On improper integrals and differential equations in ordered Banach spaces

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Abstract

In this paper we study the existence of improper integrals of vector-valued mappings. The so obtained results combined with fixed point results in partially ordered functions spaces are then applied to derive existence and comparison results for least and greatest solutions of initial- and boundary-value problems in ordered Banach spaces. The considered problems can be singular, functional, nonlocal, implicit and discontinuous. Concrete examples are also solved.

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1. Introduction

In this paper we shall first study the existence of improper integrals of a mapping $h$ from an open real interval $(a, b)$, $-\infty \leq a < b \leq \infty$, to an ordered Banach space $E$. We show, for instance, that if the order cone of $E$ is regular, an improper integral of $h$ exists if $h$ is strongly measurable and a.e. pointwise bounded from above and from below by strongly
measurable and locally Bochner integrable mappings from \((a, b)\) into \(E\) possessing the improper integrals in question.

The so obtained results and fixed point results for mappings in partially ordered function spaces are then applied to derive existence and comparison results for least and greatest solutions of first- and second-order initial value problems and second-order boundary value problems in an ordered Banach space \(E\) whose order cone is regular. The existence of local extremal solutions for corresponding problems is studied in [6] when \(E\) is a lattice-ordered Banach space. A novel feature in our study is that the right-hand sides of differential equations comprise locally integrable vector-valued functions possessing improper integrals. Similar problems containing improper integrals of real-valued functions are studied in [10].

The following special types are included in the considered problems:

- differential equations and initial/boundary conditions may be implicit;
- differential equations may be singular;
- both the differential equations and the initial or boundary conditions may depend functionally on the unknown function and/or on its derivatives;
- both the differential equations and the initial or boundary conditions may contain discontinuous nonlinearities;
- problems on infinite intervals;
- problems of random type.

When \(E\) is the sequence space \(c_0\) we obtain results for infinite systems of initial and boundary value problems, as shown in examples. Moreover, concrete finite systems are solved to illustrate the effects of improper integrals to solutions of such problems.

2. Preliminaries

Our first task in this section is to prove existence results for improper integrals of a mapping \(h : (a, b) \to E, \ -\infty \leq a < b \leq \infty\), where \(E = (E, \| \cdot \|, \leq)\) is an ordered Banach space whose order cone is regular. If \(h\) is strongly (Lebesgue) measurable and locally Bochner integrable, denote \(h \in L^1_{\text{loc}}((a, b), E)\). For the sake of completeness we shall define the improper integrals we are dealing with.

**Definition 2.1.** Given \(h \in L^1_{\text{loc}}((a, b), E)\) and \(c \in (a, b)\), we say that an improper integral \(\int_a^c h(s) \, ds\) exists if \(\lim_{x \downarrow a} \int_x^c h(s) \, ds\) exists in \(E\). Similarly, we say that an improper integral \(\int_b^c h(s) \, ds\) exists if \(\lim_{x \uparrow b} \int_x^c h(s) \, ds\) exists in \(E\).

The existence results proved in the next lemma for the above defined improper integrals are essential tools in our study of differential equations in ordered Banach spaces.

**Lemma 2.1.** Let \(h : (a, b) \to E\) be strongly measurable, \(h_{\pm} \in L^1_{\text{loc}}((a, b), E)\), and assume that \(h_-(s) \leq h(s) \leq h_+(s)\) for a.e. \(s \in (a, b)\). Then the following results hold.

(a) \(h\) is locally Bochner integrable, i.e. \(h \in L^1_{\text{loc}}((a, b), E)\).
(b) If \( \int_{a+}^{c} h_{\pm}(s) \, ds \) exists for some \( c \in (a, b) \), then \( \int_{a+}^{t} h(s) \, ds \) exists for all \( t \in (a, b) \).

(c) If \( \int_{c}^{b-} h_{\pm}(s) \, ds \) exists for some \( c \in (a, b) \), then \( \int_{t}^{b-} h(s) \, ds \) exists for all \( t \in (a, b) \).

**Proof.** (a) Since the order cone of \( E \) is regular and hence also normal, the norm of \( E \) is semimonotone, i.e. there exists such a positive constant \( M \) that

\[
0 \leq x \leq y \quad \text{in} \quad E \quad \text{implies} \quad \|x\| \leq M\|y\|.
\]

The assumption: \( h_{\pm}(s) \leq h(s) \leq h_{+}(s) \) for a.e. \( t \in (a, b) \) can be rewritten as

\[
0 \leq h(s) - h_{-}(s) \leq h_{+}(s) - h_{-}(s) \quad \text{for a.e.} \quad s \in (a, b).
\]

In view of this result and the semimonotonicity of the norm of \( E \) we obtain

\[
\|h(s) - h_{-}(s)\| \leq M\|h_{+}(s) - h_{-}(s)\| \quad \text{for a.e.} \quad s \in (a, b),
\]

whence

\[
\|h(s)\| \leq (M + 1)\|h_{-}(s)\| + M\|h_{+}(s)\| \quad \text{for a.e.} \quad s \in (a, b).
\]

This result, strong measurability of \( h \) and the assumption that \( h_{\pm} \in L^1_{\text{loc}}((a, b), E) \) imply that \( h \in L^1_{\text{loc}}((a, b), E) \).

(b) Assume that \( \int_{a+}^{c} h_{\pm}(s) \, ds \) exists for some \( c \in (a, b) \). Since \( h_{\pm} \leq h \leq h_{+} \), it follows from [8, Corollary 1.4.6] that

\[
\int_{\tau}^{c} h_{-}(s) \, ds \leq \int_{\tau}^{c} h(s) \, ds \leq \int_{\tau}^{c} h_{+}(s) \, ds \quad \text{whenever} \quad a < \tau < c.
\]

Choose a decreasing sequence \((\tau_n)\) from \((a, c)\) such that \( \tau_n \to a \) as \( n \to \infty \), and denote

\[
y_n = \int_{\tau_n}^{c} \left(h(s) - h_{-}(s)\right) \, ds, \quad n \in \mathbb{N}. \tag{2.1}
\]

The sequence \((y_n)\) is increasing by [8, Proposition 1.4.3]. Since \( \int_{a+}^{c} h_{\pm}(s) \, ds \) exist, then also \( \int_{a+}^{c} (h_{+}(s) - h_{-}(s)) \, ds \) exists, and

\[
0 \leq y_n \leq \int_{a+}^{c} (h_{+}(s) - h_{-}(s)) \, ds, \quad n \in \mathbb{N}.
\]

Since the order cone of \( E \) is regular, then \( y = \lim_{n \to \infty} y_n \) exists. If \( \tau \in (a, \tau_m) \), then

\[
0 \leq \int_{\tau}^{\tau_m} (h(s) - h_{-}(s)) \, ds \leq \int_{\tau}^{\tau_m} (h_{+}(s) - h_{-}(s)) \, ds \leq \int_{a+}^{\tau_m} (h_{+}(s) - h_{-}(s)) \, ds,
\]

so that

\[
\left\| \int_{\tau}^{\tau_m} (h(s) - h_{-}(s)) \, ds \right\| \leq M \left\| \int_{a+}^{\tau_m} (h_{+}(s) - h_{-}(s)) \, ds \right\|. \tag{2.2}
\]
It then follows from (2.1) and (2.2) that
\[
\left\| \int_{\tau}^{c} \left( h(s) - h_{-}(s) \right) ds - y \right\| \leq \| y_{m} - y \| + M \left\| \int_{a+}^{\tau_{m}} \left( h_{+}(s) - h_{-}(s) \right) ds \right\|.
\]
Since the right-hand side of the above inequality tends to 0 as \( m \to \infty \), it implies that
\[
\int_{a+}^{c} \left( h(s) - h_{-}(s) \right) ds = y.
\]
Because
\[
h = h_{-} + (h - h_{-}) \quad \text{and} \quad \int_{a+}^{c} h_{-}(s) ds
\]
exists, it follows that \( \int_{a+}^{c} h(s) ds \) exists. Since \( h \in L_{1}\text{loc}((a, b), E) \), then
\[
\int_{a+}^{t} h(s) ds \text{ exists for all } t \in (a, b).
\]
This proves (b), and the proof of (c) is similar. □

The following properties of improper integrals will be needed in the study of second-order initial and boundary value problems in Sections 4 and 5.

**Lemma 2.2.** Assume that \( h \in L_{1}\text{loc}((a, b), E) \) and \( q \in L_{1}\text{loc}((a, b), \mathbb{R}_{+}) \).

(a) If
\[
\int_{a+}^{c} h(s) ds \text{ exists and } \int_{a}^{c} q(s) ds < \infty \text{ for some } c \in (a, b),
\]
then the Bochner integral
\[
\int_{a}^{c} q(t) \left( \int_{a+}^{t} h(s) ds \right) dt \text{ exists for all } c \in (a, b).
\]

(b) If
\[
\int_{a+}^{c} h(s) ds \text{ and } \int_{c}^{b-} h(s) ds \text{ exist and } \int_{c}^{b} q(s) ds < \infty \text{ for some } c \in (a, b),
\]
then the Bochner integral
\[
\int_{c}^{b} q(t) \left( \int_{a+}^{t} h(s) ds \right) dt \text{ exists for all } c \in (a, b).
\]
Proof. To prove (b), assume that
\[
\int_{a+}^{c} h(s) \, ds \quad \text{and} \quad \int_{c}^{b-} h(s) \, ds \quad \text{exist and} \quad \int_{c}^{b} q(s) \, ds < \infty \quad \text{for some } c \in (a, b).
\]
Because of local integrability of both \( q \) and \( h \) these properties hold for all \( c \in (a, b) \).

Choose \( x \in (a, b) \) so that
\[
\left\| \int_{t}^{b-} h(s) \, ds \right\| \leq 1 \quad \text{for all } t \in [x, b).
\]
Then for all such \( t \),
\[
\left\| \int_{a+}^{t} h(s) \, ds \right\| = \left\| \int_{a+}^{b-} h(s) \, ds - \int_{t}^{b-} h(s) \, ds \right\| \leq \left\| \int_{a+}^{b-} h(s) \, ds \right\| + \left\| \int_{t}^{b-} h(s) \, ds \right\| \leq \left\| \int_{a+}^{b-} h(s) \, ds \right\| + 1,
\]
whence for all \( y \in (x, b) \),
\[
\int_{x}^{y} \left\| q(t) \left( \int_{a+}^{t} h(s) \, ds \right) \right\| \, dt \leq \left( \left\| \int_{a+}^{b-} h(s) \, ds \right\| + 1 \right) \int_{x}^{b} q(t) \, dt \to 0 \quad \text{as } x \uparrow b.
\]
This result implies that \( t \mapsto \| q(t)(\int_{a+}^{t} h(s) \, ds)\| \) is Lebesgue integrable on \((c, b)\) for all \( c \in (a, b) \), which is equivalent to the assertion of (b). The proof of (a) is similar. \( \square \)

The following fixed point result is a consequence of [4, Theorem A.2.1], or [8, Theorem 1.2.1 and Proposition 1.2.1].

Lemma 2.3. Given a partially ordered set \( P = (P, \leq) \), and its order interval \([x_-, x_+] = \{ x \in P \mid x_- \leq x \leq x_+ \} \), assume that \( G : [x_-, x_+] \rightarrow [x_-, x_+] \) is increasing, i.e., \( Gx \leq Gy \) whenever \( x_- \leq x \leq y \leq x_+ \), and that each well-ordered chain of the range \( \text{ran} G \) of \( G \) has a supremum in \( P \) and each inversely well-ordered chain of \( \text{ran} G \) has an infimum in \( P \). Then \( G \) has least and greatest fixed points, and they are increasing with respect to \( G \).

In our applications of Lemma 2.3 to differential equations we need the following result.

Lemma 2.4. Assume that \( W \) is a nonempty subset of \( L_{1, \text{loc}}((a, b), E) \), ordered a.e. pointwise, and that there exist functions \( u_{\pm} \in L_{1, \text{loc}}((a, b), E) \), \( i = 1, 2 \), such that \( W \subset [u_-, u_+] \), i.e.
\[
u_- (t) \leq u (t) \leq u_+ (t) \quad \text{for all } u \in W \text{ and for a.e. } t \in (a, b).
\]

(a) If \( W \) is well ordered, it contains an increasing sequence which converges a.e. pointwise to \( \sup W \).
(b) If $W$ is inversely well ordered, it contains a decreasing sequence which converges a.e. pointwise to $\inf W$.

**Proof.** (a) Assume that $W$ is well ordered and (2.3) holds. Choose a sequence of compact subintervals $J_n, n \in \mathbb{N},$ of $(a, b)$ such that $(a, b) = \bigcup_{n=0}^{\infty} J_n,$ and that $J_n \subset J_{n+1}$ for each $n \in \mathbb{N}.$ The given assumptions ensure that for each $n \in \mathbb{N}$ the restrictions $u|_{J_n}, u \in W,$ form a well-ordered and order-bounded chain $W_n$ in $L^1(J_n, E),$ ordered a.e. pointwise. It follows from [8, Proposition 1.3.2, Lemma 5.8.2 and Proposition 5.8.7] that for each $n \in \mathbb{N}$

$$v_n = \sup W_n$$

exists in $L^1(J_n, E),$ and there exist an increasing sequence $(u^*_n)_{n=0}^{\infty}$ of $W$ and a null-set $Z_n \subset J_n$ such that

$$v_n(t) = \lim_{k \to \infty} u^k_n(t) = \sup_{k \in \mathbb{N}} u^k_n(t) \quad \text{for each } t \in J_n \setminus Z_n. \quad (2.4)$$

Defining $v_n(t) = 0$ for $t \in (a, b) \setminus J_n$ we obtain a sequence of strongly measurable functions $v_n : (a, b) \to E.$ The sequence $(v_n)$ is also increasing since $J_n \subset J_{n+1}, n \in \mathbb{N}.$ It is also a.e. pointwise bounded by (2.3) and (2.4), whence

$$u^*(t) = \lim_{n \to \infty} v_n(t) = \sup_{n \in \mathbb{N}} v_n(t) \quad (2.5)$$

exists for a.e. $t \in (a, b).$ Defining $u^*(t) = 0$ for the remaining $t \in (a, b)$ we get a strongly measurable function $u^* : (a, b) \to E.$ Denoting

$$u_n = \max \{ u_n^j \mid 0 \leq j \leq n \}, \quad n \in \mathbb{N},$$

we obtain an increasing sequence $(u_n)$ of $W$ which satisfies

$$u^*_n(t) \leq u_n(t) \leq u^*(t)$$

for each $k = 0, \ldots, n$ and $t \in J_n \setminus Z_n.$ Moreover, by (2.3) the sets $Z_n$ can be so chosen that $(u_n(t))_{n=0}^{\infty}$ is order bounded and increasing for each $t \in (a, b) \setminus Z,$ where $Z = \bigcup_{n=0}^{\infty} Z_n.$ Thus

$$u(t) = \lim_{n \to \infty} u_n(t) = \sup_{n \in \mathbb{N}} u_n(t)$$

exists for each $t \in (a, b) \setminus Z.$ The definitions of $v_n$ and $u$ imply that

$$v_n(t) \leq u(t) \leq u^*(t) \quad \text{for each } t \in J_n \setminus Z_n.$$ 

Thus

$$u^*(t) = \lim_{n \to \infty} v_n(t) \leq u(t) \leq u^*(t)$$

for a.e. $t \in (a, b).$ This result implies that $u = u^*,$ whence $u_n(t) \to u^*(t)$ for a.e. $t \in (a, b).$ Since $(u_n)_{n=0}^{\infty}$ is a sequence of $W,$ it follows from (2.3) that

$$u_-(t) \leq u^*(t) \leq u_+(t) \quad \text{for a.e. } t \in (a, b).$$

This result and strong measurability of $u^*$ imply that $u^* \in L^1_{\text{loc}}((a, b), E).$
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It remains to prove that \( u^* = \sup W \). If \( w \in W \), then \( w|_{J_n} \leq v_n \), whence
\[
w(t) \leq v_n(t) \leq u^*(t) \quad \text{for a.e. } t \in J_n \text{ and for each } n \in \mathbb{N}.
\]

Thus \( w \leq u^* \) for each \( w \in W \), so that \( u^* \) is an upper bound of \( W \). If \( v \in L^1_{\text{loc}}((a, b), E) \) is another upper bound of \( W \), then \( w|_{J_n} \leq v|_{J_n} \) for all \( n \in \mathbb{N} \) and \( w \in W \), whence
\[
v_n(t) \leq v(t) \quad \text{for a.e. } t \in J_n \text{ and for each } n \in \mathbb{N}.
\]

This result and definition (2.5) of \( u^* \) imply that \( u^* \leq v \). Consequently, \( u^* = \sup W \) in \( L^1_{\text{loc}}((a, b), E) \).

(b) If \( W \) is inversely well ordered, then \(-W\), satisfies the hypotheses imposed on \( W \) in hypotheses (a). Thus there exists an increasing sequence \((u_n)\) in \(-W\) such that \( u_n \to u = \sup(-W) \) a.e. pointwise on \((a, b)\). Denoting \( w_n = -u_n, n \in \mathbb{N}, \) we obtain a decreasing sequence of \( W \) which converges a.e. pointwise to \(-u = \inf W \). □

3. Existence results for first-order implicit initial value problems

In this section we study initial value problems which can be represented in the form
\[
\begin{align*}
Lu(t) &:= \frac{d}{dt}(p(t)u(t)) = f(t, u, Lu) \quad \text{for almost every (a.e.) } t \in (a, b), \\
\lim_{t \to a^+} p(t)u(t) &= c(u, Lu),
\end{align*}
\]
where \(-\infty < a < b \leq \infty, f : (a, b) \times L^1_{\text{loc}}((a, b), E)^2 \to E, c : L^1_{\text{loc}}((a, b), E)^2 \to E, p : (a, b) \to \mathbb{R}_+, \) and \( E \) is an ordered Banach space with a regular order cone. We are looking for least and greatest solutions of (3.1) from the set
\[
S := \{ u \in L^1_{\text{loc}}((a, b), E) \mid p \cdot u \text{ is locally absolutely continuous and a.e. differentiable} \}.
\]

Denote
\[
X := \left\{ h \in L^1_{\text{loc}}((a, b), E) \mid \int_{a^+}^c h(t) \, dt \text{ exists for some } c \in (a, b) \right\}.
\]

We shall first convert IVP (3.1) to a system of two equation.

**Lemma 3.1.** Assume that \( \frac{1}{p} \in L^1_{\text{loc}}((a, b), \mathbb{R}_+) \), and that \( f(\cdot, u, v) \in X \) for all \( u, v \in L^1_{\text{loc}}((a, b), E) \). Then \( u \) is a solution of IVP (3.1) in \( S \) if and only if \( (u, Lu) = (u, v) \), where \( (u, v) \in L^1_{\text{loc}}((a, b), E)^2 \) is a solution of the system
\[
\begin{align*}
u(t) &= \frac{1}{p(t)}(c(u, v) + \int_{a^+}^t v(s) \, ds), \quad t \in (a, b), \\
v(t) &= f(t, u, v) \quad \text{for a.e. } t \in (a, b).
\end{align*}
\]

**Proof.** It follows from Lemma 2.1 that the improper integral in the first equation of (3.4) exists for all \( t \in (a, b) \). Assume that \( u \) is a solution of (3.1) in \( S \). Denoting
\[
v(t) = Lu(t) = \frac{d}{dt}(p(t)u(t)),
\]

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the differential equation of (3.1), definition (3.2) of \( S \) and (3.5) ensure that
\[
\int_r^s v(t) \, dt = \int_r^s \frac{d}{dt} \left( p(t)u(t) \right) \, dt = p(s)u(s) - p(r)u(r), \quad a < r \leq s < b.
\]
This result and the initial condition of (3.1) imply that the first equation of (3.4) holds. The validity of the second equation of (3.4) is a consequence of the differential equation of (3.1) and definition (3.5) of \( v \).

Conversely, let \((u, v)\) be a solution of system (3.4) in \( L^1_{\text{loc}}((a, b), E)^2 \). According to (3.4), we have
\[
p(t)u(t) = c(u, v) + \int_{a+}^{t} v(s) \, ds, \quad t \in (a, b).
\]
(3.6)
This equation implies that \( u \in S \), and by differentiation we obtain from (3.6) that
\[
v(t) = \frac{d}{dt} \left( p(t)u(t) \right) = Lu(t) \quad \text{for a.e.} \; t \in (a, b).
\]
This result, Eq. (3.6) and the second equation of (3.4) imply that \( u \) is a solution of the IVP (3.1).

To prove our main existence and comparison result for IVP (3.1), assume that \( L^1_\text{loc}((a, b), E), X \) and \( S \) are ordered a.e. pointwise, and that the functions \( p, f \) and \( c \) satisfy the following hypotheses:

(p) \( \frac{1}{p} \in L^1_\text{loc}((a, b), \mathbb{R}_+) \).

(fa) \( f(\cdot, u, v) \) is strongly measurable for all \( u, v \in L^1_\text{loc}((a, b), E) \), and there exists \( h_\pm \in X \) such that \( h_- \leq f(\cdot, u, v) \leq h_+ \) for all \( u, v \in L^1_\text{loc}((a, b), E) \).

(fb) There exists a \( \lambda \geq 0 \) such that \( f(\cdot, u_1, v_1) + \lambda v_1 \leq f(\cdot, u_2, v_2) + \lambda v_2 \) whenever \( u_i, v_i \in L^1_\text{loc}((a, b), E) \), \( i = 1, 2 \), \( u_1 \leq u_2 \) and \( v_1 \leq v_2 \).

(c) There exists \( c_\pm \in E \) such that \( c_- \leq c(u_1, v_1) \leq c(u_2, v_2) \leq c_+ \) whenever \( u_i, v_i \in L^1_\text{loc}((a, b), E) \), \( i = 1, 2 \), \( u_1 \leq u_2 \) and \( v_1 \leq v_2 \).

**Theorem 3.1.** Assume that hypotheses (p), (fa), (fb) and (c) hold. Then IVP (3.1) has least and greatest solutions in \( S \), and they are increasing with respect to \( f \) and \( c \).

**Proof.** Assume that \( P = L^1_\text{loc}((a, b), E)^2 \) is ordered componentwise. The relations
\[
x_\pm(t) := \left( \frac{1}{p(t)} \left( c_\pm + \int_{a+}^{t} h_\pm(s) \, ds \right), h_\pm(t) \right), \quad t \in (a, b),
\]
(3.7)
define functions \( x_\pm \in P \). By Lemma 2.1 \( v \in X \) whenever \( v \in L^1_\text{loc}((a, b), E) \) and \( h_- \leq v \leq h_+ \). Hence, if \((u, v) \in [x_-, x_+]\), then \( v \in X \). The given hypotheses ensure that the relations
\[ G_1(u,v)(t) := \frac{1}{p(t)} \left( c(u,v) + \int_{a+}^{t} v(s) \, ds \right), \]
\[ G_2(u,v)(t) := \frac{f(t,u,v) + \lambda v(t)}{1 + \lambda}, \quad (3.8) \]
define an increasing mapping \( G = (G_1, G_2) : [x_-, x_+] \rightarrow [x_-, x_+] \).

Let \( W \) be a well-ordered chain in \( \text{ran} \, G \). These sets \( W_1 = \{ u \mid (u,v) \in W \} \) and \( W_2 = \{ v \mid (u,v) \in W \} \) are well-ordered and order-bounded chains in \( L^1_{\text{loc}}((a,b),E) \). It then follows from Lemma 2.4 that \( \sup W_1 \) and \( \sup W_2 \) exist in \( L^1_{\text{loc}}((a,b),E) \). Obviously, \( (\sup W_1, \sup W_2) \) is a supremum of \( W \) in \( P \). Similarly one can show that each inversely well-ordered chain of \( \text{ran} \, G \) has an infimum in \( P \).

The above proof shows that the operator \( G = (G_1, G_2) \) defined by (3.8) satisfies the hypotheses of Lemma 2.3, whence \( G \) has a least fixed point \( x^* = (u^*, v^*) \) and a greatest fixed point \( x^* = (u^*, v^*) \). It follows from (3.8) that \((u^*, v^*) \) and \((u^*, v^*) \) are solutions of the system (3.4). According to Lemma 3.1 \( u_* \) and \( u^* \) belong to \( S \) and are solutions of IVP (3.1).

To prove that \( u_* \) and \( u^* \) are least and greatest of all solutions of (3.1) in \( S \), let \( u \in S \) be a solution of (3.1). In view of Lemma 3.1, \((u,v) = (u, Lu)\) is a solution of system (3.4). Applying the hypotheses (fa) and (c) it is easy to show that \( x = (u,v) \in [x_-, x_+] \), where \( x_\pm \) are defined by (3.7). Thus \( x = (u,v) \) is a fixed point of \( G = (G_1, G_2) : [x_-, x_+] \rightarrow [x_-, x_+] \), defined by (3.8). Because \( x_* = (u_*, v_*) \) and \( x^* = (u^*, v^*) \) are least and greatest fixed points of \( G \), then \((u_*, v_*) \leq (u,v) \leq (u^*, v^*) \). In particular, \( u_* \leq u \leq u^* \), whence \( u_* \) and \( u^* \) are least and greatest of all solutions of IVP (3.1).

The last assertion is an easy consequence of the last conclusion of Lemma 2.3 and the definition of \( G \). \( \square \)

As a special case we obtain an existence result for the IVP:

\[ \frac{d}{dt} \left( p(t)u(t) \right) = g \left( t, u(t), \frac{d}{dt} \left( p(t)u(t) \right) \right) \quad \text{for a.e. } t \in (a,b), \]
\[ \lim_{t \rightarrow a^+} p(t)u(t) = c. \quad (3.9) \]

**Proposition 3.1.** Let the hypothesis (p) hold, and let \( g : (a,b) \times E \times E \rightarrow E \) satisfy the following hypotheses:

\( (ga) \) \( g(\cdot, u(\cdot), v(\cdot)) \) is strongly measurable and \( h_- \leq g(\cdot, u(\cdot), v(\cdot)) \leq h_+ \) for all \( u,v \in L^1_{\text{loc}}((a,b),E) \) and for some \( h_\pm \in X \).

\( (gb) \) There exists \( \lambda \geq 0 \) such that \( g(t,x,z) + \lambda z \leq g(t,y,w) + \lambda w \) for a.e. \( t \in (a,b) \) and whenever \( x \leq y \) and \( z \leq w \) in \( E \).

Then IVP (3.9) has for each choice of \( c \in E \) least and greatest solutions in \( S \). Moreover, these solutions are increasing with respect to \( g \) and \( c \).
Solution. If $c \in E$, IVP (3.9) is reduced to (3.1) when we define
\begin{align*}
\begin{cases}
f(t, u, v) = g(t, u(t), v(t)), & t \in (a, b), \ u, v \in L^1_{\text{loc}}((a, b), E), \\
c(u, v) \equiv c, & u, v \in L^1_{\text{loc}}((a, b), E).
\end{cases}
\end{align*}
(3.10)
The hypotheses (ga) and (gb) imply that $f$ satisfies the hypotheses (fa) and (fb). The hypothesis (c) is also valid, whence (3.1), with $f$ and $c$ defined by (3.10), and hence also (3.9), has by Theorem 3.1 least and greatest solutions. The last assertion follows from the last assertion of Theorem 3.1. \qed

Example 3.1. Determine least and greatest solutions of the following system of IVPs:
\begin{align*}
\begin{cases}
L_1 u_1(t) := \frac{d}{dt}(\sqrt{t} u_1(t)) = -\frac{1}{t} \sin \frac{1}{t} + \frac{\int_1^2 (u_2(s) + L_2 u_2(s)) \, ds}{1 + \int [\int_1^2 (u_2(s) + L_2 u_2(s)) \, ds]} \\
& \text{a.e. in } (0, \infty), \\
L_2 u_2(t) := \frac{d}{dt}(\sqrt{t} u_2(t)) = \frac{1}{t} \sin \frac{1}{t} + \frac{\int_1^2 (u_1(s) + L_1 u_1(s)) \, ds}{1 + \int [\int_1^2 (u_1(s) + L_1 u_1(s)) \, ds]}
\end{cases}
\end{align*}
(3.11)
where $[z]$ denotes the greatest integer $\leq z$.

Solution. System (3.11) is a special case of (3.1) when $E = \mathbb{R}^2$, ordered coordinatewise, $a = 0, b = \infty, p(t) = \sqrt{t}$, and
\begin{align*}
\begin{cases}
f(t, (u_1, u_2), (v_1, v_2)) = \left(-\frac{1}{t} \sin \frac{1}{t} + \frac{\int_1^2 (u_2(s) + v_2(s)) \, ds}{1 + \int [\int_1^2 (u_2(s) + v_2(s)) \, ds]}, \\
& \frac{1}{t} \sin \frac{1}{t} + \frac{\int_1^2 (u_1(s) + v_1(s)) \, ds}{1 + \int [\int_1^2 (u_1(s) + v_1(s)) \, ds]}
\end{cases}
\end{align*}
(3.12)
\begin{align*}
c((u_1, u_2), (v_1, v_2)) = \left(\frac{2[u_2(1)]}{1 + \|[u_2(1)]\|}, \frac{3[u_1(1)]}{1 + \|[u_1(1)]\|}\right).
\end{align*}
The hypotheses (fa), (fb) and (c) hold when $h_{\pm}(t) = (-\frac{1}{t} \sin \frac{1}{t} \pm 1, \frac{1}{t} \sin \frac{1}{t} \pm 1), \lambda = 0$ and $c_{\pm} = (\pm 2, \pm 3)$. Thus (3.11) has least and greatest solutions. The functions $x_-$ and $x_+$ defined by (3.7) can be calculated, and one obtains
\begin{align*}
\begin{cases}
x_-(t) = \left(\frac{-2}{\sqrt{t}} + \frac{\text{Si}(1/t)}{\sqrt{t}} - \frac{\pi}{2\sqrt{t}} - \sqrt{t}, -\frac{3}{\sqrt{t}} - \frac{\text{Si}(1/t)}{\sqrt{t}} + \frac{\pi}{2\sqrt{t}} - \sqrt{t}\right), h_-(t), \\
x_+(t) = \left(\frac{2}{\sqrt{t}} + \frac{\text{Si}(1/t)}{\sqrt{t}} - \frac{\pi}{2\sqrt{t}} + \sqrt{t}, \frac{3}{\sqrt{t}} - \frac{\text{Si}(1/t)}{\sqrt{t}} + \frac{\pi}{2\sqrt{t}} + \sqrt{t}\right), h_+(t),
\end{cases}
\end{align*}
where
\begin{align*}
\text{Si}(x) = \int_0^x \frac{\sin t}{t} \, dt
\end{align*}
is the sine integral function. According to Lemma 3.1 the least solution of (3.11) is equal to the first component of the least fixed point of $G = (G_1, G_2)$, defined by (3.8), with $f$, and $c$ given by (3.12) and $p(t) = \sqrt{t}$. By the proof of [8, Theorem 1.2.1], the least fixed point of $G$ is a maximum of a well-ordered chain of $G$-iterations of $x_-$, whose least elements are iterations $G^n x_-$. Calculating these iterations, it turns out that $G^3 x_- = G^4 x_-$. Thus $(u_{*1}, u_{*2}) = G^3_1 x_-$ is the least solution of (3.11). Similarly, one can show that $G^3 x_+ = G^4 x_+$, which implies that $(u^*_{1}, u^*_{2}) = G^3_1 x_+$ is the greatest solution of (3.11). The exact expressions of these solutions of the IVP (3.11) are:

$$(u_{*1}(t), u_{*2}(t)) = \left( -\frac{3}{2\sqrt{t}} + \frac{\text{Si}(1/t)}{\sqrt{t}} - \frac{\pi}{2\sqrt{t}} - \frac{3}{4\sqrt{t}} - \frac{\text{Si}(1/t)}{\sqrt{t}} + \frac{\pi}{2\sqrt{t}} - \frac{5}{6\sqrt{t}} ,\right),$$

$$(u^*_{1}(t), u^*_{2}(t)) = \left( \frac{4}{3\sqrt{t}} + \frac{\text{Si}(1/t)}{\sqrt{t}} + \frac{\pi}{2\sqrt{t}} + \frac{3}{4\sqrt{t}} - \frac{\text{Si}(1/t)}{\sqrt{t}} + \frac{\pi}{2\sqrt{t}} + \frac{1}{2\sqrt{t}} \right).$$

**Example 3.2.** Let $E$ be the space $c_0$ of the sequences of real numbers converging to zero, ordered componentwise and normed by the sup-norm. The mappings $h_\pm : (0, \infty) \to c_0$, defined by

$$h_\pm(t) = \left( \frac{1}{nt} \sin \frac{1}{t} \pm \frac{1}{n} \right)_{n=1}^\infty, \quad t \in (0, \infty),$$

belong to $X$, defined by (3.3). Thus these mappings are possible upper and lower boundaries for $f$ in the hypothesis (fa) of Theorem 3.1 and for $g$ in the hypothesis (ga) of Proposition 3.1 when $E = c_0$. Choosing $c_\pm = (\frac{1}{n})_{n=1}^\infty$ and $p(t) := t$, the solutions of the initial value problems

$$\frac{d}{dt} \left( p(t) u(t) \right) = h_\pm(t) \quad \text{for (a.e.) } t \in (0, \infty), \quad \lim_{t \to 0^+} p(t) u(t) = c_\pm,$$

are

$$u_\pm(t) = \left( \frac{1}{nt} \left( \frac{1}{2} \pi - \text{Si}\left( \frac{1}{t} \right) \right) \pm (t + 1) \right)_{n=1}^\infty.$$

In particular, the infinite system of initial value problems

$$\begin{cases} L_n u_n(t) := \frac{d}{dt}(t u_n(t)) = \frac{1}{n} \left( \frac{1}{t} \sin \frac{1}{t} + f_n(u, Lu) \right) \quad \text{for a.e. } t \in (0, \infty), \\ \lim_{t \to 0^+} (t u_n(t)) = \frac{c_n}{n}, \quad n = 1, 2, \ldots, \end{cases}$$

where $u = (u_n)_{n=1}^\infty$, $Lu = (L_n u_n)_{n=1}^\infty$, each $f_n : L_{loc}^1((0, \infty), c_0) \times L_{loc}^1((0, \infty), c_0) \to \mathbb{R}$, is increasing with respect to both arguments and $-1 \leq c_n, f_n(u, v) \leq 1$ for all $u, v \in L_{loc}^1((0, \infty), c_0)$ and $n = 1, 2, \ldots$, has least and greatest solutions $u_* = (u_n*)_{n=1}^\infty$ and $u^* = (u_n*)_{n=1}^\infty$, and they belong to the order interval $[u_-, u_+]$, where $u_\pm$ are given by (3.15).

**Remark 3.1.** No component of the mappings $h_\pm$ defined in (3.13) belongs to $L^1((0, t), \mathbb{R}_+)$ for any $t > 0$. Consequently, the mappings $h_\pm$ do not belong to $L^1((0, t), c_0)$ for any $t > 0$. Notice also that if $f$ in Theorem 3.1 and $g$ in Proposition 3.1 are norm-bounded by a function $h_0$ which belongs to $L^1((a, t), \mathbb{R}_+)$ for every $t \in (a, b)$, as assumed in [6], then the mappings $f(\cdot, u, v)$ and $g(\cdot, u(\cdot), v(\cdot))$ belong to $L^1((a, t), E)$ for all $t \in (a, b)$. 

4. Existence results for second-order initial value problems

Next we study initial value problems of the form
\[
\begin{cases}
Lu(t) := \frac{d}{dt}(p(t)u'(t)) = f(t, u, u', Lu) & \text{for a.e. } t \in (a, b), \\
\lim_{t \to a^+} p(t)u'(t) = c(u, u', Lu), & \lim_{t \to a^+} u(t) = d(u, u', Lu),
\end{cases}
\tag{4.1}
\]
where $-\infty \leq a < b \leq \infty$, $f : (a, b) \times L^1_{\text{loc}}((a, b), E)^3 \to E$, $c, d : L^1_{\text{loc}}((a, b), E)^3 \to E$ and $p : (a, b) \to \mathbb{R}_+$. Now we are looking for least and greatest solutions of (4.1) from set
\[
Y := \{ u : (a, b) \to E \mid u \text{ and } p \cdot u' \text{ are locally absolutely continuous and} \\
a.e. \text{ differentiable} \}.
\tag{4.2}
\]
Denote, as in Section 3,
\[
X := \left\{ h \in L^1_{\text{loc}}((a, b), E) \mid \int_{a}^{c} h(t) \, dt \text{ exists for some } c \in (a, b) \right\}.
\tag{4.3}
\]
IVP (4.1) can be converted to a system of equations which do not contain derivatives.

**Lemma 4.1.** Assume that $\frac{1}{p} \in L^1_{\text{loc}}((a, b), \mathbb{R}_+)$, that $\int_{a}^{t} 1/p(s) \, ds < \infty$ for some $t \in (a, b)$, and that $f(\cdot, u, v, w) \in X$ for all $u, v, w \in L^1_{\text{loc}}((a, b), E)$. Then $u$ is a solution of IVP (4.1) in $Y$ if and only if $(u, u', Lu) = (u, v, w)$, where $(u, v, w) \in L^1_{\text{loc}}((a, b), E)^3$ is a solution of the system
\[
\begin{cases}
u(t) = d(u, v, w) + \int_{a}^{t} v(s) \, ds, & t \in (a, b), \\
v(t) = \frac{1}{p(t)}(c(u, v, w) + \int_{a}^{t} w(s) \, ds), & t \in (a, b), \\
w(t) = f(t, u, v, w) & \text{for a.e. } t \in (a, b).
\end{cases}
\tag{4.4}
\]

**Proof.** The results of Lemmas 2.1 and 2.2 ensure that the improper integrals of (4.4) exist for all $t \in (a, b)$. Assume that $u$ is a solution of (4.1) in $Y$, and denote
\[
w(t) = Lu(t) = \frac{d}{dt}(p(t)u'(t)), \quad v(t) = u'(t).
\tag{4.5}
\]
The differential equation and the second initial condition of (4.1), definition (4.2) of $Y$ and notations (4.5) ensure that first and third equations of (4.4) hold, and that
\[
\int_{r}^{s} w(t) \, dt = \int_{r}^{s} \frac{d}{dt}(p(t)v(t)) \, dt = p(s)v(s) - p(r)v(r), \quad a < r \leq s < b.
\]
This result and the first initial condition of (4.1) imply that the second equation of (4.4) holds. Obviously, $(u, u', Lu) \in L^1_{\text{loc}}((a, b), E)^3$.

Conversely, let $(u, v, w)$ be a solution of system (4.4) in $L^1_{\text{loc}}((a, b), E)^3$. The first equation of (4.4) implies that $v = u'$, that $u$ is locally absolutely continuous, and that the second
initial condition of (4.1) holds. Since \( v = u' \), it follows from the second equation of (4.4) that

\[
p(t)u'(t) = c(u, u', w) + \int_{a+}^{t} w(s) \, ds, \quad t \in (a, b).
\]

(4.6)

This equation implies that \( p \cdot u' \) is locally absolutely continuous and a.e. differentiable, and thus \( u \in Y \). By differentiation we obtain from (4.6) that

\[
w(t) = \frac{d}{dt} \left( p(t)u'(t) \right) = Lu(t) \quad \text{for a.e. } t \in (a, b).
\]

(4.7)

This result and (4.6) imply that the first initial condition of (4.1) holds. The validity of the differential equation of (4.1) is a consequence of the third equation of (4.4), Eq. (4.7), and the fact that \( v = u' \). \( \Box \)

Assume that \( L_{loc}^1((a,b),E) \) and \( X \) are ordered a.e. pointwise, that \( Y \) is ordered pointwise, and that the functions \( p, f, c \) and \( d \) satisfy the following hypotheses:

\begin{itemize}
  \item[(p0)] \( \frac{1}{p} \in L_{loc}^1((a,b),\mathbb{R}^+) \) and \( \int_{a+}^{t} \frac{ds}{p(s)} < \infty \) for some \( t \in (a,b) \).
  \item[(f0)] \( f(\cdot, u, v, w) \) is strongly measurable and \( X \supseteq h_- \leq f(\cdot, u, v, w) \leq h_+ \in X \) for all \( u, v, w \in L_{loc}^1((a,b),E) \).
  \item[(f1)] There exists a \( \lambda \geq 0 \) such that \( f(\cdot, u_1, v_1, w_1) + \lambda w_1 \leq f(\cdot, u_2, v_2, w_2) + \lambda w_2 \) whenever \( u_i, v_i, w_i \in L_{loc}^1((a,b),E), i = 1, 2, u_1 \leq u_2, v_1 \leq v_2 \) and \( w_1 \leq w_2 \).
  \item[(c0)] \( c_\pm \in E, \) and \( c_- \leq c(u_1, v_1, w_1) \leq c(u_2, v_2, w_2) \leq c_+ \) whenever \( u_i, v_i, w_i \in L_{loc}^1((a,b),E), i = 1, 2, u_1 \leq u_2, v_1 \leq v_2 \) and \( w_1 \leq w_2 \).
  \item[(d0)] \( d_- \in E, \) and \( d_- \leq d(u_1, v_1, w_1) \leq d(u_2, v_2, w_2) \leq d_+ \) whenever \( u_i, v_i, w_i \in L_{loc}^1((a,b),E), i = 1, 2, u_1 \leq u_2, v_1 \leq v_2 \) and \( w_1 \leq w_2 \).
\end{itemize}

Our main existence and comparison result for IVP (4.1) reads as follows.

**Theorem 4.1.** Assume that the hypotheses (p0), (f0), (f1), (c0) and (d0) hold. Then IVP (4.1) has least and greatest solutions in \( Y \), and they are increasing with respect to \( f, c \) and \( d \).

**Proof.** Assume that \( P = L_{loc}^1((a,b),E)^3 \) is ordered componentwise. The relations

\[
x_\pm(t) := \left( d_\pm + \int_{a+}^{t} \frac{1}{p(s)} \left( c_\pm + \int_{a+}^{s} h_\pm(\tau) \, d\tau \right) \, ds, \frac{1}{p(t)} \left( c_\pm + \int_{a+}^{t} h_\pm(s) \, ds \right), h_\pm(t) \right)
\]

(4.8)

define functions \( x_\pm \in P \). If \( (u, v, w) \in [x_- , x_+] \), then \( w \in [h_- , h_+] \), whence \( w \in X \). Hence, it is easy to show, by applying the given hypotheses, that the relations

\[
\begin{align*}
G_1(u, v, w)(t) &:= d(u, v, w) + \int_{a+}^{t} v(s) \, ds, \quad t \in (a, b), \\
G_2(u, v, w)(t) &:= \frac{1}{p(t)} \left( c(u, v, w) + \int_{a+}^{t} w(s) \, ds \right), \quad t \in (a, b), \\
G_3(u, v, w)(t) &:= \frac{f(t, u, v, w) + \lambda w(t)}{1 + \lambda}, \quad t \in (a, b)
\end{align*}
\]

(4.9)
define an increasing mapping \( G = (G_1, G_2, G_3) : [x_-, x_+] \to [x_-, x_+] \).

Let \( W \) be a well-ordered chain in \( \text{ran} \ G \). The sets

\[
W_1 = \{ u \mid (u, v, w) \in W \}, \quad W_2 = \{ v \mid (u, v, w) \in W \} \quad \text{and} \quad W_3 = \{ w \mid (u, v, w) \in W \}
\]

are well-ordered and order-bounded chains in \( L^1_{\text{loc}}((a, b), E) \). It then follows from Lemma 2.4 that the suprema of \( W_1, W_2 \) and \( W_3 \) exist in \( L^1_{\text{loc}}((a, b), E) \). Obviously, \( (\sup W_1, \sup W_2, \sup W_3) \) is a supremum of \( W \) in \( P \). Similarly one can show that each inversely well-ordered chain of \( \text{ran} \ G \) has an infimum in \( P \).

The above proof shows that the operator \( G = (G_1, G_2, G_3) \) defined by (4.9) satisfies the hypotheses of Lemma 2.3, whence \( G \) has a least fixed point \( x_* = (u_*, v_*, w_*) \) and a greatest fixed point \( x^* = (u^*, v^*, w^*) \). It follows from (4.9) that \( (u_*, v_*, w_*) \) and \( (u^*, v^*, w^*) \) are solutions of system (4.4). According to Lemma 4.1 \( u_* \) and \( u^* \) belong to \( Y \) and are solutions of IVP (4.1).

To prove that \( u_* \) and \( u^* \) are least and greatest of all solutions of (4.1) in \( Y \), let \( u \in Y \) be a solution of (4.1). In view of Lemma 4.1, \( (u, v, w) = (u, u', Lu) \) is a solution of system (4.4). Applying hypotheses (f0), (c0) and (d0), it is easy to show that \( x = (u, v, w) \in [x_-, x_+] \), where \( x_\pm \) are defined by (4.8). Thus \( x = (u, v, w) \) is a fixed point of \( G = (G_1, G_2, G_3) : [x_-, x_+] \to [x_-, x_+] \), defined by (4.9). Because \( x_* = (u_*, v_*, w_*) \) and \( x^* = (u^*, v^*, w^*) \) are least and greatest fixed points of \( G \), then \( (u_*, v_*, w_*) \leq (u, v, w) \leq (u^*, v^*, w^*) \). In particular, \( u_* \leq u \leq u^* \), whence \( u_* \) and \( u^* \) are least and greatest of all solutions of IVP (4.1).

The last assertion is an easy consequence of the last conclusion of Lemma 2.3 and definition (4.9) of \( G = (G_1, G_2, G_3) \). □

As a special case we obtain an existence result for the IVP

\[
\begin{align*}
\frac{d}{dt}(p(t)u'(t)) &= g(t, u(t), u'(t), \frac{d}{dt}(p(t)u'(t))) \quad \text{for a.e. } t \in (a, b), \\
\lim_{t \to a+} p(t)u'(t) &= c, \quad \lim_{t \to a+} u(t) = d.
\end{align*}
\]

\tag{4.10}

**Corollary 4.1.** Let the hypothesis (p0) hold, and let \( g : (a, b) \times E \times E \times E \to E \) satisfy the following hypotheses:

1. \( g(\cdot, u(\cdot), v(\cdot), w(\cdot)) \) is strongly measurable and \( h_\pm \leq g(\cdot, u(\cdot), v(\cdot), w(\cdot)) \leq h_\pm \) for all \( u, v, w \in L^1_{\text{loc}}((a, b), E) \) and for some \( h_\pm \in X \).
2. There exists a \( \lambda \geq 0 \) such that \( g(t, x_1, x_2, x_3) + \lambda x_3 \leq g(t, y_1, y_2, y_3) + \lambda y_3 \) for a.e. \( t \in (a, b) \) and whenever \( x_1 \leq y_1 \leq E, i = 1, 2, 3 \).

Then IVP (4.10) has for each choice of \( c, d \in E \) least and greatest solutions in \( Y \). Moreover, these solutions are increasing with respect to \( g, c \) and \( d \).

**Proof.** If \( c, d \in E \), IVP (4.10) is reduced to (4.1) when we define

\[
\begin{align*}
f(t, u, v, w) &= g(t, u(t), v(t), w(t)), \quad t \in (a, b), \quad u, v, w \in L^1_{\text{loc}}((a, b), E), \\
c(u, v, w) &= c, \quad d(u, v, w) &= d, \quad u, v, w \in L^1_{\text{loc}}((a, b), E).
\end{align*}
\]
The hypotheses (g0) and (g1) imply that \( f \) satisfies the hypotheses (f0) and (f1). The hypotheses (c0) and (d0) are also valid, whence (4.1), with \( f, c \) and \( d \) defined above, and hence also (4.10), has by Theorem 4.1 least and greatest solutions. The last assertion follows from the last assertion of Theorem 4.1. \( \Box \)

**Example 4.1.** Determine least and greatest solutions of the following system of the implicit singular IVPs

\[
\begin{align*}
L_1u_1(t) &:= \frac{d}{dt} \left( \sqrt{t} u_1'(t) \right) = \frac{1}{t} \sin \frac{1}{t} + \frac{\left[ f_1^2(u_2(s) + u_2'(s) + L_2u_2(s)) ds \right]}{1 + \left[ f_1^2(u_2(s) + u_2'(s) + L_2u_2(s)) ds \right]}, \\
& \quad \text{a.e. in } (0, \infty), \\
L_2u_2(t) &:= \frac{d}{dt} \left( \sqrt{t} u_2'(t) \right) = -\frac{1}{t} \sin \frac{1}{t} + \frac{\left[ f_1^2(u_1(s) + u_1'(s) + L_1u_1(s)) ds \right]}{1 + \left[ f_1^2(u_1(s) + u_1'(s) + L_1u_1(s)) ds \right]}, \\
& \quad \text{a.e. in } (0, \infty),
\end{align*}
\]

\[
\begin{align*}
\lim_{t \to 0^+} \sqrt{t} u_1'(t) &= \frac{[u_2'(1)]}{1 + [[u_2(1)]]}, \\
\lim_{t \to 0^+} \sqrt{t} u_2'(t) &= \frac{[u_1'(1)]}{1 + [[u_1(1)]]},
\end{align*}
\]

(4.11)

**Solution.** System (4.11) is a special case of (4.1) when \( E = \mathbb{R}^2, a = 0, b = \infty, p(t) = \sqrt{t} \), and

\[
\begin{align*}
f(t, (u_1, u_2), (v_1, v_2), (w_1, w_2)) &= \left( \frac{1}{t} \sin \frac{1}{t} + \frac{\left[ f_1^2(u_2(s) + v_2(s) + w_2(s)) ds \right]}{1 + \left[ f_1^2(u_2(s) + v_2(s) + w_2(s)) ds \right]}, \\
& \quad -\frac{1}{t} \sin \frac{1}{t} + \frac{\left[ f_1^2(u_1(s) + v_1(s) + w_1(s)) ds \right]}{1 + \left[ f_1^2(u_1(s) + v_1(s) + w_1(s)) ds \right]})
\end{align*}
\]

(4.12)

\[
\begin{align*}
c((u_1, u_2), (v_1, v_2), (w_1, w_2)) &= \begin{pmatrix} [v_2(1)] / (1 + [v_2(1)]) & [v_1(1)] / (1 + [v_1(1)]) \\ [u_2(1)] / (1 + [u_2(1)]) & [u_1(1)] / (1 + [u_1(1)]) \end{pmatrix}, \\
d((u_1, u_2), (v_1, v_2), (w_1, w_2)) &= \begin{pmatrix} [v_2(1)] / (1 + [v_2(1)]) & [v_1(1)] / (1 + [v_1(1)]) \\ [u_2(1)] / (1 + [u_2(1)]) & [u_1(1)] / (1 + [u_1(1)]) \end{pmatrix}.
\end{align*}
\]

The hypotheses (f0), (c0) and (d0) hold when

\[
h_{\pm}(t) = \left( \frac{1}{t} \sin \frac{1}{t} \pm 1, -\frac{1}{t} \sin \frac{1}{t} \pm 1 \right),
\]

\( \lambda = 0 \) and \( c_{\pm} = d_{\pm} = (\pm 1, \pm 1) \). Thus (4.11) has least and greatest solutions. Functions \( x_0 \) and \( x_1 \) defined by (4.8) can be calculated, and their first components are \( (u_1^+, u_2^+) \) and \( (u_1^+, u_2^+) \), where
\[ u^-_1(t) = -1 - 2\sqrt{2\pi} - 2\sqrt{t} - 2\sqrt{t} \text{Si}\left(\frac{1}{t}\right) - 4\sin\left(\frac{1}{t}\right)\sqrt{t} \]
\[ + 4\sqrt{2\pi} \text{FresnelC}\left(\sqrt{\frac{2}{\pi t}}\right) + \pi \sqrt{t} - \frac{2}{3}t\sqrt{t}, \]
\[ u^-_2(t) = -1 + 2\sqrt{2\pi} - 2\sqrt{t} - 2\sqrt{t} \text{Si}\left(\frac{1}{t}\right) + 4\sin\left(\frac{1}{t}\right)\sqrt{t} \]
\[ - 4\sqrt{2\pi} \text{FresnelC}\left(\sqrt{\frac{2}{\pi t}}\right) - \pi \sqrt{t} - \frac{2}{3}t\sqrt{t}, \]
\[ u^+_1(t) = 1 - 2\sqrt{2\pi} + 2\sqrt{t} - 2\sqrt{t} \text{Si}\left(\frac{1}{t}\right) - 4\sin\left(\frac{1}{t}\right)\sqrt{t} + 4\sqrt{2\pi} \text{FresnelC}\left(\sqrt{\frac{2}{\pi t}}\right) \]
\[ + \pi \sqrt{t} + \frac{2}{3}t\sqrt{t}, \]
\[ u^+_2(t) = 1 + 2\sqrt{2\pi} + 2\sqrt{t} + 2\sqrt{t} \text{Si}\left(\frac{1}{t}\right) + 4\sin\left(\frac{1}{t}\right)\sqrt{t} - 4\sqrt{2\pi} \text{FresnelC}\left(\sqrt{\frac{2}{\pi t}}\right) \]
\[ - \pi \sqrt{t} + \frac{2}{3}t\sqrt{t}, \]

where \[ \text{FresnelC}(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt \]
is the Fresnel cosine integral. According to Lemma 4.1 the least solution of (4.11) is equal to the first component of the least fixed point of \( G = (G_1, G_2, G_3) \), defined by (4.9), with \( f, c \) and \( d \) given by (4.12) and \( p(t) = \sqrt{t} \). Calculating the iterations \( G^n x_- \) it turns out that \( G^2 x_- = G^3 x_- \), whence \( (u^*_1, u^*_2) = G^2_1 x_- \) is the least solution of (4.11). Similarly, one can show that \( (u^*_1, u^*_2) = G^4_1 x_+ \) is the greatest solution of (4.11). The exact expressions of these solutions are:

\[ u^*_{11}(t) = \frac{3}{4} - 2\sqrt{2\pi} - \frac{3}{2}\sqrt{t} - 2\sqrt{t} \text{Si}\left(\frac{1}{t}\right) - 4\sqrt{t} \sin\left(\frac{1}{t}\right) \]
\[ + 4\sqrt{2\pi} \text{FresnelC}\left(\sqrt{\frac{2}{\pi t}}\right) + \pi \sqrt{t} - \frac{16}{27} t\sqrt{t}, \]
\[ u^*_{12}(t) = -\frac{3}{4} + 2\sqrt{2\pi} - \frac{4}{3}\sqrt{t} + 2\sqrt{t} \text{Si}\left(\frac{1}{t}\right) + 4\sqrt{t} \sin\left(\frac{1}{t}\right) \]
\[ - 4\sqrt{2\pi} \text{FresnelC}\left(\sqrt{\frac{2}{\pi t}}\right) - \pi \sqrt{t} - \frac{5}{9} t\sqrt{t}, \]
\[ u^*_1(t) = \frac{1}{2} - 2\sqrt{2\pi} - 2\sqrt{t} \text{Si}\left(\frac{1}{t}\right) - 4\sqrt{t} \sin\left(\frac{1}{t}\right) + 4\sqrt{2\pi} \text{FresnelC}\left(\sqrt{\frac{2}{\pi t}}\right) \]
\[ + \pi \sqrt{t} + \frac{1}{2} t\sqrt{t}, \]
\[ u^*_2(t) = \frac{1}{2} + 2\sqrt{2\pi} + \sqrt{t} + 2\sqrt{t} \text{Si} \left( \frac{1}{t} \right) + 4\sqrt{t} \sin \left( \frac{1}{t} \right) - 4\sqrt{2\pi t} \text{FresnelC} \left( \sqrt{\frac{2}{\pi t}} \right) \]
\[ - \pi \sqrt{t} + \frac{8}{15} t \sqrt{t}. \]

**Example 4.2.** Let \( E \) be the space \((c_0)\), ordered coordinatewise and normed by the sup-norm. Mappings \( h_{\pm} : (0, \infty) \to c_0 \), defined by

\[ h_{\pm}(t) = \left( \frac{1}{n} \sin \frac{1}{t} \pm \frac{1}{n} \right)_{n=1}^{\infty}, \quad t \in (0, \infty), \tag{4.13} \]

belong to \( X \), defined by (4.3). Thus these mappings are possible upper and lower boundaries for \( f \) in the hypothesis (fa) of Theorem 4.1 and for \( g \) in the hypothesis (ga) of Corollary 4.1 when \( E = c_0 \).

Choosing \( c_{\pm} = \left( \pm \frac{1}{n} \right)_{n=1}^{\infty} \), \( d_{\pm} = \left( \pm \frac{1}{n} \right)_{n=1}^{\infty} \) and \( p(t) := t^{1/2} \), the solutions of the initial value problem

\[ \frac{d}{dt} \left( \sqrt{tu'}(t) \right) = h_{\pm}(t) \quad \text{for a.e. } t \in (0, \infty), \]
\[ \lim_{t \to 0^+} \sqrt{tu'}(t) = c_{\pm}, \quad \lim_{t \to 0^+} u(t) = d_{\pm} \tag{4.14} \]

are:

\[ u_+(t) = \left( \frac{1}{n} \left( 1 - 2\sqrt{2\pi} - 2\sqrt{t} \text{Si} \left( \frac{1}{t} \right) - 4\sqrt{t} \sin \left( \frac{1}{t} \right) + 4\sqrt{\pi t} \text{FresnelC} \left( \sqrt{\frac{2}{\pi t}} \right) \right) + (\pi + 2)\sqrt{t} + \frac{2}{3} t \sqrt{t} \right)_{n=1}^{\infty}, \]
\[ u_-(t) = \left( \frac{1}{n} \left( -1 - 2\sqrt{2\pi} - 2\sqrt{t} \text{Si} \left( \frac{1}{t} \right) - 4\sqrt{t} \sin \left( \frac{1}{t} \right) + 4\sqrt{\pi t} \text{FresnelC} \left( \sqrt{\frac{2}{\pi t}} \right) \right) + (\pi - 2)\sqrt{t} - \frac{2}{3} t \sqrt{t} \right)_{n=1}^{\infty}. \tag{4.15} \]

In particular, the infinite system of initial value problems

\[
\begin{cases}
L_n u_n(t) := \frac{d}{dt}(\sqrt{tu_n'}(t)) = \frac{1}{n} \left( \frac{1}{t} \sin \frac{1}{t} + f_n(u, u', Lu) \right) & \text{for a.e. } t \in (0, \infty), \\
\lim_{t \to 0^+} (\sqrt{tu_n'}(t)) = \frac{c_n}{n}, & \lim_{t \to 0^+} u_n(t) = \frac{d_n}{n}, \quad n = 1, 2, \ldots, \tag{4.16}
\end{cases}
\]

where \( u = (u_n)_{n=1}^{\infty}, Lu = (L_n u_n)_{n=1}^{\infty} \), each \( f_n : L_{1 \text{loc}}(\mathbb{R}) \to \mathbb{R} \) is increasing with respect to every argument \(-1 \leq c_n, d_n, f_n(u, v, w) \leq 1\) for all \( u, v, w \in L_{1 \text{loc}}(\mathbb{R}) \) and \( n = 1, 2, \ldots, \) has least and greatest solutions \( u_* = \left( u_{*n} \right)_{n=1}^{\infty} \) and \( u^* = \left( u^*_n \right)_{n=1}^{\infty} \), and they belong to the order interval \([u_-, u_+]\), where \( u_{\pm} \) are given by (4.15).
5. Existence results for second-order boundary value problems

This section is devoted to the study of boundary value problems of the form

\[
\begin{aligned}
  &\begin{cases}
    Lu(t) := -\frac{d}{dt}(p(t)u'(t)) = f(t, u, u', Lu) & \text{for a.e. } t \in (a, b), \\
    \lim_{t \to a^+} p(t)u'(t) = c(u, u', Lu), & \\
    \lim_{t \to b^-} u(t) = d(u, u', Lu),
  \end{cases}
  \\
  &\lim_{t \to a^+} p(t)u'(t) = c(u, u', Lu), \\
  &\lim_{t \to b^-} u(t) = d(u, u', Lu),
\end{aligned}
\tag{5.1}
\]

where \(-\infty < a < b < \infty, f : (a, b) \times L^1_{\text{loc}}((a, b), E)^3 \to E, c, d : L^1_{\text{loc}}((a, b), E)^3 \to E\) and \(p : (a, b) \to \mathbb{R}^+\). Denote

\[
Z := \left\{ h \in X \left| \int_r^b h(t) \, dt \text{ exists for some } r \in (a, b) \right. \right\},
\tag{5.2}
\]

where \(X\) is defined by (4.3).

As in Section 3 we shall first convert BVP (5.1) to a system of three equations.

**Lemma 5.1.** Assume that

\[
\frac{1}{p} \in L^1_{\text{loc}}((a, b), \mathbb{R}^+), \\
\int_a^b \frac{1}{p(s)} \, ds < \infty \text{ for some } t \in (a, b), \text{ and } \\
f(\cdot, u, v, w) \in Z \text{ for all } u, v, w \in L^1_{\text{loc}}((a, b), E).
\]

Then \(u\) is a solution of IVP (5.1) in \(Y\), defined by (4.2) if and only if \((u, u', Lu) = (u, v, w)\), where \((u, v, w) \in L^1_{\text{loc}}((a, b), E)^3\) is a solution of the system

\[
\begin{aligned}
  &\begin{cases}
    u(t) = d(u, v, w) - \int_t^b v(s) \, ds, & \text{for a.e. } t \in (a, b), \\
    u'(t) = \frac{1}{p(t)} \left( c(u, v, w) - \int_{a^+}^t w(s) \, ds \right), & \text{for a.e. } t \in (a, b), \\
    w(t) = f(t, u, v, w)
  \end{cases}
  \\
  &\int_r^s w(t) \, dt = -\int_r^s \frac{d}{dt}(p(t)v(t)) \, dt = p(r)v(r) - p(s)v(s), \quad a < r \leq s < b.
\end{aligned}
\tag{5.3}
\]

**Proof.** The results of Lemmas 2.1 and 2.2 ensure that the improper integrals of (5.3) exist for all \(t \in (a, b)\). Assume that \(u\) is a solution of (5.1) in \(Y\), and denote

\[
w(t) = Lu(t) = -\frac{d}{dt}(p(t)u'(t)), \quad v(t) = u'(t), \quad t \in (a, b).
\tag{5.4}
\]

The differential equation and the second initial condition of (5.1), definition (4.2) of \(Y\) and notations (5.4) ensure that first and third equations of (5.3) hold, and that

\[
\int_r^s w(t) \, dt = -\int_r^s \frac{d}{dt}(p(t)v(t)) \, dt = p(r)v(r) - p(s)v(s), \quad a < r \leq s < b.
\]

This result and the first initial condition of (5.1) imply that the second equation of (5.3) holds.

Conversely, let \((u, v, w)\) be a solution of system (5.3) in \(L^1_{\text{loc}}((a, b), E)^3\). The first equation of (5.3) implies that \(v = u'\), that \(u\) is locally absolutely continuous, and that the second
initial condition of (5.1) holds. Since \( v = u' \), it follows from the second equation of (5.3) that
\[
 p(t)u'(t) = c(u, u', w) - \int_{a+}^{t} w(s) \, ds, \quad t \in (a, b). \tag{5.5}
\]
This equation implies that \( p \cdot u' \) is locally absolutely continuous and a.e. differentiable, and thus \( u \in Y \). It follows from (5.5) by differentiation that
\[
 w(t) = -\frac{d}{dt} (p(t)u'(t)) = Lu(t) \quad \text{for a.e. } t \in (a, b). \tag{5.6}
\]
This result and (5.5) imply that the first initial condition of (5.1) holds. The validity of the differential equation of (5.1) is a consequence of the third equation of (5.3), Eq. (5.6), and the fact that \( v = u' \). □

Assuming that \( L_{1}^{1}((a, b), E) \) and \( Z \) are ordered a.e. pointwise, we shall impose the following hypotheses for the functions \( p, f, c \) and \( d \).

(\( p_1 \)) \( \frac{1}{p} \in L_{1}^{1}((a, b), \mathbb{R}^{+}) \) and \( \int_{a}^{b} \frac{ds}{p(s)} < \infty \) for some \( t \in (a, b) \).

(\( f_0 \)) \( f(\cdot, u, v, w) \) is strongly measurable and \( Z \ni h_{-} \leq f(\cdot, u, v, w) \leq h_{+} \in Z \) for all \( u, v, w \in L_{1}^{1}((a, b), E) \).

(\( f_1 \)) There exists a \( \lambda > 0 \) such that \( f(\cdot, u_1, v_1, w_1) + \lambda w_1 \leq f(\cdot, u_2, v_2, w_2) + \lambda w_2 \) whenever \( u_1, v_1, w_1 \in L_{1}^{1}((a, b), E), i = 1, 2, u_1 \leq u_2, v_1 \geq v_2 \text{ and } w_1 \leq w_2 \).

(\( c_1 \)) \( c_{\pm} \in E, \text{ and } c_{-} \leq c(u_2, v_2, w_2) \leq c(u_1, v_1, w_1) \leq c_{+} \text{ whenever } u_1, v_1, w_1 \in L_{1}^{1}((a, b), E), i = 1, 2, u_1 \leq u_2, v_1 \geq v_2 \text{ and } w_1 \leq w_2 \).

(\( d_1 \)) \( d_{\pm} \in E, \text{ and } d_{-} \leq d(u_1, v_1, w_1) \leq d(u_2, v_2, w_2) \leq d_{+} \text{ whenever } u_1, v_1, w_1 \in L_{1}^{1}((a, b), E), i = 1, 2, u_1 \leq u_2, v_1 \geq v_2 \text{ and } w_1 \leq w_2 \).

The next theorem is our main existence and comparison result for BVP (5.1).

**Theorem 5.1.** Assume that hypotheses (\( p_1 \)), (\( f_0 \)), (\( f_1 \)), (\( c_1 \)) and (\( d_1 \)) hold. Then BVP (5.1) has least and greatest solutions in \( Y \), and they are increasing with respect to \( f \) and \( d \) and decreasing with respect to \( c \).

**Proof.** Assume that \( P = L_{1}^{1}((a, b), E)^{3} \) is ordered by
\[
 (u_1, v_1, w_1) \leq (u_2, v_2, w_2) \quad \text{ if and only if } \quad u_1 \leq u_2, v_1 \geq v_2 \text{ and } w_1 \leq w_2. \tag{5.7}
\]
The relations
\[
x_{-}(t) := \left( d_{-} - \int_{a+}^{b-} \frac{1}{p(s)} \left( c_{+} - \int_{a+}^{s} h_{-}(\tau) \, d\tau \right) \, ds, \quad \frac{1}{p(t)} \left( c_{+} - \int_{a+}^{t} h_{-}(s) \, ds \right), \, h_{-}(t) \right),
\]
\[
x_{+}(t) := \left( d_{+} - \int_{a+}^{b-} \frac{1}{p(s)} \left( c_{-} - \int_{a+}^{s} h_{+}(\tau) \, d\tau \right) \, ds, \quad \frac{1}{p(t)} \left( c_{-} - \int_{a+}^{t} h_{+}(s) \, ds \right), \, h_{+}(t) \right), \tag{5.8}
\]
define functions $x_\pm \in P$, and $x_- \leq x_+$. Moreover, it is easy to show, by applying the given hypotheses, that the relations

$$
\begin{align*}
G_1(u, v, w)(t) &:= d(u, v, w) - \int_t^{b-} v(s) \, ds, \quad t \in (a, b), \\
G_2(u, v, w)(t) &:= \frac{1}{p(t)} \left( c(u, v, w) - \int_{a+}^t w(s) \, ds \right), \quad t \in (a, b), \\
G_3(u, v, w)(t) &:= \frac{f(t, u, v, w) + \lambda w(t)}{1 + \lambda}, \quad t \in (a, b)
\end{align*}
$$

(5.9)
define an increasing mapping $G = (G_1, G_2, G_3) : [x_-, x_+] \to [x_-, x_+]$.

Let $W$ be a well-ordered chain in $\text{ran} \, G$. The sets $W_1 = \{ u \mid (u, v, w) \in W \}$ and $W_3 = \{ w \mid (u, v, w) \in W \}$ are well ordered, $W_2 = \{ v \mid (u, v, w) \in W \}$ is inversely well ordered, and all three are order-bounded in $L^1_{\text{loc}}((a, b), E)$. It then follows from Lemma 2.4 that the suprema of $W_1$ and $W_3$ and an infimum of $W_2$ exist in $L^1_{\text{loc}}((a, b), E)$. Obviously, $(\sup W_1, \inf W_2, \sup W_3)$ is a supremum of $W$ in $(P, \preceq)$. Similarly one can show that each inversely well-ordered chain of ran $G$ has an infimum in $(P, \preceq)$.

The above proof shows that the operator $G = (G_1, G_2, G_3)$ defined by (5.9) satisfies the hypotheses of Lemma 2.3, whence $G$ has a least fixed point $x_\ast = (u_\ast, v_\ast, w_\ast)$ and a greatest fixed point $x^\ast = (u^\ast, v^\ast, w^\ast)$. It follows from (5.9) that $(u_\ast, v_\ast, w_\ast)$ and $(u^\ast, v^\ast, w^\ast)$ are solutions of system (5.3). According to Lemma 5.1 $u_\ast$ and $u^\ast$ belong to $Y$ and are solutions of IVP (5.1).

To prove that $u_\ast$ and $u^\ast$ are least and greatest of all solutions of (5.1) in $Y$, let $u \in Y$ be a solution of (5.1). In view of Lemma 5.1, $(u, v, w) = (u, u', Lu)$ is a solution of system (5.3). Applying the hypotheses (f1), (c1) and (d1) it is easy to show that $x = (u, v, w) \in [x_-, x_+]$, where $x_\pm$ are defined by (5.8). Thus $x = (u, v, w)$ is a fixed point of $G = (G_1, G_2, G_3) : [x_-, x_+] \to [x_-, x_+]$, defined by (5.9). Because $x_\ast = (u_\ast, v_\ast, w_\ast)$ and $x^\ast = (u^\ast, v^\ast, w^\ast)$ are least and greatest fixed points of $G$, then $(u_\ast, v_\ast, w_\ast) \preceq (u, v, w) \preceq (u^\ast, v^\ast, w^\ast)$. In particular, $u_\ast \preceq u \preceq u^\ast$, whence $u_\ast$ and $u^\ast$ are least and greatest of all solutions of IVP (5.1).

The last assertion is an easy consequence of the last conclusion of Lemma 2.3 and definition (5.9) of $G = (G_1, G_2, G_3)$. \qed

As a special case we obtain an existence result for BVP

$$
\begin{align*}
\frac{d}{dt} (p(t)u'(t)) = g(t, u(t), u'(t), -\frac{d}{dt} (p(t)u'(t))) & \quad \text{for a.e. } t \in (a, b), \\
\lim_{t \to a^+} p(t)u'(t) = c, \quad \lim_{t \to b^-} u(t) = d
\end{align*}
$$

(5.10)

**Corollary 5.1.** Let the hypothesis $(p_1)$ hold, and let $g : (a, b) \times E \times E \to E$ satisfy the following hypotheses:

$(g_0)$ $g(\cdot, u(\cdot), v(\cdot), w(\cdot))$ is Lebesgue measurable and $h_- \leq g(\cdot, u(\cdot), v(\cdot), w(\cdot)) \leq h_+$ for all $u, v, w \in L^1_{\text{loc}}((a, b), E)$ and for some $h_\pm \in Z$.

$(g_1)$ There exists $\lambda \geq 0$ such that $g(t, x_1, y_1, z_1) + \lambda z_1 \leq g(t, x_2, y_2, z_2) + \lambda z_2$ for a.e. $t \in (a, b)$ and whenever $x_1 \leq x_2$, $y_1 \geq y_2$ and $z_1 \leq z_2$ in $E$.

Then BVP (5.10) has for each choice of $c, d \in E$ least and greatest solutions in $Y$. Moreover, these solutions are increasing with respect to $g$ and $d$ and decreasing with respect to $c$. 

Proof. If \( c, d \in E \), BVP (5.10) is reduced to (5.1) when we define

\[
\begin{align*}
f(t, u, v, w) &= g(t, u(t), v(t), w(t)), \quad t \in (a, b), \quad u, v, w \in L^1_{\text{loc}}((a, b), E), \\
c(u, v, w) &\equiv c, \quad d(u, v, w) \equiv d, \quad u, v, w \in L^1_{\text{loc}}((a, b), E).
\end{align*}
\]

(5.11)

The hypotheses \((g_0)\) and \((g_1)\) imply that \( f \) satisfies the hypotheses \((f_0)\) and \((f_1)\). The hypotheses \((c_1)\) and \((d_1)\) is also valid, whence (5.1), with \( f, c \) and \( d \) defined by (5.11), and hence also (5.10), has by Theorem 5.1 least and greatest solutions. The last assertion follows from the last assertion of Theorem 5.1. \( \square \)

Example 5.1. Determine least and greatest solutions of the following system of BVPs:

\[
\begin{align*}
L_1u_1(t) &:= -\frac{d}{dt}(\sqrt{t}u_1'(t)) = \frac{1}{t} \sin \frac{1}{t} + \left[ 10 \tanh \left( \frac{1}{100} \int_1^2 (3u_2(s) - 2u'_2(s) + L_2u_2(s)) \, ds \right) \right] \quad \text{a.e. in } (0, 3), \\
L_2u_2(t) &:= -\frac{d}{dt}(\sqrt{t}u_2'(t)) = -\frac{1}{t} \sin \frac{1}{t} + \left[ 10 \arctan \left( \frac{1}{100} \int_1^2 (2u_1(s) - u'_1(s) + 3L_1u_1(s)) \, ds \right) \right] \quad \text{a.e. in } (0, 3), \\
\lim_{t \to 0^+} \sqrt{t}u_1'(t) &= \frac{[u_2'(1)]}{1 + [[u_2'(1)]]}, \quad u_1(3) = \frac{[u_2(1)]}{1 + [[u_2(1)]]}, \\
\lim_{t \to 0^+} \sqrt{t}u_2'(t) &= \frac{[u_1'(1)]}{1 + [[u_1'(1)]]}, \quad u_2(3) = \frac{[u_1(1)]}{1 + [[u_1(1)]]}.
\end{align*}
\]

(5.12)

Solution. (5.12) is a special case of (5.1) when \( E = \mathbb{R}^2 \), \( a = 0, b = 3 \), \( p(t) = \sqrt{t} \), and

\[
\begin{align*}
f(t, (u_1, u_2), (v_1, v_2), (w_1, w_2)) &= \left( \frac{1}{t} \sin \frac{1}{t} + \left[ 10 \tanh \left( \frac{1}{100} \int_1^2 (3u_2(s) - 2v_2(s) + w_2(s)) \, ds \right) \right] \right), \\
&\quad -\frac{1}{t} \sin \frac{1}{t} + \left[ 10 \arctan \left( \frac{1}{100} \int_1^2 (2u_1(s) - v_1(s) + 3w_1(s)) \, ds \right) \right], \\
c((u_1, u_2), (v_1, v_2), (w_1, w_2)) &= \left( \frac{[v_2(1)]}{1 + [[v_2(1)]]}, \frac{[v_1(1)]}{1 + [[v_1(1)]]} \right), \\
d((u_1, u_2), (v_1, v_2), (w_1, w_2)) &= \left( \frac{[u_2(1)]}{1 + [[u_2(1)]]}, \frac{[u_1(1)]}{1 + [[u_1(1)]]} \right).
\end{align*}
\]

(5.13)
The hypotheses \((f_0), (f_1), (c_1)\) and \((d_1)\) hold when
\[
h_{\pm}(t) = \left( \frac{1}{t} \sin \frac{1}{t} \pm 10, \frac{1}{t} \sin \frac{1}{t} \pm 16 \right).
\]
\(\lambda = 0\) and \(c_{\pm} = d_{\pm} = (\pm 1, \pm 1).\) Thus (5.12) has least and greatest solutions. The first components of the functions \(x_-\) and \(x_+\) defined by (5.8) are \((u_{1-}, u_{2-})\) and \((u_{1+}, u_{2+}),\) where
\[
u_{1-}(t) = -1 + 2\sqrt{t} + 2\sqrt{t} \text{Si}\left(\frac{1}{t}\right) + 4\sin\left(\frac{1}{t}\right)\sqrt{t} - 4\sqrt{2}\sqrt{\pi} \text{FresnelC}\left(\frac{\sqrt{2}}{\sqrt{t}\sqrt{\pi}}\right)
- \pi\sqrt{t} + \frac{20}{3}t^{3/2} - 22\sqrt{3} - 2\sqrt{3} \text{Si}\left(\frac{1}{3}\right) - 4\sin\left(\frac{1}{3}\right)\sqrt{3}
+ 4\sqrt{2}\sqrt{\pi} \text{FresnelC}\left(\frac{\sqrt{2}\sqrt{\pi}}{\sqrt{t}}\right) + \pi\sqrt{3},
\]
\[
u_{2-}(t) = -1 + 2\sqrt{t} - 2\sqrt{t} \text{Si}\left(\frac{1}{t}\right) - 4\sin\left(\frac{1}{t}\right)\sqrt{t} + 4\sqrt{2}\sqrt{\pi} \text{FresnelC}\left(\frac{\sqrt{2}}{\sqrt{t}\sqrt{\pi}}\right)
+ \pi\sqrt{t} - \frac{32}{3}t^{3/2} - 34\sqrt{3} + 2\sqrt{3} \text{Si}\left(\frac{1}{3}\right) + 4\sin\left(\frac{1}{3}\right)\sqrt{3}
- 4\sqrt{2}\sqrt{\pi} \text{FresnelC}\left(\frac{\sqrt{2}\sqrt{\pi}}{\sqrt{t}}\right) - \pi\sqrt{3},
\]
\[
u_{1+}(t) = 1 - 2\sqrt{t} + 2\sqrt{t} \text{Si}\left(\frac{1}{t}\right) + 4\sin\left(\frac{1}{t}\right)\sqrt{t} - 4\sqrt{2}\sqrt{\pi} \text{FresnelC}\left(\frac{\sqrt{2}}{\sqrt{t}\sqrt{\pi}}\right)
- \pi\sqrt{t} - \frac{20}{3}t^{3/2} + 22\sqrt{3} - 2\sqrt{3} \text{Si}\left(\frac{1}{3}\right) - 4\sin\left(\frac{1}{3}\right)\sqrt{3}
+ 4\sqrt{2}\sqrt{\pi} \text{FresnelC}\left(\frac{\sqrt{2}\sqrt{\pi}}{\sqrt{t}}\right) + \pi\sqrt{3},
\]
\[
u_{2+}(t) = 1 - 2\sqrt{t} - 2\sqrt{t} \text{Si}\left(\frac{1}{t}\right) - 4\sin\left(\frac{1}{t}\right)\sqrt{t} + 4\sqrt{2}\sqrt{\pi} \text{FresnelC}\left(\frac{\sqrt{2}}{\sqrt{t}\sqrt{\pi}}\right)
+ \pi\sqrt{t} - \frac{32}{3}t^{3/2} + 34\sqrt{3} + 2\sqrt{3} \text{Si}\left(\frac{1}{3}\right) + 4\sin\left(\frac{1}{3}\right)\sqrt{3}
- 4\sqrt{2}\sqrt{\pi} \text{FresnelC}\left(\frac{\sqrt{2}\sqrt{\pi}}{\sqrt{t}}\right) - \pi\sqrt{3}.
\]
According to Lemma 5.1 the least solution of (5.12) is equal to the first component of the least fixed point of \(G = (G_1, G_2, G_3),\) defined by (5.9), with \(f, c\) and \(d\) given by (5.13) and \(p(t) = \sqrt{t} .\) Calculating the first iterations \(G^n x_-\), it turns out that \(G^4 x_- = G^5 x_- .\) Thus \((u_{1+}, u_{2+}) = G_1^4 x_-\) is the least solution of (5.12). Similarly, one can show that \(G^6 x_+ = G^7 x_+ ,\) whence \((u_{1+}, u_{2+}) = G_1^6 x_+\) is the greatest solution of (5.12). The exact expressions of these solutions are:
\[
u_{1+}(t) = \frac{9}{10} - \frac{8}{5}\sqrt{t} + 2\sqrt{t} \text{Si}\left(\frac{1}{t}\right) + 4\sin\left(\frac{1}{t}\right)\sqrt{t} - 4\sqrt{2}\sqrt{\pi} \text{FresnelC}\left(\frac{\sqrt{2}}{\sqrt{t}\sqrt{\pi}}\right)
\]
\[-\pi \sqrt{t} - 2t^{3/2} + \frac{38}{5} \sqrt{3} - 2\sqrt{3} \sin\left(\frac{1}{3}\right) - 4 \sin\left(\frac{1}{3}\right) \sqrt{3} \]

\[+ 4\sqrt{2} \sqrt{\pi} \text{FresnelC}\left(\frac{1}{3}\sqrt{\pi}\right) + \pi \sqrt{3}, \]

\[u_2(t) = \frac{11}{12} - \frac{5}{3} \sqrt{t} - 2\sqrt{t} \sin\left(\frac{1}{3}\right) - 4 \sin\left(\frac{1}{3}\right) \sqrt{t} + 4\sqrt{2} \sqrt{\pi} \text{FresnelC}\left(\frac{1}{3}\sqrt{\pi}\right) \]

\[+ \pi \sqrt{t} - 2t^{3/2} + \frac{23}{3} \sqrt{3} + 2\sqrt{3} \sin\left(\frac{1}{3}\right) + 4 \sin\left(\frac{1}{3}\right) \sqrt{3} \]

\[- 4\sqrt{2} \sqrt{\pi} \text{FresnelC}\left(\frac{1}{3}\sqrt{\pi}\right) - \pi \sqrt{3}, \]

\[u_1(t) = -\frac{21}{22} + \frac{7}{4} \sqrt{t} + 2\sqrt{t} \sin\left(\frac{1}{3}\right) + 4 \sin\left(\frac{1}{3}\right) \sqrt{t} - 4\sqrt{2} \sqrt{\pi} \text{FresnelC}\left(\frac{1}{3}\sqrt{\pi}\right) \]

\[+ \pi \sqrt{t} + \frac{14}{3} t^{3/2} - \frac{63}{4} \sqrt{3} - 2\sqrt{3} \sin\left(\frac{1}{3}\right) - 4 \sin\left(\frac{1}{3}\right) \sqrt{3} \]

\[+ 4\sqrt{2} \sqrt{\pi} \text{FresnelC}\left(\frac{1}{3}\sqrt{\pi}\right) + \pi \sqrt{3}, \]

\[u_2(t) = -\frac{21}{22} + \frac{7}{4} \sqrt{t} - 2\sqrt{t} \sin\left(\frac{1}{3}\right) - 4 \sin\left(\frac{1}{3}\right) \sqrt{t} + 4\sqrt{2} \sqrt{\pi} \text{FresnelC}\left(\frac{1}{3}\sqrt{\pi}\right) \]

\[+ \pi \sqrt{t} + 4t^{3/2} - \frac{55}{4} \sqrt{3} + 2\sqrt{3} \sin\left(\frac{1}{3}\right) + 4 \sin\left(\frac{1}{3}\right) \sqrt{3} \]

\[- 4\sqrt{2} \sqrt{\pi} \text{FresnelC}\left(\frac{1}{3}\sqrt{\pi}\right) + \pi \sqrt{3}. \]

**Example 5.2.** Let \( E \) be the space \((c_0), \) ordered coordinatewise and normed by the sup-norm. The mappings \( h_\pm : (0, 3) \to c_0, \) defined by

\[ h_\pm(t) = \left( \frac{1}{n} \left( \sin \frac{1}{t} - \frac{5t^2 - 12}{2n\sqrt{3} - t} \pm \frac{1}{n} \right) \right)_{n=1}^\infty, \quad t \in (0, 3), \]

belong to \( Z, \) defined by (5.2). Thus these mappings are possible upper and lower boundaries for \( f \) in the hypothesis \((f_0)\) of Theorem 5.1 and for \( g \) in the hypothesis \((g_0)\) of Corollary 5.1 when \( E = c_0. \) Choosing \( c_\pm = (\frac{1}{n})_{n=1}^\infty, \) \( d_\pm = (\pm \frac{1}{n})_{n=1}^\infty \) and \( p(t) := \sqrt{t}, \) the solutions of the boundary value problems

\[-\frac{d}{dt}\left(\sqrt{t}u(t)\right) = h_\pm(t) \quad \text{for a.e. } t \in (0, b), \]

\[\lim_{t \to 0^+} \sqrt{t}u(t) = c_\pm, \quad \lim_{t \to 3^-} u(t) = d_\pm \]

are

\[ u_\pm(t) = \left( \frac{1}{n} \left( -\frac{t\sqrt{3}t - t^2}{4} + \frac{t^2\sqrt{3}t - t^2}{3} - \frac{9\sqrt{3}t - t^2}{8} + \frac{27 \arcsin(\frac{2}{3}t - 1)}{16} - \frac{27\pi}{32} \right) \right)_{n=1}^\infty, \]
In particular, the infinite system of initial value problems
e.g., in [7, Section 2.2] and [8, Section 5.8]. For instance, spaces
Remark 5.1.

\[
\begin{align*}
-2\sqrt{t} \text{Si} \left( \frac{1}{t} \right) - 4\sqrt{t} \sin \left( \frac{1}{t} \right) + 4\sqrt{2\pi} \text{ FresnelC} \left( \sqrt{\frac{2}{\pi t}} \right) + \frac{2t\sqrt{t}}{3} \\
+ (\pi + 2)\sqrt{t} + 2\sqrt{3} \text{Si} \left( \frac{1}{3} \right) + 4\sqrt{3} \sin \left( \frac{1}{3} \right) - 4\sqrt{2\pi} \text{ FresnelC} \left( \sqrt{\frac{2}{3\pi}} \right) \\
- 4\sqrt{3} - \pi \sqrt{3} + 1 \right) & \sum_{n=1}^{\infty}, \\
\end{align*}
\]

\[
\begin{align*}
u_-(t) &= \left( \frac{1}{n} \right) \left( -\frac{t \sqrt{3t - t^2}}{4} + \frac{t^2 \sqrt{3t - t^2}}{3} - \frac{9\sqrt{3t - t^2}}{8} + \frac{27 \arcsin \left( \frac{t}{\sqrt{3}} - 1 \right)}{16} - \frac{27\pi}{32} \\
- 2\sqrt{t} \text{Si} \left( \frac{1}{t} \right) - 4\sqrt{t} \sin \left( \frac{1}{t} \right) + 4\sqrt{2\pi} \text{ FresnelC} \left( \sqrt{\frac{2}{\pi t}} \right) - \frac{2t\sqrt{t}}{3} \\
+ (\pi - 2)\sqrt{t} + 2\sqrt{3} \text{Si} \left( \frac{1}{3} \right) + 4\sqrt{3} \sin \left( \frac{1}{3} \right) - 4\sqrt{2\pi} \text{ FresnelC} \left( \sqrt{\frac{2}{3\pi}} \right) \\
+ 4\sqrt{3} - \pi \sqrt{3} - 1 \right) & \sum_{n=1}^{\infty}. \\
\end{align*}
\]

(5.16)

In particular, the infinite system of initial value problems

\[
\begin{align*}
\begin{cases}
L_n u_n(t) := -\frac{d}{dt} (\sqrt{t} u'_n(t)) = \frac{1}{n} \left( \frac{1}{t} \sin \frac{1}{t} + f_n(u, u', Lu) \right) \text{ for a.e. } t \in (0, b), \\
\lim_{t \to 0+} (\sqrt{t} u'_n(t)) = \frac{c_n}{n}, \quad \lim_{t \to b-} u_n(t) = \frac{d_n}{n}, \quad n = 1, 2, \ldots,
\end{cases}
\end{align*}
\]

(5.17)

where \( u = (u_n)_{n=1}^{\infty}, Lu = (L_n u_n)_{n=1}^{\infty}, \) each \( f_n : L^1_{\text{loc}}((0, b), c_0) \to \mathbb{R} \) is increasing with respect to the first and third arguments and decreasing with respect to the second argument, and \( -1 \leq c_n, d_n, f_n(u, v, w) \leq 1 \) for all \( u, v, w \in L^1_{\text{loc}}((0, b), c_0) \) and \( n = 1, 2, \ldots, \) has least and greatest solutions \( u_* = (u_{n,*})_{n=1}^{\infty} \text{ and } u^* = (u_{n,*})_{n=1}^{\infty}, \) and they belong to the order interval \([u_-, u_+],\) where \( u_{\pm} \) are given by (5.16).

**Remark 5.1.** Examples of ordered Banach spaces whose order cones are regular are given, e.g., in [7, Section 2.2] and [8, Section 5.8]. For instance, spaces \( \mathbb{R}^m, m = 1, 2, \ldots, \) ordered coordinatewise and normed by any norm, spaces \( L^p, p \in [1, \infty), \) and \( c_0, \) ordered componentwise and normed by their usual norms, and spaces \( L^p(\Omega, \mathbb{R}), \) where \( p \in [1, \infty) \) and \( \Omega = (\Omega, \mathcal{A}, \mu) \) is a measure space, equipped with \( p \)-norm and a.e. pointwise ordering, have regular order cones. In particular, we can choose \( E \) to be one of these spaces in the above considerations.

Problems of the form (3.1), (4.1) and (5.1) include many kinds of special types. For instance, they can be:

- singular, because case \( \lim_{t \to a+} p(t) = 0 \) is allowed, and since limits \( \lim_{t \to a+} f(t, u, v) \) and/or \( \lim_{t \to b-} f(t, u, v) \) need not to exist;
- functional, because the functions \( c, d \) and \( f \) may depend functionally on \( u, u' \) and/or \( Lu; \)
– discontinuous, because the dependencies of $c$, $d$ and $f$ on $u$, $u'$ and/or $Lu$ can be discontinuous;
– problems on unbounded intervals, because cases $a = -\infty$ and/or $b = \infty$ are included;
– finite systems when $E = \mathbb{R}^m$;
– infinite systems when $E$ is $l^p$ or $c_0$-space;
– of random type when $E = L^p(\Omega)$ and $\Omega$ is a probability space.

Problems which include some of the types listed above when $E = \mathbb{R}$ are studied, e.g., in [1–4,9–13]. Initial and boundary value problems in ordered Banach spaces are studied, e.g., in [5–8].

The solutions of examples have been calculated by using Maple 9 and simple Maple programming.

References