a Fourier transform in the Gelfand-Shilov-Ehrenpreis sense [11; Chapter 7], we can conclude from Theorem 2 that $\mathfrak{N}$ possesses a frequency-domain description whenever $\mathfrak{N}$ satisfies P1 through P4. Its system function is defined as the Fourier transform $\tilde{y}$ of y, and the convolution $y * v$ is transformed into the product $\tilde{y}\tilde{v}$ at least whenever $v \in \mathscr{E}'$ [11; Section 7.9]. Postulates P1 through P4 appear to be the weakest set of assumptions under which one can arrive at such a frequency-domain description for $\mathfrak{N}$.

In case $\mathfrak{N}$ does not satisfy P4, the same sort of unique extension as above can be made via the kernel representation: $\mathfrak{N} = y \cdot$. Let's be more precise. Assume that $\mathscr{A}$ is a Fréchet space of smooth functions on $\mathscr{R}$ or a countable strict inductive limit of such spaces [14; p. 85 and p. 126]. Let $\mathscr{A}'$ be the weak dual of $\mathscr{A}$. Suppose that the following four conditions are satisfied.

I. $\mathscr{D}$ is a dense subset of $\mathscr{A}$, and the topology of $\mathscr{D}$ is stronger than the topology induced on $\mathscr{D}$ by $\mathscr{A}$.

II. If $\psi \in \mathscr{A}$ and if λ is a smooth function on $\mathscr{R}$ such that, for each non-negative integer k, $\lambda^{(k)}$ is bounded on $\mathscr{R}$, then $\lambda\psi \in \mathscr{A}$.

III. $\mathscr{D}$ is dense in $\mathscr{A}'$.

IV. For certain (but not necessarily all) $w \in \mathscr{D}'$, the operator $w \cdot$ is defined on $\mathscr{A}'$ and is a continuous linear mapping of $\mathscr{A}'$ into $\mathscr{D}'$ [13; pp. 224-225].

Condition I implies that $\mathscr{A}'$ is a subspace of $\mathscr{D}'$ and that the topology of $\mathscr{A}'$ is stronger than that induced on $\mathscr{A}'$ by $\mathscr{D}'$. We will use condition II in a subsequent proof. It is conditions III and IV that allow us to extend $\mathfrak{N} = y \cdot$ onto $\mathscr{A}'$ if y happens to be one of the w indicated in IV. From now on we shall always assume that $\mathfrak{N}$ has been extended through the right-hand side of $\mathfrak{N} = y \cdot$ onto every such space of distributions $\mathscr{A}'$ that satisfies the above stated conditions with $y = w$. Condition III implies that this extension is unique in the aforementioned sense. We also assume that $\mathfrak{N}$ is extended no further.

Definition of Causality. Let v_1 and v_2 be distributions in the domain of an operator $\mathfrak{U}$ mapping $\mathscr{D}'$ or a subset of $\mathscr{D}'$ into $\mathscr{D}'$, and let $u_1 = \mathfrak{U}v_1$ and $u_2 = \mathfrak{U}v_2$. $\mathfrak{U}$ is said to be causal (or to satisfy causality) if the condition $v_1(t) = v_2(t)$ on $-\infty < t < t_0$ implies that $u_1(t) = u_2(t)$ on $-\infty < t < t_0$ and if this property holds for all real values of t_0.

The equalities herein are understood in the sense of equality in $\mathscr{D}'$. Also, a causal operator is clearly single-valued.

We can characterize causality in the following way:

THEOREM 3. *Let $\mathfrak{N}$ satisfy* P1, P2, *and* P3; *then, (the extended) $\mathfrak{N}$ is causal if and only if the support of $y = y(t, \tau)$ is contained in the half-plane $\Lambda = \{(t, \tau) : t \geqslant \tau\}$. Next, assume in addition that $\mathfrak{N}$ satisfies* P4; *then, (the extended) $\mathfrak{N}$ is causal if and only if the support of $y = y(t)$ is contained in the interval $0 \leqslant t < \infty$.*

PROOF. The second sentence follows from the first one because under P4 the kernel has the form $y(t - \tau)$ where y is a distribution on $\mathscr{R}$. To prove the first sentence, let $v \in \mathscr{D}$ be such that $v(\tau) \equiv 0$ for $\tau < t_0$, and let $\phi \in \mathscr{D}$ be such that supp $\phi \subset (-\infty, t_0)$. Then, the support of $v(\tau)\phi(t)$ is contained in the half-plane $\{(t, \tau) : t < \tau\}$. Moreover,

$$\langle \mathfrak{N}v, \phi \rangle = \langle y \cdot v, \phi \rangle = \langle y(t, \tau), v(\tau)\phi(t) \rangle$$

The only way that the right-hand side can equal zero for every such v and ϕ is that supp $y(t, \tau) \subset \Lambda$. This proves the "only if" part of the first sentence.

This also proves that $u(t) = (\mathfrak{N}v)(t) = 0$ distributionally for $t < t_0$ whenever supp $y(t, \tau) \subset \Lambda$, $v \in \mathscr{D}$, and $v(t) \equiv 0$ on $-\infty < t < t_0$. We shall show that the same result holds when v is any distribution in the domain of $\mathfrak{N}$ that is distributionally equal to zero on $-\infty < t < t_0$. Indeed, by the way $\mathfrak{N}$ was extended, v is a member of some space of distributions $\mathscr{A}'$ on which $\mathfrak{N}$ is a continuous linear mapping into $\mathscr{D}'$ and in which $\mathscr{D}$ is dense. Given any $\epsilon > 0$, choose a sequence $\{v_n\}$ with $v_n \in \mathscr{D}$ such that $v_n \to v$ in $\mathscr{A}'$ as $n \to \infty$ and supp $v_n \subset (t_0 - \epsilon, \infty)$ for all n.

That the supports of the v_n can be so restricted is a consequence of condition II above in the following way. Let $\{f_n\}$ be a sequence of elements in $\mathscr{D}$ with $f_n \to v$ in $\mathscr{A}'$ as $n \to \infty$. Also, let λ be a smooth function identically equal to 1 on a neighborhood of $[t_0, \infty)$ and identically equal to zero on $(-\infty, t_0)$. Then, for any $\theta \in \mathscr{A}$, $\lambda\theta$ is also in $\mathscr{A}$ according to condition II, and in addition

$$\langle \lambda f_n, \theta \rangle = \langle f_n, \lambda\theta \rangle \to \langle v, \lambda\theta \rangle = \langle v, \theta \rangle.$$

Upon setting $v_n = \lambda f_n$, we obtain a sequence $\{v_n\}$ possessing the desired properties.

By a previous result, $u = \mathfrak{N}v_n = 0$ distributionally on $-\infty < t < t_0 - \epsilon$. But, $u_n \to u = \mathfrak{N}v$ in $\mathscr{D}'$ since $\mathfrak{N}$ is continuous from $\mathscr{A}'$ into $\mathscr{D}'$. Therefore,

$u = 0$ distributionally on $-\infty < t < t_0 - \epsilon$. Since $\epsilon > 0$ was arbitrary, this is true on $-\infty < t < t_0$.

The "if" part of the first sentence now follows from the linearity of N.

Q.E.D.

Theorem 3 and its proof show that, if $\mathfrak{N}$ satisfies P1, P2, and P3 and in addition is causal on the subset $\mathscr{D}$ of its domain, then it must be causal on all of its domain. Thus, we are lead to a fifth postulate:

P5*. *$\mathfrak{N}$ is causal on $\mathscr{D}$.*

We shall refer to the five postulates stated so far as the P^* postulates. They are appropriate for single-valued linear systems that may be active. The identity operator is an example of a system satisfying the P^* postulates; this verifies their consistency. (In fact, the consistency of any of the other sets of postulates appearing subsequently is also established in this way by the identity operator.)

We mention in passing that any system that satisfies the P^* postulates can be characterized in the frequency domain by the fact that its system function, when regularized in a certain way, becomes an entire function satisfying certain growth conditions [15].

The next postulate coupled with P1, P2, and P3 implies that $\mathfrak{N}$ is causal (see Theorem 4 below), and hence we number it as a replacement for P5*.

P5. *$\mathfrak{N}$ is passive on $\mathscr{D}$.*

This means that, for every $v \in \mathscr{D}$, $u = \mathfrak{N}v$ is locally integrable (i.e., Lebesgue integrable on every bounded interval), and in addition, for every real finite number t, we have that

$$\int_{-\infty}^{t} v(x)\, u(x)\, dx \geqslant 0. \tag{1}$$

Note that, if $\mathfrak{N}$ satisfies P1 through P4, then, for any $v \in \mathscr{D}$, $u = y * v$ is smooth and therefore locally integrable.

By the "P postulates" we mean the set P1, P2, P3, P4, and P5.

An obvious result that we shall need later on is

LEMMA 1. *If $\mathfrak{N}$ is a causal mapping of $\mathscr{D}_R$ into $\mathscr{D}'$ and is passive on $\mathscr{D}$, then, for all $v \in \mathscr{D}_R$, $u = \mathfrak{N}v$ is locally integrable and* (1) *holds for all $t < \infty$ (i.e., $\mathfrak{N}$ is passive on $\mathscr{D}_R$ as well).*

A remarkable fact discovered by Youla, Castriota, and Carlin [3] states essentially that linearity and passivity imply causality. For one-ports possessing kernel representations, this fact can be stated as follows.

THEOREM 4. *If $\mathfrak{N}$ satisfies* P1, P2, P3, *and* P5, *then $\mathfrak{N}$ is causal.*

For a proof, see [11; pp. 301-303]. (That proof requires now a modification to make it applicable to kernel representations; the arguments needed are given in the proof of Theorem 3 in this paper.)

Theorem 4 shows that the P postulates imply the P^* postulates. However, the converse is not true, as is indicated by the example $\mathfrak{N} = -\delta *$.

The next theorem characterizes in six different ways one-ports that satisfy the P postulates.

THEOREM 5. *If $\mathfrak{N}$ satisfies the P postulates, then $y = \mathfrak{N}\delta$ (the extended $\mathfrak{N}$ is understood here) satisfies the following six equivalent conditions. Conversely, if $y \in \mathscr{D}'$ satisfies any one of the following conditions, then $\mathfrak{N} = y*$ satisfies the P postulates.*

1. *The Laplace transform Y of y is a positive-real function.*
2. *y has the representation:*

$$y(t) = \alpha\delta^{(1)}(t) + 1_+(t)\int_{-\infty}^{\infty}(1+\eta^2)\cos\eta t \, dH(\eta), \tag{2}$$

where α is a real nonnegative number, $H(\eta)$ is a real nondecreasing bounded function on $-\infty < \eta < \infty$, and

$$1_+(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2} & t = 0 \\ 1 & t > 0. \end{cases}$$

3. *y also has the representation:*

$$y(t) = \alpha\delta^{(1)}(t) + \beta 1_+(t) + p(t)\, 1_+(t) + \frac{d^2}{dt^2}\{1_+(t)\,[p(0) - p(t)]\}. \tag{3}$$

Here, α and β are real nonnegative numbers, and

$$p(t) = \int_{-\infty}^{\infty} \cos\eta t \, dM(\eta), \tag{4}$$

where $M(\eta)$ is a real nondecreasing bounded function that is continuous at the origin.

4. *$y = \alpha\delta^{(1)} + y_0$ where α is a real nonnegative number, and y_0 is a member of $\mathscr{D}'_+$ and a distribution of zero C-order; moreover, the even part of y, namely;*

$$y_e(t) = \tfrac{1}{2}\,[y(t) + y(-t)]$$

is a nonnegative-definite distribution.

5. *$y \in \mathscr{D}'_+$, and, for every real nonnegative number c and every $\phi \in \mathscr{D}$,*

$$\langle y(t)\, e^{-ct}, \phi(t) * \phi(-t)\rangle \geqslant 0. \tag{5}$$

6. *Let $\tilde{y}$ denote the Fourier transform of y. Then,*

$$\tilde{y}(\omega) = i\omega \left[\alpha + \int_{-\infty}^{\infty} dH(\eta)\right] + (1 + \omega^2) \left[\pi H^{(1)}(\omega) - iH^{(1)}(\omega) * Pv \frac{1}{\omega}\right], \tag{6}$$

where α and H are restricted as in condition 2 and $Pv\ 1/\omega$ is the standard pseudo-function arising from Cauchy's principal value.

REMARKS. Let us make some explanations about each of these conditions.

CONDITION 1. A positive-real function $W(p)$ is a complex-valued function that is defined on the open right-half plane $\{p : \operatorname{Re} p > 0\}$ and satisfies there the following conditions:

(i) $W(p)$ is analytic.

(ii) $W(p)$ is real whenever p is real.

(iii) $\operatorname{Re} W(p) \geqslant 0$.

For a proof of this part of the theorem, see [11; Chapter 10].

CONDITION 2. The integral in (2) must be interpreted in the generalized sense since it doesn't converge in general in the conventional sense. In particular, the second term on the right-hand side of (2) is a distribution of slow growth and the value that it assigns to any $\phi \in \mathscr{S}$ can be shown to be equal to

$$\int_0^{\infty} dt \int_{-\infty}^{\infty} \phi(t) \cos \eta t \, dH(\eta) + \int_0^{\infty} dt \int_{-\infty}^{\infty} \phi^{(2)}(t) (1 - \cos \eta t) \, dH(\eta)$$

Again, see [11; Chapter 10].

CONDITION 3. This representation is due to König and Meixner [2]. Generalized differentiation is understood in (3), as well as in (6).

CONDITION 4. This representation is proven in [16]. By the C-order of a distribution we mean the least nonnegative integer r for which the $(r + 2)$th-order primitives of y_0 are continuous functions. Also, a distribution f is said to be nonnegative-definite if, for every testing function ϕ in $\mathscr{D}$, $\langle f, \phi * \check{\phi} \rangle \geqslant 0$ where $\check{\phi}(t) = \phi(-t)$.

CONDITION 5. This was established by Wohlers and Beltrami [5; p. 168].

CONDITION 6. This condition is also due to Beltrami and Wohlers [10; pp. 86-89]. It can be viewed as a generalized Kronig-Kramers-Bode equation characterizing positive-real functions. These authors also show that $\tilde{y}(\omega)$ is the limit as $\sigma \to 0+$ of the Laplace transform $Y(\sigma + i\omega)$ of y in the sense of convergence in the space $\mathscr{S}'$.

(Before leaving this section, it is worth pointing out a remarkable result due to H. König [19]. It states that a linear time-invariant passive operator is continuous, where now some of these properties of the operator are defined differently than they are in the present paper).

III. The Scattering Formulism

We turn now to the scattering formulism for $\mathfrak{N}$. This is obtained by defining two new variables as follows:

$$a = \tfrac{1}{2}(v + u) \tag{7}$$

$$b = \tfrac{1}{2}(v - u) \tag{8}$$

a and b can be physically interpreted as incident and reflected waves. If $\mathfrak{N}$ satisfies P1, P2, P3, and P4, then $u = \mathfrak{N}v = y * v$, and

$$a = \tfrac{1}{2}(\delta + y) * v,$$

$$b = \tfrac{1}{2}(\delta - y) * v.$$

If, in addition, $\mathfrak{N}$ satisfies P5, then $\delta + y$ possesses an inverse in the convolution algebra $\mathscr{D}'_R$. That is, there exists a unique member of $\mathscr{D}'_R$, which we denote by $(\delta + y)^{*-1}$, such that

$$(\delta + y)^{*-1} * (\delta + y) = \delta.$$

Indeed, the Laplace transform of $\delta + y$ is $1 + Y(p)$, which is positive-real. Consequently, $[1 + Y(p)]^{-1}$ is also positive-real and therefore the Laplace transform of a unique member of $\mathscr{D}'_R$, namely, $(\delta + y)^{*-1}$.

These results show that, if $\mathfrak{N}$ satisfies the P postulates, then, for every $v \in \mathscr{D}'_R$ and $u = \mathfrak{N}v$ and for a and b given by (7) and (8), we have

$$b = s * a, \tag{9}$$

where

$$s = (\delta - y) * (\delta + y)^{*-1}. \tag{10}$$

Equation (9) is the scattering representation, and s is called the scattering parameter for $\mathfrak{N}$. Wohlers and Beltrami [5; p. 168] have pointed out that the representation (9) contains all $a \in \mathscr{D}'_R$ in its domain because for every $a \in \mathscr{D}'_R$ there exists a unique $v \in \mathscr{D}'_R$ such that $2a = v + \mathfrak{N}v$. We can establish this by again taking Laplace transforms and invoking the positive-reality of $1 + Y(p)$ as above.

Actually, we can arrive at a scattering formulism in another way and indeed obtain greater generality if we employ a different set of postulates. The postulates, which we now present, are a modified form of those suggested by Youla, Castriota, and Carlin [3] and are very similar to those used by Newcomb [17].

Q1. *$\mathfrak{N}$ is a multivalued mapping of a subset of $\mathscr{D}'$ into $\mathscr{D}'$.*

(As was mentioned above, our use of the word "multivalued" allows "single-valued" as a special case.)

We have not as yet specified the domain of $\mathfrak{N}$. A rather restricted domain for $\mathfrak{N}$ is implied by the next postulate Q2.

In contrast to postulate P1, Q1 allows $\mathfrak{N}$ to have more than one response $u \in \mathscr{D}'$ to any given $v \in \mathscr{D}'$ in the domain of $\mathfrak{N}$. An example of a multivalued $\mathfrak{N}$ that is not single-valued is the short circuit. Its domain contains only one voltage, the zero distribution; but, it can respond with any current in $\mathscr{D}'$. The short circuit was prohibited under postulate P1.

Q2. *$\mathfrak{N}$ is uniquely solvable from $\mathscr{D}$ into $\mathscr{D}'_R$ with solutions in $\mathscr{D}_R$.*

By this we mean that, given any $e \in \mathscr{D}$, there exists a unique $v \in \mathscr{D}'_R$ which satisfies

$$e = v + \mathfrak{N}v$$

and that this v will be a member of $\mathscr{D}_R$. If we view the system as a one-port, the last equation signifies that a unit resistor has been connected in series with the port and that e is the voltage applied to the resulting series circuit; this is illustrated in Fig. 2.

Postulate Q2 implicitly specifies a rather restricted domain for $\mathfrak{N}$ and also restricts the range values. In particular, let $C(v, u)$ denote the set of all pairs v, u appearing as solutions of the equations:

$$e = v + \mathfrak{N}v, \qquad u = \mathfrak{N}v$$

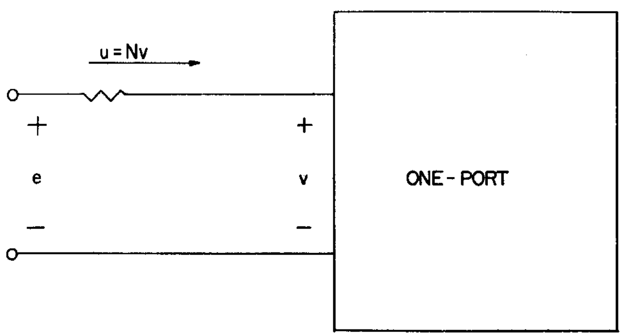

FIG. 2.

as e traverses $\mathscr{D}$. In symbols,

$$C(v, u) = \{v, u : e = v + \mathfrak{N}v, u = \mathfrak{N}v, e \in \mathscr{D}\}.$$

By Q2, every such v, u is a pair of elements in $\mathscr{D}_R$. (When $\mathfrak{N}$ is the ideal short circuit, v is always the zero distribution, whereas u can be any member of $\mathscr{D}$ because in this case $u = e \in \mathscr{D}$.) We will assume for the moment that the domain of $\mathfrak{N}$ is restricted to the set of v's appearing in $C(v, u)$ and that the range of $\mathfrak{N}$ is also restricted in accordance with $C(v, u)$. Subsequently, the domain and range of $\mathfrak{N}$ will be extended by means of a kernel representation for the socalled "augmented operator" $\mathfrak{N}_a$.

As is suggested by Fig. 2 and Q2, this new operator $\mathfrak{N}_a$ is defined on any $e \in \mathscr{D}$ by $u = \mathfrak{N}_a e$ where $u = \mathfrak{N}v = e - v$, $v \in \mathscr{D}_R$. Moreover, $\mathfrak{N}_a$ turns out to be a single-valued mapping of $\mathscr{D}$ into $\mathscr{D}'_R$ with its range in $\mathscr{D}_R$. Indeed, by Q2, given any $e \in \mathscr{D}$, v and therefore $u = \mathfrak{N}v = e - v$ are unique members of $\mathscr{D}'_R$ that are both in $\mathscr{D}_R$, which verifies our assertion.

Because $\mathfrak{N}_a$ is single-valued from $\mathscr{D}$ into $\mathscr{D}'_R$, we can define it as a single-valued operator from $\mathscr{D}$ into $\mathscr{D}'$ simply by prohibiting $\mathfrak{N}_a e$ (for any $e \in \mathscr{D}$) from having any other values in $\mathscr{D}'$. Henceforth, we adopt this convention. The physical significance of this is that we are requiring that the system $\mathfrak{N}_a$ be initially at rest. For example, consider the network of Fig. 3. We have that

$$e = v + u = v + v^{(1)}.$$

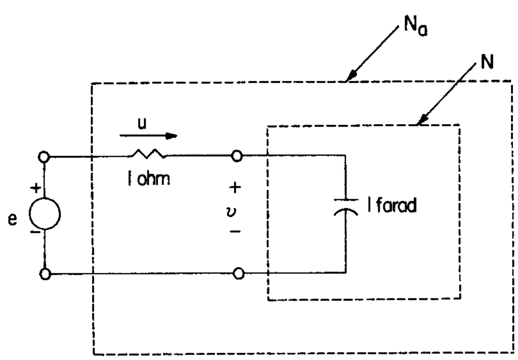

FIG. 3.

Every free oscillation of this network is of the form: $e = 0$, $v = ce^{-t}$, where c is any constant. It follows that $\mathfrak{N}$ satisfies Q2 because the only solution of $0 = v + v^{(1)}$ in $\mathscr{D}'_R$ is the zero distribution (i.e., if $c \neq 0$, then $ce^{-t} \notin \mathscr{D}'_R$). On the other hand, $0 = v + v^{(1)}$ has an infinity of solutions in $\mathscr{D}'$, namely, $v = ce^{-t}$ with c arbitrary. Hence, $\mathfrak{N}_a$ is not single-valued as a mapping of $\mathscr{D}$ into $\mathscr{D}'$. But, if we add the additional condition that $\mathfrak{N}_a$ be initially at rest (so that $c = 0$), then, $\mathfrak{N}_a$ becomes a single-valued mapping of $\mathscr{D}$ into $\mathscr{D}'$.

Q3. *$\mathfrak{N}$ is linear on $C(v, u)$.*

In other words, if v_1, u_1 and v_2, u_2 are two pairs in $C(v, u)$ and if α and β are real numbers, then $\alpha v_1 + \beta v_2$, $\alpha u_1 + \beta u_2$ is another pair in $C(v, u)$.

It readily follows from Q2 and Q3 that $\mathfrak{N}_a$ is linear on $\mathscr{D}$; that is, $\mathfrak{N}_a$ satisfies $P2$, as well as P1.

Q4. *$\mathfrak{N}$ is passive on $C(v, u)$.*

This means that, for every pair v, u in $C(v, u)$, (1) holds for all finite t; that is,

$$\int_{-\infty}^{t} v(x)\, u(x)\, dx \geqslant 0 \tag{1}$$

A useful result can now be established: Under the preceding four postulates, $\mathfrak{N}_a$ is a continuous mapping of $\mathscr{D}$ into $\mathscr{D}'$. Indeed, let n be any positive integer. For any $e_n \in \mathscr{D}$, $e_n = v_n + \mathfrak{N}v_n$, and $u_n = \mathfrak{N}v_n$, Q2 implies that both v_n and u_n are members of $\mathscr{D}_R$. Hence, all the integrals in the following equation exist for each finite value of t.

$$\int_{-\infty}^{t} e_n^2\, dx = \int_{-\infty}^{t} v_n^2\, dx + \int_{-\infty}^{t} u_n^2\, dx + 2\int_{-\infty}^{t} v_n u_n\, dx. \tag{11}$$

Furthermore, the last term is nonnegative according to Q4, and so are all the others. Assume now that $e_n \to 0$ in $\mathscr{D}$ as $n \to \infty$. The left-hand side of (11) tends to zero as $n \to \infty$. Therefore, each term on the right-hand side does too. Next, let $\phi \in \mathscr{D}$ with supp $\phi \subset [\tau, \rho]$, $-\infty < \tau < \rho < \infty$. Then,

$$|\langle u_n, \phi\rangle| = \left|\int_{\tau}^{\rho} u_n\phi\, dx\right| \leqslant \left[\int_{-\infty}^{\rho} u_n^2\, dx \int_{\tau}^{\rho} \phi^2\, dx\right]^{1/2} \to 0 \qquad n \to \infty,$$

which verifies that $\mathfrak{N}_a$ satisfies P3.

Moreover, when $\mathfrak{N}$ satisfies Q1 through Q4, $\mathfrak{N}_a$ also satisfies P5 since for any $e \in \mathscr{D}$, we have

$$\int_{-\infty}^{t} eu\, dx = \int_{-\infty}^{t} vu\, dx + \int_{-\infty}^{t} u^2\, dx \geqslant 0.$$

We have just seen that the augmentation technique allows us to start with an operator $\mathfrak{N}$ which is neither single-valued now continuous and to obtain from it an operator $\mathfrak{N}_a$ which is. Moreover, even though $\mathfrak{N}$ may not contain all of $\mathscr{D}$ in its domain, $\mathfrak{N}_a$ will. Furthermore, we have

THEOREM 6. *If $\mathfrak{N}$ satisfies* Q1, Q2, Q3, *and* Q4, *then $\mathfrak{N}_a$ satisfies* P1, P2, P3, *and* P5. *Moreover, $\mathfrak{N}_a$ is causal and has a kernel representation $\mathfrak{N}_a = y_a \cdot$ where* $y_a = y_a(t, \tau) \in \mathscr{D}' \mid_{t,\tau}$.

Proof. We have already established the first statement; we remind the reader that the satisfaction of P1 by $\mathfrak{N}_a$ is a consequence of our convention concerning the single-valued definition of the range of $\mathfrak{N}_a$ in $\mathscr{D}'$. The second statement follows from Theorems 1 and 4.

y_a is called the augmented admittance of $\mathfrak{N}$.

As was done in the admittance formulism, we can now extend $\mathfrak{N}_a$ via the kernel representation $\mathfrak{N}_a = y_a \cdot$ onto every space of distributions $\mathscr{A}'$ that satisfies the conditions stated in the preceding section. This extension is unique in the aforementioned sense, and we henceforth assume that it has been made. This automatically extends the operator $\mathfrak{N}$, under the constraint that it has been augmented, into a mapping from a subset of $\mathscr{D}'$ into $\mathscr{D}'$ according to

$$u = \mathfrak{N}(e - u)$$

where $u = \mathfrak{N}_a e = y_a \cdot e$.

We add one more postulate:

Q5. *$\mathfrak{N}$ is time-invariant on $C(v, u)$.*

That is, if v, u is a pair in $C(v, u)$ and if σ_x is the shifting operator as before, then $\sigma_x v, \sigma_x u$ is also a pair in $C(v, u)$, whatever be the value of x. Since $u = \mathfrak{N}v$, this implies that $\sigma_x \mathfrak{N} v = \mathfrak{N} \sigma_x v$ for every pair v, $\mathfrak{N}v$ in $C(v, u)$.

If $\mathfrak{N}$ satisfies Q1, Q2, and Q5, then $\mathfrak{N}_a$ satisfies P4 (i.e., $\mathfrak{N}_a$ is time-invariant on $\mathscr{D}$). To show this, we start from $e = v + \mathfrak{N}v$, $e \in \mathscr{D}$. Thus,

$$\sigma_x e = \sigma_x v + \sigma_x \mathfrak{N} v = \sigma_x v + \mathfrak{N} \sigma_x v,$$

and this decomposition is unique by Q2. Therefore, for all $e \in \mathscr{D}$,

$$\sigma_x \mathfrak{N}_a e = \sigma_x \mathfrak{N} v = \mathfrak{N} \sigma_x v = \mathfrak{N}_a \sigma_x e,$$

which verifies our assertion.

Henceforth, when referring to the "Q postulates," we mean all the postulates from Q1 through Q5.

Under the additional postulate Q5, Theorem 6 becomes

Theorem 7. *If $\mathfrak{N}$ satisfies the Q postulates, then $\mathfrak{N}_a$ satisfies the P postulates. Moreover, $\mathfrak{N}_a$ is causal and has a convolution representation $\mathfrak{N}_a = y_a *$ where $y_a = y_a(t) \in \mathscr{D}' |_t$.*

The scattering formulism for an operator $\mathfrak{N}$ that satisfies Q1, Q2, Q3, and Q4 is derived as follows: We have that

$$u = y_a \cdot e = y_a \cdot (v + u).$$

Therefore,

$$y_a \cdot v = (\delta - y_a) \cdot u.$$

[Here, δ denotes the kernel $\delta(t - \tau)$, so that $\delta \cdot u = u$.] Now, substitute the quantities: $v = a + b$, $u = a - b$. This yields

$$b = s \cdot a, \tag{12}$$

where

$$a = \tfrac{1}{2}(v + u), \qquad b = \tfrac{1}{2}(v - u)$$

and s is the kernel:

$$s = \delta - 2y_a . \tag{13}$$

The time-variable scattering parameter for $\mathfrak{N}$ is s.

THEOREM 8. *If* $\mathfrak{N}$ *satisfies* Q1,Q2, Q3, *and* Q4, *then* $\mathfrak{N}$ *has the scattering formulism* (12) *and* (13), *where* y_a *is the augmented admittance of* $\mathfrak{N}$. *Moreover,* supp $s(t, \tau) \subset \{(t, \tau) : t \geqslant \tau\}$. *If, in addition,* $\mathfrak{N}$ *satisfies* Q5, *then*

$$b = s * a \tag{14}$$

where $s = s(t) \in \mathscr{D}'|_t$ *and* supp $s \subset [0, \infty)$.

PROOF. Invoke Theorems 3, 6, and 7.

As one of our principal conclusions, we have that the P postulates imply the Q postulates. Indeed, assume that $\mathfrak{N}$ satisfies the P postulates. Then, $\mathfrak{N}$ obviously satisfies Q1. That it satisfies Q2 can be established through an argument due to Wohlers and Beltrami [5; p. 168]. Q3 and Q5 are also clearly satisfied since $u = \mathfrak{N}v = y * v$ where $y \in \mathscr{D}'_+$. This also shows that $\mathfrak{N}$ is a causal mapping of $\mathscr{D}_R$ into $\mathscr{D}_R$, and it now follows from Lemma 1 and P5 that $\mathfrak{N}$ satisfies Q4.

Turning now to active systems, we obtain a suitable set of Q-type postulates by replacing the postulate Q4 by the postulates Q4* and Q6*. We shall refer to the collection: Q1, Q2, Q3, Q4*, Q5, and Q6* as the Q* postulates.

Q4*. $\mathfrak{N}$ *is continuously solvable for* $C(v, u)$.

By this we mean that $\mathfrak{N}$ fulfills Q2 and that the following statement is true: Whenever the sequence $\{v_n , u_n\}_{n=1}^{\infty}$ of pairs in $C(v, u)$ is such that $v_n + u_n$ converges in $\mathscr{D}$ to, say, the limit e, it is also true that v_n converges in $\mathscr{D}'$ to v where $e = v + \mathfrak{N}v$ (or, equivalently, that u_n converges in $\mathscr{D}'$ to $u = \mathfrak{N}v$).

Q4* means that $\mathfrak{N}_a$ is continuous from $\mathscr{D}$ into $\mathscr{D}'$. Newcomb [17; p. 24] refers to those $\mathfrak{N}$ that satisfy this kind of property as being "completely solvable." We prefer the phrase "continuously solvable" as being more descriptive.

Q6*. $\mathfrak{N}$ *has a causal augmentation for* $C(v, u)$.

This is taken to mean that, whatever be the choice of the real number t_0, if the pairs v_1, u_1 and v_2, u_2 in $C(v, u)$ are such that $v_1 + u_1 = v_2 + u_2$ for

$t < t_0$, then $u_1 = u_2$ for $t < t_0$. In other words, we are simply assuming that $\mathfrak{N}_a$ is causal on $\mathscr{D}$.

We have already seen that the Q postulates imply Q4* (see the discussion after Q4) as well as Q6* (see Theorem 6). Thus, the assertion $Q \Rightarrow Q^*$ in Fig. 1 is valid.

Moreover, we can revise Theorem 6 as follows:

Corollary 6a. *Theorem 6 remains true when* Q4 *is replaced by* Q4* *and* Q6* *together, and in addition* P5 *is replaced by* P5*.

Indeed, that $\mathfrak{N}_a$ satisfies P1 and P2 follows as before from Q1, Q2, and Q3. Q4* states that $\mathfrak{N}_a$ satisfies P3, and Q6* states that $\mathfrak{N}_a$ satisfies P5*. By Theorem 1, $\mathfrak{N}_a$ has a kernel representation. Also, by P5* and the argument in the first paragraph of the proof of Theorem 3, supp $y_a(t, \tau) \subset \{(t, \tau) : t \geqslant \tau\}$ so that $\mathfrak{N}_a$ is causal.

As an immediate consequence, we can revise Theorems 7 and 8 as follows:

Corollary 7a. *Theorem 7 remains true when the symbols Q and P are replaced by Q^* and P^*, respectively.*

Corollary 8a. *Theorem 8 remains true when* Q4 *is replaced by* Q4* *and* Q6* *together.*

Let us now list some of the properties of the operator $\mathfrak{M} = s*$ assuming that $\mathfrak{N}$ satisfies the Q postulates. In view of Theorem 8 and the properties of distributional convolution, we have the following:

S1. *$\mathfrak{M}$ is a single-valued mapping of $\mathscr{D}$ into $\mathscr{D}'$.*

S2. *$\mathfrak{M}$ is linear on $\mathscr{D}$.*

S3. *$\mathfrak{M}$ is continuous from $\mathscr{D}$ into $\mathscr{D}'$.*

S4. *$\mathfrak{M}$ is time-invariant on $\mathscr{D}$.*

S5. *$\mathfrak{M}$ is causal on $\mathscr{D}$.*

S6. *$\mathfrak{M}$ is weakly passive on $\mathscr{D}$.*

Property S5 means that the causality property defined in the preceding section holds for $\mathfrak{M}$ whenever the elements a in the domain of $\mathfrak{M}$ are restricted to $\mathscr{D}$.

Property S6 means that, given any $a \in \mathscr{D}$, the integral:

$$\int_{-\infty}^{\infty} vu\, dt = \int_{-\infty}^{\infty} (a^2 - b^2)\, dt \tag{15}$$

exists and is nonnegative. That $\mathfrak{M} = s^*$ truly satisfies S6 can be shown as follows. For any $a \in \mathscr{D}$, we have that $b = \mathfrak{M}a = s * a \in \mathscr{D}_R$ since $s \in \mathscr{D}'_+$.

Hence, $v = a + b \in \mathscr{D}_R$ and $u = a - b \in \mathscr{D}_R$. Also, v, u is a pair in $C(v, u)$ since $v + u = 2a \in \mathscr{D}$. By Q4 and the fact that $vu = a^2 - b^2$, we have that for all finite t

$$\int_{-\infty}^{t} a^2\, dx \geqslant \int_{-\infty}^{t} b^2\, dx \geqslant 0.$$

The left-hand side converges as $t \to \infty$, and therefore the middle term does too. Hence, (15) exists and is nonnegative.

Wohlers and Beltrami [5] have proposed that S-type properties be used as postulates on the operator $\mathfrak{M}$ relating the reflected wave b to the incident wave a and have shown that a theory for linear systems can be derived therefrom. In the present case, S1 through S4 read precisely the same as P1 through P4. Therefore, S1 through S3 imply that $\mathfrak{M}$ has a kernel representation $\mathfrak{M} = s\ \cdot$. Moreover, S1 through S4 imply that $\mathfrak{M}$ has a convolution representation $\mathfrak{M} = s*$, as well as a system function $\tilde{s}$. The additional postulate S5 and the argument in the first paragraph of the proof of Theorem 3 show that s satisfies the same restrictions on its support as does y in that theorem. We can now extend $\mathfrak{M}$ onto larger spaces of distributions via those representations and this extension must be unique as explained before.

We refer to S1 through S5 as the S^* postulates and to S1 through S6 as the S postulates. The S^* (and S) postulates are suitable for active (respectively, passive) systems that may be multivalued in the sense that more than one $u = \frac{1}{2}(a - b)$ may correspond to a given $v = \frac{1}{2}(a + b)$. The frequency-domain characterization [15] for $\mathfrak{N}$ under the P^* postulates holds also for $\mathfrak{M}$ under the S^* postulates.

Theorem 3 and its proof show that, if $\mathfrak{M}$ satisfies S1, S2, S3, and S4 and is also passive in the sense that

$$\int_{-\infty}^{t} (a^2 - b^2)\, dx \geqslant 0 \qquad a \in \mathscr{D}, \qquad -\infty < t < \infty, \tag{16}$$

then $\mathfrak{M}$ is causal. It is also weakly passive on $\mathscr{D}$. Conversely, Wohlers and Beltrami [5; p. 167] have shown that the S postulates imply that $\mathfrak{M}$ is passive in the sense of (16). Thus, assuming that S1 through S4 are satisfied, we can conclude that postulates S5 and S6 are equivalent to the single assumption that $\mathfrak{M}$ is passive in the sense of (16). However, using S5 and S6 is preferable since causality and weak passivity are independent assumptions. Indeed, that S5 does not imply S6 follows from the example $s = 2\delta$. That S6 does not imply S5 follows from the example $s(t) = \delta(t + c)$, where c is a positive number.

By itself, postulate S6 allows one-ports that up to finite instances of time have emitted more energy than they have received but will ultimately (i.e., as $t \to \infty$) absorb at least as much energy as they have emitted.

It is worth noting that, when $\mathfrak{N}$ satisfies P1 through P4, the nonnegativity of (15) for all $v \in \mathscr{D}$ and the causality of $\mathfrak{N}$ do not imply the passivity of $\mathfrak{N}$ according to P5. In regard to this, consider the example $\mathfrak{N} = -\delta^{(1)} *$.

It may also be argued, especially when the one-port is a part of a wave propagation system, that the basic physical variables are the incident wave a and the reflected wave b. If this point of view is accepted, then the axioms that ought to be used are the S^* or S postulates and not the P^*, P, Q, or Q^* postulates.

Operators $\mathfrak{M}$ that satisfy the S postulates can be characterized as follows:

THEOREM 9. *If $\mathfrak{M}$ satisfies the S postulates, then $s = \mathfrak{M}\delta$ (as always, the extended $\mathfrak{M}$ is understood) satisfies the following three equivalent conditions. Conversely, if $s \in \mathscr{D}'$ satisfies any one of the following conditions, then $\mathfrak{M} = s *$ satisfies the S postulates.*

1. *The Laplace transform $\mathfrak{S}(p)$ of s is a bounded-real function. [$\mathfrak{S}(p)$ is called bounded-real if on the open right-half p plane $\{p : \operatorname{Re} p > 0\}$ we have that $\mathfrak{S}(p)$ is analytic, $\mathfrak{S}(p)$ is real whenever p is real, and $|\mathfrak{S}(p)| \leqslant 1$.]*

2. *$s \in \mathscr{D}'_+ \cap \mathscr{D}'_{L_2}$, and, for every $\phi \in \mathscr{D}$,*

$$\langle \delta - s * \check{s}, \phi * \check{\phi} \rangle \geqslant 0,$$

where $\check{f}(t) = f(-t)$. (For a definition of $\mathscr{D}'_{L_2}$, see [12; Vol. II, pp. 55-56].)

3. *Let $\tilde{s}$ denote the Fourier transform of s. Then, $\tilde{s}(\omega)$ is a conventional function such that $|\tilde{s}(\omega)| \leqslant 1$ almost everywhere on $-\infty < \omega < \infty$, $\tilde{s}(-\omega)$ is equal to the complex conjugate of $\tilde{s}(\omega)$, and*

$$\tilde{s}^{(1)}(\omega) = \frac{1}{i\pi} \tilde{s}^{(1)}(\omega) * Pv \frac{1}{\omega},$$

where $Pv\ 1/\omega$ is again the standard pseudofunction arising from Cauchy's principal value.

This theorem is proven in [10; pp. 89-93] and [18]. The third condition presents another example of a generalized Kronig-Kramers-Bode equation.

We have seen that the P postulates imply the P^* postulates as well as the Q postulates, and that the Q postulates imply the S postulates, which in turn obviously imply the S^* postulates. We have also noted that the Q postulates imply the Q^* postulates. That the Q^* postulates imply the S^* postulates follows directly from Corollary 8a. To complete Fig. 1, we have to show that $S^* \Rightarrow Q^*$ and that $S \Rightarrow Q$.

Under the S^* postulates, we have that $b = s * a$ for any $a \in \mathscr{D}$ where $s \in \mathscr{D}'_+$. Therefore, corresponding to the input:

$$v = a + b = (\delta + s) * a \in \mathscr{D}_R, \tag{17}$$

we have the output:

$$\mathfrak{N}v = u = a - b = (\delta - s) * a \in \mathscr{D}_R . \tag{18}$$

So truly, the mapping $\mathfrak{N} : v \to u$ satisfies Q1.

Moreover, a degenerate case arises when $s = -\delta$. Under the restriction that $a = \frac{1}{2}(v + u) \in \mathscr{D}$, this requires that v be the zero element in $\mathscr{D}$, whereas u can be any member of $\mathscr{D}$. In this case, it is obvious that the Q^* (as well as the Q) postulates are satisfied. As was pointed out before, when $\mathfrak{N}$ is a one-port, this case corresponds to the short circuit.

For other $s \in \mathscr{D}'_+$ (i.e., for $s \neq -\delta$), we can show that $\mathfrak{N}$ satisfies the other Q^* postulates by using the equation:

$$v - u = s * (v + u) \qquad v + u \in \mathscr{D}. \tag{19}$$

For example, to demonstrate Q2, we first note that $e = v + \mathfrak{N}v$ is solvable for any $e \in \mathscr{D}$; its solution is given by (17) with $a = \frac{1}{2} e$. We want to show that it is uniquely solvable. Suppose that v_1 is another solution in $\mathscr{D}'_R$, and set $u_1 = e - v_1$. Then, $v_1 + u_1 = e = v + u$, so that, by (19), $v_1 - u_1 = v - u$. Consequently, $v_1 = v$ and $u_1 = u$.

It is equally straightforward to prove from (19) that $\mathfrak{N}$ satisfies the other Q^* postulates whenever $\mathfrak{N}$ satisfies the S^* postulates and $s \neq -\delta$. We omit the details.

Finally, consider $S \Rightarrow Q$. Condition 1 of Theorem 9 and a Theorem of Youla, Castriota, and Carlin [3; pp. 116-117] assert that the S postulates imply their form of the Q postulates. Under the present formulation we can get from the S postulates to the Q postulates in the following way.

We mentioned before that the Q postulates are obviously satisfied in the degenerate case where $s = -\delta$. So, assume that $s \neq -\delta$ and that the S postulates are satisfied. The Laplace transform $Y(p)$ of the unit impulse response y of the operator $\mathfrak{N} : v \to u$ exists and is equal to

$$Y(p) = \frac{1 - \mathfrak{S}(p)}{1 + \mathfrak{S}(p)} \tag{20}$$

for at least $\operatorname{Re} p > 0$. Indeed, by Theorem 9, $\mathfrak{S}(p)$ is bounded-real; also, $\mathfrak{S}(p) \not\equiv -1$ for $\operatorname{Re} p > 0$. By the maximum-modulus theorem, $1 + \mathfrak{S}(p) \neq 0$ at every point p such that $\operatorname{Re} p > 0$. Since (20) maps the unit circle in the $\mathfrak{S}$-plane onto the right-half Y-plane, $Y(p)$ is positive-real and therefore a Laplace transform in the Schwartz sense. We can now verify our assertion by setting $v = \delta$, $u = y$ and taking the Laplace transform of (19). Moreover, Theorem 5 shows that $\mathfrak{N}$ satisfies the P postulates. Consequently, $\mathfrak{N}$ satisfies the Q postulates as well. Our proof of the assertion $S \Rightarrow Q$ is finished.

We can also conclude at this time that the only possible multivalued one-port satisfying the Q postulates is the short circuit. On the impedance basis, it would be the open circuit.

[In the case of n-ports ($n > 1$), however, there are many operators $\mathfrak{N}$ that exhibit multivaluedness and yet satisfy the n-port analogues to the Q and S postulates. When the n-port $\mathfrak{N}$ is treated on an admittance basis as above, it is multivalued when and only when the $n \times n$ matrix $1_n + \mathfrak{S}(p)$ is singular everywhere in the right-half plane $\operatorname{Re} p > 0$. Here, 1_n denotes the $n \times n$ unit matrix. On the impedance basis, $\mathfrak{N}$ is multivalued when and only when the $n \times n$ matrix $1_n - \mathfrak{S}(p)$ is singular everywhere in the half-plane $\operatorname{Re} p > 0$.]

Finally, it is worth noting that none of the following assertions are true.

$P^* \nRightarrow P$. Counterexample: $\mathfrak{N} = -\delta *$.

$Q^* \nRightarrow Q$. Counterexample: $\mathfrak{N} = -2\delta *$.

$S^* \nRightarrow S$. Counterexample: $\mathfrak{M} = 2\delta *$.

$Q^* \nRightarrow P^*$. Counterexample: $\mathfrak{N}$ = the short circuit.

$Q \nRightarrow P$. Counterexample: $\mathfrak{N}$ = the short circuit.

$S^* \nRightarrow P^*$. Counterexample: $\mathfrak{N}$ = the short circuit.

$P^* \nRightarrow Q^*$. Counterexample: $\mathfrak{N} = -\delta *$.

$P^* \nRightarrow S^*$. Counterexample: $\mathfrak{N} = -\delta *$.

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