g-Elements, finite buildings and higher Cohen–Macaulay connectivity

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Abstract

Chari proved that if Δ is a $(d-1)$-dimensional simplicial complex with a convex ear decomposition, then $h_0 \leqslant \cdots \leqslant h_{\lfloor d/2 \rfloor}$ [M.K. Chari, Two decompositions in topological combinatorics with applications to matroid complexes, Trans. Amer. Math. Soc. 349 (1997) 3925–3943]. Nyman and Swartz raised the problem of whether or not the corresponding g-vector is an M-vector [K. Nyman, E. Swartz, Inequalities for h- and flag h-vectors of geometric lattices, Discrete Comput. Geom. 32 (2004) 533–548]. This is proved to be true by showing that the set of pairs (ω, Θ), where Θ is a l.s.o.p. for $k[\Delta]$, the face ring of Δ, and ω is a g-element for $k[\Delta]/\Theta$, is nonempty whenever the characteristic of k is zero.

Finite buildings have a convex ear decomposition. These decompositions point to inequalities on the flag h-vector of such spaces similar in spirit to those examined in [K. Nyman, E. Swartz, Inequalities for h- and flag h-vectors of geometric lattices, Discrete Comput. Geom. 32 (2004) 533–548] for order complexes of geometric lattices. This also leads to connections between higher Cohen–Macaulay connectivity and conditions which insure that $h_0 < \cdots < h_i$ for a predetermined i.
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0. Introduction

One of the most basic combinatorial invariants of a (finite) simplicial complex is its f-vector, or equivalently, its h-vector. In order to analyze h-vectors of matroid independence complexes Chari introduced the notion of a convex ear decomposition [7]. He showed that $(d-1)$-di-

mensional complexes which have such a decomposition satisfy $h_i \leqslant h_{d-i}$ and $h_i \leqslant h_{i+1}$ for all $i \leqslant \lfloor d/2 \rfloor$. In addition, he proved that independence complexes of matroids have a PS-ear decomposition, a special type of convex ear decomposition. Spaces with a PS-ear decomposition satisfy the additional condition that their g-vector, $(g_0, g_1, \ldots, g_{\lceil d/2 \rceil})$, where $g_i = h_i - h_{i-1}$, is an M-vector [27]. Our main result, Theorem 3.9, is that this holds for all spaces with a convex ear decomposition.

In Section 2 we introduce a convex ear decomposition for finite buildings. In addition to the enumerative conclusions above, this will allow an analysis of the flag h-vectors of such complexes. We end with an examination of a connection between higher Cohen–Macaulay connectivity and increasing h-vectors.

Throughout, Δ is a finite $(d-1)$-dimensional abstract simplicial complex with vertex set V, $|V| = n$. A maximal face (under inclusion) of Δ is a *facet*. The f-*vector* of Δ is $(f_0, \ldots, f_d)$, where f_i is the number of faces of Δ of cardinality i. (Note: Our f_i is frequently denoted by f_{i-1}.) The h-*vector* of Δ is $(h_0, \ldots, h_d)$ where

$$h_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_j(\Delta). \tag{1}$$

Equivalently,

$$f_j(\Delta) = \sum_{i=0}^{j} \binom{d-i}{d-j} h_i(\Delta). \tag{2}$$

We use $\Delta - v$ for the complex consisting of Δ with all of the faces containing the vertex v removed. Similarly, if $A \subseteq V$, then $\Delta - A$ is Δ with all of the faces which contain any vertex in A deleted.

The *order complex* of a poset P is the simplicial complex whose faces are the chains in P. However, if P contains a maximal element $\hat{1}$ or a minimal element $\hat{0}$, then we will always assume that the order complex refers to the poset $P - \{\hat{1}, \hat{0}\}$.

1. Convex ear decompositions

A *convex ear decomposition* of Δ is an ordered sequence $\Delta_1, \ldots, \Delta_m$ of pure $(d-1)$-dimensional subcomplexes of Δ such that:

(1) Δ_1 is the boundary complex of a simplicial d-polytope. For each $j = 2, \ldots, m$, Δ_j is a $(d-1)$-ball which is a proper subcomplex of the boundary of a simplicial d-polytope.
(2) For $j \geqslant 2$, $\Delta_j \cap (\bigcup_{i=1}^{j-1} \Delta_i) = \partial \Delta_j$.
(3) $\bigcup_{j=1}^{m} \Delta_j = \Delta$.

The *initial* subcomplex is Δ_1. Each Δ_j, for $j \geqslant 2$, is an *ear* of the decomposition.

Convex ear decompositions were originally introduced by Chari. His original example of a convex ear decomposition was the independence complex of a matroid. In fact, he proved that the independence complex of a matroid has a special type of convex ear decomposition, a *PS-ear decomposition*. In a PS-ear decomposition the initial subcomplex is a join of boundaries of simplices, and each ear is a join of a simplex and boundaries of simplices [7].

Using an idea of Björner [3], Nyman and Swartz showed that order complexes of geometric lattices have a convex ear decomposition [18]. In this case, the initial complex is the first

barycentric subdivision of the boundary of a simplex and each ear is a shellable ball which is a subcomplex of such a space. In addition to the enumerative conclusions of Theorem 3.9 below, this approach led to several inequalities for the flag h-vector of such complexes.

Definition 1.1. A *balanced* complex is a $(d-1)$-dimensional simplicial complex Δ and a map $\phi : V \to S$, $|S| = d$, such that $\phi(v) \neq \phi(w)$ for any pair of distinct vertices v and w which are contained in a face of Δ.

Equivalently, the one-skeleton of Δ is properly d-colorable. Our balanced complexes were called completely balanced in [21]. A common example of a balanced complex is the order complex of a ranked poset, with $\phi(x)$ the rank of x. For balanced complexes there is a natural refinement of the f- and h-vectors.

Definition 1.2. Let (Δ, ϕ) be a balanced complex. Let $A \subseteq S$.

$$\Delta_A = \big\{ \rho \in \Delta : \forall v \in \rho, \ \phi(v) \in A \big\}.$$
$$f_A = f_{|A|}(\Delta_A).$$
$$h_A = \sum_{B \subseteq A} (-1)^{|A-B|} f_B.$$

Proposition 1.3. [25, pp. 95–96] *Let* (Δ, ϕ) *be a pure balanced complex. Then*

$$f_i = \sum_{|A|=i} f_A, \qquad h_i = \sum_{|A|=i} h_A.$$

The inequalities for the flag h-vector of the order complex of a rank $d+1$ geometric lattice in [18] can be described in terms of the weak order (also known as the weak Bruhat order) on the symmetric group S_{d+1}. Let π be permutation in S_{d+1}. The *inversion* set of π is the set of all pairs $\mathrm{inv}(\pi) = \{(i, j) : 1 \leqslant i < j \leqslant d + 1, \ \pi^{-1}(i) > \pi^{-1}(j)\}$.

Definition 1.4. The *weak* order on S_{d+1} is defined by

$$\pi \leqslant \pi' \quad \leftrightarrow \quad \mathrm{inv}(\pi) \subseteq \mathrm{inv}(\pi').$$

The *descent set* of $\pi \in S_{d+1}$ is $\mathrm{des}(\pi) = \{i : \pi(i) > \pi(i+1)\}$. Obviously, $\mathrm{des}(\pi) \subseteq [d]$. Given $A \subseteq [d]$, the *descent class* of A is $D(A) = \{\pi \in S_{d+1} : \mathrm{des}(\pi) = A\}$. Finally, for A and B subsets of $[d]$ we say A *dominates* B if there exists an injection $\psi : D(B) \hookrightarrow D(A)$ such that $\pi \leqslant \psi(\pi)$ for all $\pi \in D(B)$.

Theorem 1.5. [18] *Let* Δ *be the order complex of a rank* $d+1$ *geometric lattice. If* A *and* B *are subsets of* $[d]$ *such that* A *dominates* B, *then*

$$h_B \leqslant h_A.$$

Several order complexes of posets have convex ear decompositions. These include rank-selected subposets of geometric lattices, supersolvable lattices with nonzero Möbius function on every interval and their rank-selected subposets, and d-divisible partition lattices ($d \geqslant 3$) [20].

The flag h-vectors of rank $(d + 1)$ supersolvable lattices with nonzero Möbius function on every interval also satisfy the conclusion of Theorem 1.5.

There is a general construction which includes all of the above examples and the buildings in the next section. Let Σ be a contractible, shellable d-polytopal complex. For polytopal shellings see, for instance, [30, Chapter 8]. Let $\Sigma_1, \ldots, \Sigma_m$ be a shelling order of the facets and assume that all of the facets are simplicial d-polytopes. Removing all the open d-cells leaves a $(d - 1)$-dimensional simplicial complex with a convex ear decomposition. The initial subcomplex is $\partial(\Sigma_1)$. For $2 \leqslant j \leqslant m$, the ear Δ_j is the closure of $\partial(\Sigma_j) - \bigcup_{i=1}^{j-1} \partial(\Sigma_i)$.

2. Finite buildings

Finite buildings have a convex ear decomposition. This is an immediate consequence of [11, Lemma 3.5]. We will use this decomposition when examining the complementary h-vector in Section 4. However, for reasons we will make clear below (see Theorem 2.4), we prefer another proof here.

There are several standard references on buildings. We mention [4,6,28], as all of the facts we use can be found in those references. Let (W, S) be a finite Coxeter system with associated Coxeter complex $\Sigma(W, S)$. Specifically, W is a finite group generated by reflections of (linear) hyperplanes in U, a d-dimensional real vector space, and S is a generating set of reflections defined below. The collection of hyperplanes is assumed to be essential, that is their intersection is the origin, and contains all of the hyperplanes of the reflections in W. The intersection of the unit sphere of U with the hyperplane arrangement results in a (spherical) simplicial complex $\Sigma(W, S)$ which is a triangulation of the $(d - 1)$-sphere.

The group W acts transitively and freely on the facets (also called chambers) of $\Sigma(W, S)$. Let σ be a fixed facet of $\Sigma(W, S)$. The simply transitive action of W on the facets allows us to identify $w \in W$ with the facet $w \cdot \sigma$. The linear span of each $(d - 2)$-face of the boundary of σ is one of the hyperplanes in the arrangement. The corresponding set of reflections is $S = \{s_1, \ldots, s_{d-1}\}$ and generates W.

Given $w \in W$, the minimal ℓ such that $w = s_{i_1} \cdots s_{i_\ell}$ is $\ell(w)$, the *length* of w. The *weak* order on W (also known as the weak Bruhat order) is defined by $w < w'$ if there exists $s_1, \ldots, s_j \in S$ such that $w \cdot s_1 \cdots s_j = w'$ and $\ell(w') = \ell(w) + j$. An equivalent formulation given by the geometry of $\Sigma(W, S)$ is as follows. A *path* (also called a gallery) in $\Sigma(W, S)$ is a sequence, $(\sigma_0, \ldots, \sigma_t)$, of facets such that for each i the intersection of σ_i and σ_{i+1} is a $(d - 2)$-face (usually called a wall). The length of the path is t. A *geodesic* is a path of minimal length among all paths beginning and ending with the same facets. The weak order on (W, S) is equivalent to $w \leqslant w'$ if and only if there is a geodesic from σ to $w' \cdot \sigma$ which contains $w \cdot \sigma$.

Example 2.1. Let W be the group generated by the collection of hyperplanes $H_{ij} = \{(x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} : x_i = x_j\}, i < j$. This is not an essential arrangement as it contains the line $x_1 = \cdots = x_{d+1}$. Intersecting with the d-dimensional subspace $U = \{(x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} : x_1 + \cdots + x_{d+1} = 0\}$ does give an essential arrangement. The group generated by the corresponding reflections in U is the symmetric group S_{d+1}. If we choose σ to be the facet which contains those $(x_1, \ldots, x_{d+1})$ such that $x_1 < \cdots < x_{d+1}$, then S corresponds to the transpositions $(i, i + 1)$ and the weak order on (W, S) is the same as defined in Section 1.

The *descent* set of $w \in W$ is $\mathrm{des}(w) = \{s \in S : \ell(w \cdot s) < \ell(w)\}$. The descent set of w can also be defined via $\Sigma(W, S)$. Let $V_{(W,S)}$ be the vertices of the Coxeter complex $\Sigma(W, S)$. For each

$(d - 2)$-face τ of $\partial\sigma$, set $\psi(\tau)$ to be the corresponding hyperplane in S. Define $\psi : V_{(W,S)} \to S$ by first defining ψ on the vertices of σ to be ψ of the opposite face. Then extend ψ to a labeling of all of $V_{\Sigma(W,S)}$ in the only way possible that insures that $\Sigma(W, S)$ and ψ form a balanced complex. This also labels the $(d - 2)$-faces of $\Sigma(W, S)$. Simply assign $\psi(\tau)$ to be $\psi(v)$, where v is any vertex such that $\tau \cup \{v\}$ is a facet. With this definition of ψ on the $(d - 2)$-faces of $\Sigma(W, S)$, the descent set of w is the set of all s such that there exists a geodesic $(\sigma, \dots, \sigma', w \cdot \sigma)$ with $\psi(\sigma' \cap (w \cdot \sigma)) = s$.

Definition 2.2. Let $\Sigma(W, S)$ be a finite Coxeter complex. A *finite building* of type (W, S) is a (finite) simplicial complex Δ which is the union of subcomplexes Σ, called *apartments*, such that:

- Each apartment Σ is isomorphic to $\Sigma(W, S)$.
- For any two faces ρ_1 and ρ_2 in Δ, there is an apartment Σ containing both of them.
- If Σ and Σ' are two apartments containing ρ_1 and ρ_2, then there is an isomorphism $\Sigma \to \Sigma'$ fixing ρ_1 and ρ_2 pointwise.

For the rest of this section Δ is a finite building of type (W, S). Let τ be a facet of Δ and let ρ be any face of Δ. A geodesic from τ to ρ is a geodesic $(\tau = \tau_0, \dots, \tau_t)$ such that $\rho \not\subseteq \tau_i$ for any $i < t$ and $\rho \subseteq \tau_t$. There exists a unique facet $p_\rho(\tau)$, the *projection* of τ on ρ, such that every geodesic from τ to ρ ends with $p_\rho(\tau)$ [28, 3.18–3.19].

Let σ be any fixed base facet. A facet σ_i is *opposite* σ if it is maximally distant from σ. Let $\sigma_1, \dots, \sigma_m$ be the facets opposite σ. For each σ_j, there is a unique apartment, Σ_j, which contains σ and σ_j. It is the union of all geodesics from σ to σ_j. Finally, set $\Delta_1 = \Sigma_1$ and for $j \geqslant 2$, define Δ_j to be the union of the facets of Σ_j not contained in any Σ_i, $i < j$. Since $\Delta = \bigcup \Sigma_j$, it is immediate that $\Delta = \bigcup \Delta_j$.

Theorem 2.3. *Let $\Delta, \Delta_1, \dots, \Delta_m$ be as above. Then $\Delta_1, \dots, \Delta_m$ is a convex ear decomposition of Δ.*

Proof. We begin by proving that for $j \geqslant 2$, Δ_j is a shellable subcomplex of Σ_j, and hence a ball. Since σ_j is not in any other Σ_i, $\sigma_j \in \Delta_j$. What other facets of Σ_j are contained in Δ_j? A facet $\tau \in \Sigma_j$ is in Δ_j if and only if for all $i < j$, τ it is not contained in any geodesic from σ to σ_i.

Let $\tau_1, \dots, \tau_t$ be an ordering of the facets in Δ_j which is a linear extension of the order dual of the weak order restricted to Δ_j. Specifically, if $\tau < \tau'$, $\tau = \tau_k$, and $\tau' = \tau_l$, then $l < k$. From the above discussion we know that if $\tau < \tau'$ and $\tau \in \Delta_j$, then $\tau' \in \Delta_j$. Thus, $\tau_1, \dots, \tau_t$ is an initial segment of a linear extension of the order dual of the weak order on all of Σ_j. By [4, Theorem A.1], $\tau_1, \dots, \tau_t$ is an initial segment of a shelling of Σ_j and hence Δ_j is a ball.

In order to see that $\Delta_j \cap \bigcup_{i=1}^{j-1} \Delta_i = \partial \Delta_j$, we first note that a face ρ is in $\partial \Delta_j$ if and only if ρ is contained in facets τ_1 and τ_2 with $\tau_1 \in \Delta_j$ and $\tau_2 \in \Sigma_j - \Delta_j$. Now suppose ρ is a face in Δ_j and Δ_i with $i < j$. Let $\tau = p_\rho(\sigma_i)$, the projection of τ on ρ. Since $\rho \in \Sigma_i \cap \Sigma_j$, τ must also be in $\Sigma_i \cap \Sigma_j$. Hence, $\rho \subseteq \tau$ with τ a facet of Σ_j not in Δ_j. Thus $\rho \subseteq \partial \Delta_j$. The other inclusion is obvious. $\quad\square$

The above construction suggests inequalities for the flag h-vector of a finite building similar in spirit to Theorem 1.5. For $A \subseteq S$ the *descent class* of A is $D(A) = \{w \in W : \text{des } w = A\}$.

If $A, B \subseteq S$, then A *dominates* B if there exists an injection $\psi : D(B) \hookrightarrow D(A)$ such that $w < \psi(w)$ for all $w \in D(B)$.

Theorem 2.4. *Let (W, S) be a finite Coxeter system. Let A and B be subsets of S and assume A dominates B. If Δ is a finite building of type (W, S), then*

$$h_B \leqslant h_A.$$

Proof. Let $\psi : D(B) \hookrightarrow D(A)$ be an injection such that $w < \psi(w)$ for all $w \in D(B)$. Let σ be a fixed base facet. For any $B \subseteq S$, h_B is the number of facets τ in Δ whose descent set is B, where the descent set is computed in any apartment containing σ and τ [4]. By the proof of Theorem 2.3, each τ with descent set B is in exactly one Δ_j, and if τ corresponds to w in Σ_j, then the facet τ' which corresponds to $\psi(w)$ in Σ_j is also in Δ_j and has descent set A. Hence, there is an injection from facets with descent set B to facets with descent set A. □

For buildings associated to $GF(q)$, the above theorem follows easily from the formula [4]

$$h_A = \sum_{w \in D(A)} q^{\ell(w)}. \tag{3}$$

One possible approach to looking for a combinatorial proof of some of the inequalities implied by Corollary 3.10 for finite buildings would be to consider the following problems.

Problem 2.5. Let (W, S) be a finite Coxeter system and let $d - 1$ be the dimension of the Coxeter complex associated to (W, S).

(1) For which subsets A, B of S does A dominate B?
(2) Suppose $i \leqslant d/2$. Does there exist an injection ι from i-subsets of S to $(i + 1)$-subsets of S such that $\iota(B)$ dominates B for all i-subsets B?
(3) Is there an injection $\alpha : \bigcup_{|B|=i} D(B) \hookrightarrow \bigcup_{|A|=i+1} D(A)$ such that $w < \alpha(w)$?
(4) Is there a bijection $\beta : \bigcup_{|B|=i} D(B) \hookrightarrow \bigcup_{|A|=d-i} D(A)$ such that $w < \beta(w)$?

Some of these problems were explored for the symmetric group in [18].

3. Face rings

One of the most powerful tools for studying enumerative properties of simplicial complexes, especially Cohen–Macaulay complexes, is the *face ring*, also known as the *Stanley–Reisner ring*. Let k be any field and set $R = k[x_1, \ldots, x_n]$. For any homogeneous ideal I of R we use $(R/I)_i$ to represent the degree i component of R/I.

Definition 3.1. The *face ring* of Δ is

$$k[\Delta] = R/I_\Delta,$$

where $I_\Delta = \langle \{x_{i_1} \cdots x_{i_j} : \{v_{i_1}, \ldots, v_{i_j}\} \notin \Delta\} \rangle$.

Let $\Theta = \{\theta_1, \ldots, \theta_d\}$ be a set of one-forms in R. We will also use Θ to represent the ideal $\langle \Theta \rangle$ in R or $k[\Delta]$, relying on context to make it clear what is intended. Write each $\theta_i = \theta_{i1}x_1 +$

$\cdots + \theta_{in} x_n$ and let T be the matrix (θ_{ij}). To each facet σ in Δ let $T|_\sigma$ be the submatrix of T, $(\theta_{ij})_{v_j \in \sigma}$, consisting of the columns of T corresponding to the vertices of σ. If $T|_\sigma$ has rank $|\sigma|$ for every facet σ, then Θ is a *linear set of parameters* (l.s.o.p.) for $k[\Delta]$. If k is infinite, then it is always possible to choose Θ such that every set of d columns of T is independent.

Two of the most useful facts concerning $k[\Delta]$ are the following.

Theorem 3.2. [19] *Let $k[\Delta]$ be the face ring of Δ. Then $k[\Delta]$ is a Cohen–Macaulay ring if and only if for all faces σ,*

$$\tilde{H}_j(\operatorname{lk} \sigma; k) = 0 \quad \textit{for all } j < d - |\sigma| - 1.$$

We call Δ a *Cohen–Macaulay* complex if $k[\Delta]$ is a Cohen–Macaulay ring. It can be shown that the property of being Cohen–Macaulay is a purely topological property [16].

A Mayer–Vietoris argument applied inductively to the number of ears shows that spaces with a convex ear decomposition are Cohen–Macaulay. As we shall see in Section 4, they are doubly Cohen–Macaulay.

Theorem 3.3. [23] *If Δ is a Cohen–Macaulay complex, then for any l.s.o.p. Θ,*

$$h_i(\Delta) = \dim_k \big(k[\Delta]/\Theta \big)_i.$$

An immediate consequence of this result is that $h_i \geqslant 0$ for any Cohen–Macaulay complex. In addition, the h-vector of a CM complex is an M-vector. A sequence $(h_0, h_1, \ldots, h_d)$ is an *M-vector* if it is the Hilbert function of a homogeneous quotient of a polynomial ring. A purely arithmetic criterion is the following description due to Macaulay.

Given positive integers h and i there is a unique way of writing

$$h = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

so that $a_i > a_{i-1} > \cdots > a_j \geqslant j \geqslant 1$. Define

$$h^{\langle i \rangle} = \binom{a_i + 1}{i + 1} + \binom{a_{i-1} + 1}{i} + \cdots + \binom{a_j + 1}{j + 1}.$$

Theorem 3.4. [25, Theorem 2.2] *A sequence of nonnegative integers $(h_0, \ldots, h_d)$ is an M-vector if and only if $h_0 = 1$ and $h_{i+1} \leqslant h_i^{\langle i \rangle}$ for all $1 \leqslant i \leqslant d - 1$.*

We denote the *canonical module* of $k[\Delta]$ by $\Omega(k[\Delta])$. The canonical module is an R-module and can be defined using homological methods. In fact, I_Δ is contained in the annihilator of $\Omega(k[\Delta])$, so the canonical module is also a $k[\Delta]$-module. The only properties of $\Omega(k[\Delta])$ that we will use are in the following theorem. As usual, $k[\Theta]$ is the ring $k[\theta_1, \ldots, \theta_d]$.

Theorem 3.5. [25, Sections I.12, II.7] *Let Δ be a $(d-1)$-dimensional Cohen–Macaulay complex and let $\Theta = \{\theta_1, \ldots, \theta_d\}$ be a l.s.o.p. for $k[\Delta]$.*

(1) $\Omega(k[\Delta]) \cong \operatorname{Hom}_{k[\Theta]}(k[\Delta], k[\Theta])$, *where the R-module structure is given by $(f\phi)(p) = \phi(f \cdot p)$, $f \in R$, $\phi \in \Omega(k[\Delta])$ and $p \in k[\Delta]$.*

(2) *There is a grading of $\Omega(k[\Delta])$ so that as an $\mathbb{N}$-graded R-module*

$$\dim_k\big(\Omega\big(k[\Delta]\big)/\Theta\,\Omega\big(k[\Delta]\big)\big)_i = \dim_k\big(k[\Delta]/\Theta\big)_{d-i}.$$

The grading of $\Omega(k[\Delta])$ is such that if $f \in \Omega(k[\Delta])$ maps degree i elements of $k[\Delta]$ to degree $i - d$ elements of $k[\Theta]$, then f has degree 0. Note that this is a shift of the usual grading of $\Omega(k[\Delta])/\Theta\,\Omega(k[\Delta])$ whose first nonzero module is in degree $-d$ instead of 0.

Let Δ be a Cohen–Macaulay complex and let Θ be a l.s.o.p. for $k[\Delta]$. A *g-element* for $k(\Delta) = k[\Delta]/\Theta$ is a one-form $\omega \in R$ such that multiplication

$$\omega^{d-2i} : k(\Delta)_i \to k(\Delta)_{d-i}$$

is an injection for every $i, 0 \leqslant i \leqslant \lfloor d/2 \rfloor$. When the multiplication maps are isomorphisms ω is usually called a Lefschetz element.

Let $G(\Delta)$ be the set of all pairs $(\omega, \Theta) \subseteq k^{n(d+1)}$ such that Θ is a l.s.o.p. for $k[\Delta]$ and ω is a *g*-element for $k(\Delta)$. While the following is well known, we include it for the sake of completeness.

Proposition 3.6. *If Δ is a Cohen–Macaulay complex, then $G(\Delta)$ is a Zariski open set.*

Proof. Given $i < d/2$, let G_i be the set of pairs (ω, Θ) such that $\omega^{d-2i} : k(\Delta)_i \to k(\Delta)_{d-i}$ is an injection and Θ is a l.s.o.p. for $k[\Delta]$. As $G(\Delta)$ is the intersection of all of the G_i, it is sufficient to show that each G_i is a Zariski open set.

Since Δ is Cohen–Macaulay, it is a pure complex, so for each facet $\sigma \in \Delta$, $T|_\sigma$ is a square matrix. Let f_σ be its determinant. Each f_σ is polynomial in the θ_{ij}. Therefore, Θ is a l.s.o.p. for $k[\Delta]$ if and only if the product of all the f_σ is nonzero. Denote this product by f_Δ.

Let $\mathcal{M}$ be the collection of all subsets U of monomials in R of degree less than or equal to d such that the number of monomials of degree i in U is h_i. The monomials in U form a k-basis of $k(\Delta)$ if and only if for each j the collection of all the monomials of the form $u_s \cdot v_t$, with u_s a monomial of degree s in $k[\Theta]$, v_t a monomial of degree t in U, and $s + t = j$, form a basis of $k[\Delta]_j$. Hence, there is a polynomial, f_U, a product of $d + 1$ determinants (one for each degree) in the θ_{ij}-variables, such that $f_\Delta f_U$ is nonzero if and only if Θ is a l.s.o.p. for $k[\Delta]$ and the monomials in U are a k-basis of $k(\Delta)$.

Fix $U \in \mathcal{M}$. For each j, let U_j be the degree j monomials in U. Now we attempt to compute the matrix for multiplication $\omega^{d-2i} : k(\Delta)_i \to k(\Delta)_{d-i}$ using the "basis" U_i for $k(\Delta)_i$, and the "basis" U_{d-i} for $k(\Delta)_{d-i}$. If Θ is a l.s.o.p. for $k[\Delta]$ and U_{d-i} is a basis for $k(\Delta)_{d-i}$, then we could compute the coefficients of $\omega^{d-2i} \cdot u$ for each $u \in U_i$ in the U_{d-i}-basis using Cramer's rule. These coefficients are rational functions in the ω and Θ variables, with the denominator equal to the determinant which indicates whether or not U_{d-i} is a basis of $k(\Delta)_{d-i}$. Instead, use Cramer's rule without the divisor. When U_{d-i} is a basis for $k(\Delta)_{d-i}$ the matrix for the linear transformation $\omega^{d-2i} : k(\Delta)_i \to k(\Delta)_{d-i}$ we obtain will be a nonzero scalar multiple of the correct matrix. In any event, the coefficients of the matrix are polynomials in the ω and Θ variables.

Let A be a subset of U_{d-i} of cardinality h_i. If $h_i > h_{d-i}$, then there are no such A. However, this implies that $G_i = \emptyset$, a Zariski open set. So we may assume that there are such sets. Let f_A be the determinant of the corresponding $h_i \times h_i$ minor of the matrix determined by our modified Cramer's rule. If $f_A f_U f_\Delta(\omega, \Theta) \neq 0$ for some U and A, then $(\omega, \Theta) \in G_i$. Conversely, suppose $(\omega, \Theta) \in G_i$. Since there exists some $U \in \mathcal{M}$ such that the monomials in U form a k-basis of

$k(\Delta)$, there must be some h_i-subset A of some U_{d-i} so that $f_A f_U f_\Delta(\omega, \Theta) \neq 0$. As there are only finitely many polynomials $f_A f_U f_\Delta$, G_i is a Zariski open set. $\square$

Using the above proposition, the necessary part of the g-theorem for simplicial polytopes can be stated in the following form.

Theorem 3.7. [14,24] *Let Δ be the boundary complex of a simplicial polytope and let k be a field of characteristic zero. Then $G(\Delta)$ is not empty.*

Kalai and Stanley used Theorem 3.7 to establish restrictions on the h-vectors of balls which were full dimensional subcomplexes of the boundary of a simplicial polytope.

Theorem 3.8. [13,22] *Suppose Δ is homeomorphic to a $(d-1)$-ball and is a subcomplex of Σ, where Σ is the boundary of a simplicial d-polytope. Then for any $\omega \in G(\Sigma)$, multiplication $\omega^{d-2i} : k(\Delta)_i \to k(\Delta)_{d-i}$ is surjective.*

Theorem 3.9. *If Δ has a convex ear decomposition and the characteristic of k is zero, then $G(\Delta)$ is not empty.*

Proof. The proof is by induction on m, the number of ears. Theorem 3.7 is $m = 1$. Let $\Sigma = \bigcup_{j=1}^{m-1} \Delta_j$. Let I be the kernel in the short exact sequence of R-modules

$$0 \to I \to k[\Delta] \to k[\Sigma] \to 0.$$

Evidently I is the ideal of $k[\Delta]$ generated by the interior faces of Δ_m. As an R-module, I is also the kernel in the short exact sequence

$$0 \to I \to k[\Delta_m] \to k[\partial\Delta_m] \to 0.$$

By a theorem of Hochster [25, Theorem II.7.3], I is isomorphic to $\Omega(k[\Delta_m])$ as a $\mathbb{Z}$-graded module.

Dividing out by Θ gives the short exact sequence of R-modules

$$0 \to I/(I \cap \Theta) \to k(\Delta) \to k(\Sigma) \to 0.$$

For each i, $\dim_k(I/(I \cap \Theta))_i = h_i(\Delta) - h_i(\Sigma) = h_i(\Delta_m) - g_i(\partial\Delta_m)$. This is $h_{d-i}(\Delta_m)$ [22]. Now, $(I/\Theta I)_i \cong (\Omega(k[\Delta_m])/\Theta\Omega(k[\Delta_m]))_i$, so $\dim_k(I/\Theta I)_i = h_{d-i}(\Delta_m)$. Since $\Theta I \subseteq I \cap \Theta$ we must have

$$\Omega(k[\Delta_m])/\Theta\Omega(k[\Delta_m]) \cong I/\Theta I \cong I/(I \cap \Theta).$$

Consider the commutative diagram,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I/(I \cap \Theta)_i & \longrightarrow & k(\Delta)_i & \longrightarrow & k(\Sigma)_i & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \omega^{d-2i}} & & \downarrow{\scriptstyle \omega^{d-2i}} & & \downarrow{\scriptstyle \omega^{d-2i}} & & \\
0 & \longrightarrow & I/(I \cap \Theta)_{d-i} & \longrightarrow & k(\Delta)_{d-i} & \longrightarrow & k(\Sigma)_{d-i} & \longrightarrow & 0
\end{array}
$$

As $\Omega(k[\Delta_m])/\Theta\Omega(k[\Delta_m]) \cong \mathrm{Hom}_k(k(\Delta_m), k)$, $I/(I \cap \Theta)$ must also be isomorphic to $\mathrm{Hom}_k(k(\Delta_m), k)$. By Theorem 3.8, multiplication $\omega^{d-2i} : k(\Delta_m)_i \to k(\Delta_m)_{d-i}$ is a surjection for the nonempty Zariski open set $G(\partial P_m)$, where P_m is a simplicial d-polytope such that $\Delta_m \subset \partial P_m$. As the left-hand vertical arrow is the k-dual for this map, it must be an injection

for pairs $(\omega, \Theta) \in G(\partial P_m)$. The induction hypothesis provides another nonempty Zariski open subset of pairs such that the right-hand vertical arrow is an injection. The intersection of these two sets is a nonempty Zariski open subset such that the middle vertical arrow is an injection. □

We note that if at some point in the future Theorem 3.7 is extended to a more general class of homology spheres, say $\mathcal{S}$, then it would be possible to define $\mathcal{S}$ ear decompositions. In that case, the above proof would still be valid.

Corollary 3.10. *Let Δ be a $(d-1)$-dimensional simplicial complex with a convex ear decomposition.*

(a) *If $i \leqslant \lfloor d/2 \rfloor$, then $h_i \leqslant h_{d-i}$ and $h_i \leqslant h_{i+1}$.*
(b) *If $g_i = h_i - h_{i-1}$, then $(g_0, g_1, \ldots, g_{\lceil d/2 \rceil})$ is an M-vector.*

Proof. Let ω be a g-element for $k(\Delta)$ and let $i \leqslant \lfloor d/2 \rfloor$. As multiplication $\omega^{d-2i}: k(\Delta)_i \to k(\Delta)_{d-i}$ is an injection, $h_i \leqslant h_{d-i}$ and multiplication $\omega: k(\Delta)_i \to k(\Delta)_{i+1}$ is also an injection. Hence $h_i \leqslant h_{i+1}$. To see that $(g_0, g_1, \ldots, g_{\lceil d/2 \rceil})$ is an M-vector we simply note that for $i \leqslant \lceil d/2 \rceil$, $g_i = \dim_k(k(\Delta)/ <\omega>)_i$. □

The inequalities in (a) are originally due to Chari [7].

Theorem 3.9 was first proved for independence complexes of rationally represented matroids by Hausel and Sturmfels [9]. They used the theory of hyperkähler toric varieties. Swartz proved this for all matroids [27]. In [8] Hausel presents a proof for all matroids which is based on ideas from both papers. The above proof is an extension of this idea to spaces with a convex ear decomposition.

4. Higher Cohen–Macaulay connectivity

In the previous section we said that spaces with a convex ear decomposition are doubly Cohen–Macaulay. A Cohen–Macaulay complex Δ is *doubly Cohen–Macaulay* if for all vertices v in Δ, the dimension of $\Delta - v$ equals the dimension of Δ, and $\Delta - v$ is still Cohen–Macaulay. More generally, Δ is q-CM if Δ is Cohen–Macaulay and for every subset A of vertices of Δ with $|A| < q$, $\dim(\Delta - A) = \dim \Delta$ and $\Delta - A$ is still Cohen–Macaulay. The maximum q such that Δ is q-CM is the *CM connectivity* of Δ.

Theorem 4.1. *If Δ has a convex ear decomposition, then Δ is doubly Cohen–Macaulay.*

As is conjecturally the case with Theorem 3.9, this result extends to a more general construction using homology spheres and balls since we only use the homological properties of balls and spheres in the proof. Here, a *homology sphere* has the homology of a sphere and every link has the homology of a sphere of the appropriate dimension. Homology spheres are also called *Gorenstein** complexes. A *homology ball* is homologically acyclic and every link is either a homology ball or a homology sphere of the appropriate dimension. Furthermore, the faces whose links are homology balls form a subcomplex, the boundary, which is a homology sphere of one lower dimension. Removing a vertex from a homology sphere leaves a homology ball with boundary the link of the vertex in the original homology sphere.

Proof. Let v be a vertex in Δ and let σ be a face of $\Delta - v$. We must show $\mathrm{lk}_{\Delta-v}\,\sigma$ is Cohen–Macaulay. The link of σ in $\Delta - v$ is the union of the links of σ in $\tilde{\Delta}_1 - v, \ldots, \tilde{\Delta}_t - v$, where the $\tilde{\Delta}_j$ is a sequential renumbering of all the Δ_i which contain σ. The proof is by induction on t. If $t = 1$, then the link is either a homology ball or homology sphere, depending on whether or not v is in the link of σ in $\tilde{\Delta}_1$. In either case the link is Cohen–Macaulay.

When $j > 1$, σ is on the boundary of $\tilde{\Delta}_j$, hence its link in $\tilde{\Delta}_j$ is a homology ball. Let $Y = \bigcup_{j=1}^{t-1} \mathrm{lk}_{\tilde{\Delta}_j - v}\,\sigma$ and $Z = \mathrm{lk}_{\tilde{\Delta}_t - v}\,\sigma$. For the induction step there are three possibilities. In each case, Mayer–Vietoris arguments are sufficient to compute the homology of $Y \cap Z$, Z and see that their union, $\mathrm{lk}_{\Delta-v}\,\sigma$, is Cohen–Macaulay.

(1) $v \notin \mathrm{lk}_{\tilde{\Delta}_t}\,\sigma$. Then Z is a $(d - 1 - |\sigma|)$-homology ball and $Y \cap Z$ is a $d - 2 - |\sigma|$-homology sphere.
(2) v is an interior point of $\mathrm{lk}_{\tilde{\Delta}_t}\,\sigma$. Then $Y \cap Z$ and Z have the homology of a $(d - 2 - |\sigma|)$-homology sphere. For Z, this follows from the Mayer–Vietoris sequence for the homology ball $\mathrm{lk}_{\tilde{\Delta}_t}\,\sigma$ written as the union of Z and the closed star of v in $\mathrm{lk}_{\tilde{\Delta}_t}\,\sigma$. In addition, the inclusion map $Y \cap Z \to Z$ is an isomorphism in homology.
(3) v is a boundary point of $\mathrm{lk}_{\tilde{\Delta}_t}\,\sigma$. Now $Y \cap Z$ and Z are homologically acyclic.   □

Doubly Cohen–Macaulay complexes and q-CM complexes were introduced by Baclawski [2]. Spheres, and more generally homology spheres, are 2-CM, but balls are not. Baclawski proved that the order complex of a semimodular poset P is 2-CM if and only if P is a geometric lattice. Furthermore, if P is a geometric lattice, then its order complex is q-CM if and only if every line of P has at least q atoms. In his study of buildings and Coxeter complexes Björner proved that finite buildings are 2-CM and any finite building associated to $\mathrm{GF}(q)$ is $(q + 1)$-CM [4]. Welker showed that order complexes of supersolvable lattices are 2-CM if and only if the Möbius function is nonzero on every interval [29]. Since all of the above examples of 2-CM complexes either have a convex ear decomposition, and hence satisfy the conclusion of Theorem 3.9, or are conjectured to satisfy Theorem 3.9, it seems natural to ask the following question which was also suggested by Björner.

Problem 4.2. Do all 2-CM complexes satisfy the conclusion of Theorem 3.9?

Nevo has shown that for 2-CM complexes and $d \geqslant 3$, $\omega: k(\Delta)_1 \to k(\Delta)_2$ is injective for a generic set of pairs (ω, Θ) [17]. An affirmative answer to this question would also show that all homology spheres satisfy the g-theorem [25, Conjecture II.6.2]. The face ring of any 2-CM complex is a level ring (see [25, p. 94] for a discussion). Combined with [12], this implies that for any 2-CM complex

$$h_0 + h_1 + \cdots + h_i \leqslant h_d + h_{d-1} + \cdots + h_{d-i}.$$

Another property shared by all $(d - 1)$-dimensional 2-CM complexes which comes from the fact that their face rings are level, is that the reversed h-vector, $(h_d, \ldots, h_0)$ is a sum of h_d M-vectors [23]. For spaces with a convex ear decomposition, this fact expresses itself in the following formula [7]:

$$h_{d-i}(\Delta) = h_i(\Delta_1) + \cdots + h_i(\Delta_m). \tag{4}$$

When Δ has a convex ear decomposition, $h_{d-i} - h_i$ is nonnegative for $i \leqslant d/2$. So it seems natural to consider the following complementary h-vector of Δ.

Definition 4.3. The *complementary h-vector* of Δ is

$$\bar{h} = (h_d - h_0, h_{d-1} - h_1, \ldots, h_{d-\lfloor d/2 \rfloor} - h_{d-\lceil d/2 \rceil}).$$

For any homology ball Δ, $h_i(\Delta) - h_{d-i}(\Delta) = g_i(\partial\Delta)$ (see, for instance, [15]). If $(\Delta_1, \ldots, \Delta_m)$ is a convex ear decomposition for Δ, then Eq. (4) implies

$$h_{d-i}(\Delta) - h_i(\Delta) = \sum_{j=2}^{m} h_i(\Delta_j) - h_{d-i}(\Delta_j) = \sum_{j=2}^{m} g_i(\partial\Delta_j).$$

Thus, if the boundaries of the ears are known to be combinatorially equivalent to the boundaries of $(d-1)$-polytopes, then by Theorem 3.7 $\bar{h}$ is the sum of $h_d - 1$ M-vectors.

Proposition 4.4. *If Δ has a PS-ear decomposition, then $\bar{h}$ is a sum of $h_d - 1$ M-vectors.*

Proof. The boundary of each ear in a PS-ear decomposition is the join of the boundaries of simplices. □

In order to analyze $\bar{h}$ when Δ is a finite building we use von Heydebreck's convex ear decomposition.

Proposition 4.5. [11, Lemma 3.4] *Let Δ be a finite building of type (W, S). Then Δ has a convex ear decomposition such that each ear is isomorphic to $\Sigma(W, S) \cap H_1^+ \cap \cdots \cap H_t^+$, where the H_i^+'s are closed half-spaces of distinct reflecting hyperplanes of W.*

Lemma 4.6. *Let $\mathcal{A} = \{H_1, \ldots, H_s\}$ be an essential arrangement of hyperplanes in $\mathbb{R}^d$. Let P be any d-polytope whose face fan is the fan of $\mathcal{A}$. Let $H_{i_1}^+, \ldots, H_{i_t}^+$ be closed half-spaces of distinct hyperplanes in $\mathcal{A}$. If $B = \partial P \cap H_{i_1}^+ \cap \cdots \cap H_{i_t}^+$ is nonempty, then ∂B is combinatorially equivalent to the boundary of a $(d-1)$-polytope.*

Proof. For notational convenience we renumber the hyperplanes so that $H_{i_j} = H_j$. So, $\mathcal{A} = \{H_1, \ldots, H_t, H_{t+1}, \ldots, H_s\}$ and $B = \partial P \cap H_1^+ \cap \cdots \cap H_t^+$.

By a familiar vertex figure argument, it is sufficient to find P', a d-polytope whose face fan is the same as the face fan of P, and a point $y \in \mathbb{R}^d$ such that the facets of P' that can be seen from y are precisely those in $B' = \partial P' \cap H_1^+ \cap \cdots \cap H_t^+$.

Let $z_1, \ldots, z_s$ be nonzero vectors such that z_i is orthogonal to H_i, and for $i \leqslant t$, $z_i \in H_i^+$. Let Z be the zonotope $[-z_1, z_1] + \cdots + [-z_s, z_s]$, where $+$ is Minkowski sum. For an s-tuple $(\varepsilon_1, \ldots, \varepsilon_s)$, $\varepsilon_i = \pm 1$, let $x_\varepsilon = \sum_{i=1}^{s} \varepsilon_i z_i$. While some of the x_ε may be interior points of Z, all of the vertices of Z are equal to some x_ε.

Set P' to be $Z^\star$, the polar of Z. The face fan of P' is the fan of $\mathcal{A}$ and the facets of P' are of the form $F_\varepsilon = P' \cap H_\varepsilon$, where $H_\varepsilon = \{w \in \mathbb{R}^d : x_\varepsilon \cdot w = 1, x_\varepsilon \text{ a vertex of } Z\}$. If x_ε is a vertex of Z, then $x_\varepsilon \cdot w \leqslant 1$ for all $w \in P'$. Therefore, a facet F_ε of P' is visible from $y \in \mathbb{R}^d$ if and only if $x_\varepsilon \cdot y > 1$.

The facets of $B' = P' \cap H_1^+ \cap \cdots \cap H_t^+$ are those facets F_ε with $\varepsilon_i = +1$ for $i \leqslant t$. Since B is nonempty we can choose $y \in R^d$ so that $z_i \cdot y > 0$ for $i \leqslant t$. By rescaling all the z_i by positive scalars, we can choose $z_1, \ldots, z_s$ and y so that $z_i \cdot y = 1$ for $i \leqslant t$, and $\sum_{i=t+1}^{s} |z_i \cdot y| < 1/100$. Now it is easy to see that for some $\delta > 0$ the only facets of P' visible from $(1 + \delta)y$ will be the facets of B'. □

Corollary 4.7. *If Δ is a finite building, then $\bar{h}$ is the sum of $h_d - 1$ M-vectors.*

If the g-theorem was known to hold for the boundary of every ball which is a full dimensional subcomplex of the boundary of a simplicial polytope, then $\bar{h}$ of any d-dimensional space with a convex ear decomposition would be the sum of $h_d - 1$ M-vectors.

Problem 4.8. *If Δ is 2-CM, is $\bar{h}(\Delta)$ the sum of $h_d - 1$ M-vectors?*

Björner has observed that for $q \gg 0$, Eq. (3) implies that finite buildings associated to GF(q) must satisfy $h_B < h_A$ for $B \subset A$ [5]. This means that for these spaces high CM connectivity implies that the h-vector is increasing. This is part of a general phenomenon involving complexes with large links. Since the removal of $q - 1$ vertices from a q-CM complex Δ leaves a pure complex, the link of every nonfacet of Δ must contain at least q vertices. A pure complex with large links must have an increasing h-vector. To prove this we use an extension of $\tilde{h}$, the short simplicial h-vector introduced in [10] as a simplicial analogue of the short cubical h-vector in [1].

Definition 4.9.

$$\tilde{h}_i^{(m)}(\Delta) = \sum_{\substack{\sigma \in \Delta \\ |\sigma| = m}} h_i(\text{lk}\,\sigma).$$

For example, $\tilde{h}^{(0)}$ is the usual h-vector and $\tilde{h}^{(1)}$ is the short simplicial h-vector defined in [10].

Proposition 4.10. *Let Δ be a pure $(d-1)$-dimensional complex. For $m \leqslant d - 1$ and $i \leqslant d - m$,*

$$(m+1)\, \tilde{h}_{i-1}^{(m+1)} = i\tilde{h}_i^{(m)} + (d - m - i + 1)\tilde{h}_{i-1}^{(m)}. \tag{5}$$

Proof. The case $m = 0$ is [26, Proposition 2.3]. For larger m,

$$\tilde{h}_i^{(m)} = \sum_{|\sigma|=m} h_i(\text{lk}\,\sigma)$$

$$= \frac{1}{i}\left[\sum_{|\sigma|=m} \left\{ \tilde{h}_{i-1}^{(1)}(\text{lk}\,\sigma) - (d - m - i + 1)h_{i-1}(\text{lk}\,\sigma) \right\} \right]$$

$$= \frac{1}{i}\left[\sum_{|\sigma|=m} \left(\left\{ \sum_{v \in \text{lk}\,\sigma} h_{i-1}\big(\text{lk}(\sigma \cup \{v\})\big) \right\} - (d - m - i + 1)h_{i-1}(\text{lk}\,\sigma) \right) \right]$$

$$= \frac{1}{i}\big[(m+1)\, h_{i-1}^{(m+1)} - (d - m - i + 1)h_{i-1}^{(m)}\big]. \qquad \square$$

Theorem 4.11. *Fix d and $i \leqslant d$. There exists $q(i, d)$ such that if Δ is a pure $(d-1)$-dimensional complex and the link of every $(i-2)$-dimensional face of Δ has at least $q(i, d)$ vertices, then*

$$h_0 < \cdots < h_i. \tag{6}$$

Proof. The proof is by induction on i and d, with $i = 1$ being trivial. We can assume that $q(i, d) \leqslant q(i', d')$ whenever $i \leqslant i'$ and $d \leqslant d'$. Suppose that $i \geqslant 2$ and $q(i - 1, d)$ satisfies the theorem. For the induction step we only need to find q so that $h_{i-1} < h_i$ whenever Δ is pure,

$(d-1)$-dimensional and the link of every $(i-2)$-dimensional face has at least q vertices. Indeed, given such a q, $q(i, d) = \max\{q, q(i-1, d)\}$ satisfies the theorem.

Let q be the minimum number of vertices in the link of an $(i-2)$-dimensional face of Δ. In order to show that for q sufficiently large $h_{i-1} < h_i$ we argue by contradiction. So, suppose $h_{i-1} \geqslant h_i$. First we estimate $\tilde{h}_{i-j}^{(j)}$ using (5) and $h_{i-1} \geqslant h_i$. For $0 \leqslant j \leqslant i-1$,

$$\tilde{h}_{i-j}^{(j)} \leqslant p_{i,j}(d)h_{i-1}, \tag{7}$$

where $p_{i,j}(d)$ is a degree j polynomial. When $j = 0$, $p_{i,0} = 1$. For higher j, (7) is proved by induction using $\tilde{h}_{i-j-1}^{(j)} < \tilde{h}_{i-j}^{(j)}$ and

$$\tilde{h}_{i-j-1}^{(j+1)} = \frac{1}{(j+1)}\big[(i-j)\tilde{h}_{i-j}^{(j)} + (d-i+1)\tilde{h}_{i-j-1}^{(j)}\big].$$

We can also estimate $\tilde{h}_1^{(i-1)}$ using the induction hypothesis and q:

$$\tilde{h}_1^{(i-1)} = \sum_{|\sigma|=i-1} h_1(\mathrm{lk}\,\sigma) \geqslant \big(q - (d-i+1)\big)f_{i-1} > \big(q - (d-i+1)\big)h_{i-1}.$$

The last inequality follows from (2) and the fact that $h_0, \ldots, h_{i-1}$ are positive for sufficiently large q by the induction hypothesis on i. Putting these estimates together,

$$\big(q - (d-i+1)\big)h_{i-1} < p_{i,i-1}(d)h_{i-1}.$$

Hence, q is bounded.   □

The above proof gives $q(i, d)$ as a polynomial of degree $i-1$ in d. Can this be improved?

Problem 4.12. What are the minimum values of $q(i, d)$?

The dependence on d in the above theorem is essential. Indeed, h_2 of two simplices connected at one vertex is always negative once $d > 2$. What if we impose the additional condition that Δ is Cohen–Macaulay? An affirmative answer to Problem 4.2 would imply that links of size $q(i, 2i)$ would be sufficient to imply at least inequality (as opposed to strict inequality) in (6) for 2-CM complexes. For balanced q-CM complexes it is possible to remove the dependence on d.

Theorem 4.13. Let (Δ, ψ) be a balanced $q(i, i)$-CM complex. Then for any $B \subset A$ with $|A| \leqslant i$,

$$h_B < h_A.$$

Proof. Our first observation is that for any $A \subseteq S$, Δ_A is also $q(i, i)$-CM. Removing $q(i, i) - 1$ vertices from Δ_A is the same as removing $q(i, i) - 1$ vertices from Δ and then restricting to Δ_A. The combination of these operations preserve dimension and the CM property [25, Theorem III.4.5]. Hence,

$$h_B(\Delta) = h_B(\Delta_A) \leqslant h_{|B|}(\Delta_A) < h_{|A|}(\Delta_A) = h_A(\Delta).   □$$

Corollary 4.14. If Δ is a $q(i, i)$-CM balanced complex and $d \geqslant i+1$, then

$$h_0 < \cdots < h_i.$$

Proof. In order to show that $h_{j-1} < h_j$ for every $j \leqslant i$, fix $j \leqslant i$. Let ϕ be any map from $(j-1)$-subsets of S to j-subsets of S such that $B \subseteq \phi(B)$ for all B.

$$h_{j-1}(\Delta) = \sum_{|B|=j-1} h_B(\Delta) = \sum_{|B|=j-1} h_B(\Delta_{\phi(B)}) = \sum_{|A|=j} \sum_{B \in \phi^{-1}(A)} h_B(\Delta_A)$$

$$\leqslant \sum_{|A|=j} h_{j-1}(\Delta_A) < \sum_{|A|=j} h_j(\Delta_A) = \sum_{|A|=j} h_A(\Delta) = h_j(\Delta). \qquad \square$$

Acknowledgments

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