Decisive dimension and other related torsion theoretic dimensions

Jaime Castro Pérez, Francisco Raggi, José Ríos Montes

Abstract

The paper is concerned with the study of the decisive dimension defined on the category of left modules over a ring \( R \). We compare the decisive dimension with the Gabriel dimension and other dimensions recently introduced. We give module theoretic as well as lattice theoretic characterizations of rings with decisive dimension. As an application we obtain characterizations of some classes of rings.

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0. Introduction

Recently, some dimensions defined by means of the lattice \( R\text{-tors} \) have been considered to obtain information about the ring \( R \) and its category of modules. For example, in [3] we introduced a dimension called \( \mathcal{P} \)-dimension. As application, we obtained characterizations of rings with bijective Gabriel correspondence in terms of \( \mathcal{P} \)-dimension. In [4] we considered a dimension called atomic dimension. In this paper, we continue the investigation started in [3] and [4]. In particular, for every \( \tau \in R\text{-tors} \) we consider the modules \( M \) such that \( \tau \vee \xi(M) \) is an atom over \( \tau \), and there exists a decisive module \( D \) with \( \chi(M) = \chi(D) \). These modules are called \( \tau\text{-D-modules} \). In Section 1, we examine some properties of decisive modules and \( \tau\text{-D-modules} \). In Section 2, we define the concept of decisive dimension in terms of \( \tau\text{-D-modules} \). In this section we give module theoretic and also lattice theoretic characterizations of rings with decisive dimension. As an application, we obtain in Section 3 a characterization of rings with local bijective Gabriel correspondence. In Section 4, we obtain characterizations of semiartinian rings and artinian rings, in terms of the strongly irreducible elements of \( R\text{-tors} \). Examples are given to illustrate the theory.

Let \( R \) be an associative ring with unity, \( R\text{-Mod} \) be the category of unitary left \( R \)-modules, and \( R\text{-tors} \) be the frame of all hereditary torsion theories on \( R\text{-Mod} \). For a family of left \( R \)-modules \( \{M_\alpha\} \), let \( \chi(\{M_\alpha\}) \) be the maximal element

* Corresponding author.

E-mail addresses: jcastrop@itesm.mx (J. Castro Pérez), fraggi@matem.unam.mx (F. Raggi), jrios@matem.unam.mx (J. Ríos Montes).
of $R$-tors for which all the $M_\alpha$ are torsion free, and let $\xi([M_\alpha])$ denote the minimal element of $R$-tors for which all the $M_\alpha$ are torsion. $\chi([M_\alpha])$ is called the torsion theory cogenerated by the family $\{M_\alpha\}$, and $\xi([M_\alpha])$ is the torsion theory generated by the family $\{M_\alpha\}$. In particular, the maximal element of $R$-tors is denoted by $\chi$ and the minimal element of $R$-tors is denoted by $\xi$. If $\tau$ is an element of $R$-tors, $\text{gen}(\tau)$ denotes the interval $[\tau, \chi]$.

Let $\tau \in R$-tors. By $\mathbb{T}_{\tau}, \mathbb{F}_{\tau}, \mathbb{I}_{\tau}, \mathbb{C}_{\tau}$, we denote, respectively, the torsion class, the torsion free class, the torsion functor and the linear filter associated to $\tau$. For $M \in R$-Mod, $M$ is called $\tau$-cocritical if $M \in \mathbb{F}_{\tau}$ and, for all $0 \neq N \subseteq M$, we have that $M/N \in \mathbb{T}_{\tau}$. We say that $M$ is cocritical if $M$ is $\tau$-cocritical for some $\tau \in R$-tors. We say that $\tau \in R$-tors is prime if $\tau = \chi(M)$, where $M$ is cocritical. We will denote by $R$-$sp \{\chi(M) \mid M \text{ is cocritical}\}$.

A torsion theory $\tau \in R$-tors is irreducible if for $\tau', \tau'' \in R$-tors with $\tau' \land \tau'' = \tau$, we have that $\tau' = \tau$ or $\tau'' = \tau$. An element $\tau \in R$-tors is strongly irreducible if for any non-empty family $U \subseteq R$-tors such that $\land U \subseteq \tau$ there exists an element $\sigma \in U$ satisfying $\sigma \leq \tau$. Strongly irreducible torsion theories are irreducible. For $M \in R$-Mod, let $E(M)$ denote the injective hull of $M$. For all other concepts and terminology concerning torsion theories and torsion theoretic dimensions, the reader is referred to [6,7,13].

1. Decisive modules and $\mathcal{D}$-modules

Definition 1.1. A non-zero left $R$-module $M$ is decisive if for any element $\tau$ of $R$-tors, $M$ is either $\tau$-torsion or $\tau$-torsion free.

Proposition 1.2. Let $D_1$ and $D_2$ be decisive modules. The following conditions are equivalent:

1. $\chi(D_1) = \chi(D_2)$.
2. For each $\tau \in R$-tors, $D_1 \in \mathbb{T}_{\tau}$ if and only if $D_2 \in \mathbb{T}_{\tau}$.

Proof. (1) $\Rightarrow$ (2) Let $\tau \in R$-tors such that $D_1 \in \mathbb{T}_{\tau}$. Assume that $D_2 \notin \mathbb{T}_{\tau}$. Since $D_2$ is decisive, then $D_2 \in \mathbb{F}_{\tau}$. Hence $\tau \leq \chi(D_2)$. Thus by (1) we obtain $\tau \leq \chi(D_1)$, therefore $D_1 \notin \mathbb{F}_{\tau}$, which is a contradiction.

(2) $\Rightarrow$ (1) $D_1 \in \mathbb{F}_{\chi(D_1)}$, then by (2) and the fact that $D_2$ is decisive, we get $D_2 \in \mathbb{F}_{\chi(D_1)}$. Hence $\chi(D_1) \leq \chi(D_2)$. By symmetry, $\chi(D_2) \leq \chi(D_1)$. \(\square\)

Proposition 1.3. If $M$ is a decisive module and $N$ is a proper submodule of $M$ such that $M/N \in \mathbb{F}_{\chi(M)}$, then $M/N$ is decisive and $\chi(M) = \chi(M/N)$.

Proof. Let $\tau \in R$-tors. Since $M$ is decisive, then $M \in \mathbb{T}_{\tau}$ or $M \in \mathbb{F}_{\tau}$. If $M \in \mathbb{T}_{\tau}$, then $M/N \in \mathbb{T}_{\tau}$. Now suppose that $M \in \mathbb{F}_{\tau}$, then $\tau \leq \chi(M)$. Also $\chi(M) \leq \chi(M/N)$, by hypothesis. Hence $\tau \leq \chi(M/N)$. So $M/N \in \mathbb{F}_{\tau}$ which proves that $M/N$ is decisive. From Proposition 1.2 we obtain that $\chi(M) = \chi(M/N)$.

Note that the class of decisive modules is not closed under quotients. $\mathbb{Z}$ is a decisive $\mathbb{Z}$-module, but $\mathbb{Z}/6\mathbb{Z}$ is not decisive. \(\square\)

Proposition 1.4. Let $P$ be a prime ideal of $R$ and let $C$ be a cocritical left $R$-module such that $C \subseteq R/P$, then $C$ is compressible.

Proof. Let $\bar{x} \in C = I/P$, define $f_{\bar{x}} : C \to C\bar{x}$ by $f_{\bar{x}}(\bar{y}) = \bar{y}\bar{x}$. Since $C\bar{x} \subseteq C$ and $C$ is cocritical, we have that $f_{\bar{x}} = 0$ or $f_{\bar{x}}$ is a monomorphism.

Now, let $0 \neq C' \subseteq C$. Suppose that $C\bar{x} = 0$ for all $\bar{x} \in C'$. Hence $C(R\bar{x}) = 0$, then $(I/P)(R(x + P)) = 0$. So $IRx \subseteq P$. Since $P$ is a prime ideal, $I \subseteq P$ or $Rx \subseteq P$. Therefore $C = 0$ or $x = 0$, a contradiction. So there exists $0 \neq \bar{x} \in C'$ such that $C\bar{x} \neq 0$. Hence the composition $C \xrightarrow{f_{\bar{x}}} C\bar{x} \hookrightarrow R\bar{x} \hookrightarrow C'$ is a monomorphism. Therefore $C$ is compressible. \(\square\)

Examples 1.5. (1) Each compressible module is decisive.

(2) If $R$ is a commutative ring and $P$ is a prime ideal of $R$, then $R/P$ is decisive as an $R$-module and also as an $R/P$-module.

(3) $R$ is decisive if and only if $\tau \leq \chi(R)$ for every $\tau \in R$-tors, $\tau \neq \chi$. See [6].

(4) Let $V$ be an infinite dimensional vector space over a field $K$. Let $R = \text{End}_K(V)$ and let $\tau_{sp}$ be the element of $R$-tors whose torsion class is $\mathbb{T}_{\tau_{sp}} = \{M \in R$-mod $\mid M$ is semisimple and projective$\}$. Then $0 \neq \tau_{sp}(R) = \text{soc}(R) \neq R$, hence $R$ is not decisive. On the other hand, $R$ is a prime ring. So, it is not in general true that if $P$ is a prime ideal of $R$, then $R/P$ is decisive.
Let $\tau \in R$-tors. Then $\tau$ is strongly irreducible if and only if there exists a decisive module $M$ such that $\tau = \chi(M)$. (See [6, Proposition 32.7]).

(6) A left $R$-module is decisive if and only if $\xi(M) = \xi(N)$ for every non-zero submodule $N$ of $M$.

Definition 1.6. Let $\tau \in R$-tors, $\tau \neq \chi$ and $M \in R$-Mod. We say that $M$ is a $\tau$-$A$-module if $M \in \mathbb{F}_\tau$ and $\tau \vee \xi(M)$ is an atom in $\text{gen}(\tau)$. (See [4] for details about these modules.)

Let us denote $D = \{\chi(D) \mid D$ is decisive\}.

A module $M$ is called an $A$-module if there exists $\tau \in R$-tors such that $M$ is a $\tau$-$A$-module. Notice that $D \subseteq A = \{\chi(M) \mid M$ is an $A$-module\}. (See [3].)

Definition 1.7. Let $\tau \in R$-tors and $M \in R$-Mod; $M$ is called a $\tau$-$D$-module if $M$ is a $\tau$-$A$-module and $\chi(M) \in D$.

Example 1.8. (a) If $M$ is a simple module, then $M$ is a $\tau$-$D$-module for all $\tau$ such that $M \in \mathbb{F}_\tau$.

(b) If $M$ is decisive, then $M$ is a $\chi(M)$-$D$-module.

(c) Let $F$ be a field and let $R = \left\{ \begin{pmatrix} a & y \\ 0 & z \end{pmatrix} \mid x, y \in F \right\}$. Now, let $M = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \mid a, b \in F \right\}$ and let $S = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \mid a \in F \right\}$.

Then $M$ is a left $R$-module and $S$ is a submodule of $M$. Notice that $M/S \cong S$, then $M$ is a decisive module but $M$ is not a cocritical module.

As a consequence of the theory developed in [4] for $\tau$-$A$-modules, we have the following results for $\tau$-$D$-modules.

Proposition 1.9. If $M$ is a $\tau$-$D$-module, then:

(1) Every non-zero submodule $N$ of $M$ is a $\tau$-$D$-module.

(2) If $\sigma \in \text{gen}(\tau)$ and $M \in \mathbb{F}_\sigma$, then $M$ is a $\sigma$-$D$-module.

(3) Let $N$ be a proper submodule of $M$ such that $M/N \in \mathbb{F}_\tau$, then $M/N$ is a $\tau$-$D$-module and $\chi(M) = \chi(M/N)$.

(4) If $\sigma \in R$-tors and $M \in \mathbb{T}_\sigma$, then $M$ is a $(\tau \land \sigma)$-$D$-module.

(5) $M(X)$ is a $\tau$-$D$-module for any non-empty set $X$.

Proposition 1.10. If $\{M_\alpha\}_{\alpha \in I}$ is a family of $\tau$-$D$-modules such that $\chi(M_\alpha) = \chi(M_\beta) \forall \alpha, \beta \in I$, then $\oplus_{\alpha \in I} M_\alpha$ is a $\tau$-$D$-module.

Proof. For any $\beta \in I$, $\chi(M_\beta) = \wedge_{\alpha \in I} \chi(M_\alpha) = \chi(\oplus_{\alpha \in I} M_\alpha)$ by hypothesis. Hence $\chi(\oplus_{\alpha \in I} M_\alpha) \in D$. By [4, Corollary 2.16], we have that $\tau \vee \xi(M_\alpha) = \tau \vee \xi(M_\beta) \forall \alpha, \beta \in I$. Hence $\forall \beta \in I, \tau \vee \xi(M_\beta) = \vee_{\alpha \in I}(\tau \vee \xi(M_\alpha)) = \tau \vee (\vee_{\alpha \in I} \xi(M_\alpha)) = \tau \vee \xi(\oplus_{\alpha \in I} M_\alpha)$. So, $\oplus_{\alpha \in I} M_\alpha$ is a $\tau$-$A$-module. Thus $\oplus_{\alpha \in I} M_\alpha$ is a $\tau$-$D$-module. \hfill \square

Proposition 1.11. Let $M \in R$-Mod. Then the following conditions are equivalent.

(1) $M$ is a $\xi$-$D$-module.

(2) There exists a simple left $R$-module $S$ such that $\xi(M) = \xi(S)$ and $\chi(M) = \chi(S)$.

Proof. (1) $\Rightarrow$ (2) If $M$ is a $\xi$-$D$-module, then $\xi(M)$ is an atom of $R$-tors. Hence there exists a simple left $R$-module $S$ such that $\xi(M) = \xi(S)$. Now, by [4, Corollary 2.16] we have that $\chi(M) = \chi(S)$.

(2) $\Rightarrow$ (1) It is clear. \hfill \square

Definition 1.12. A left $R$-module $M$ is called a $D$-module if there exists $\tau \in R$-tors such that $M$ is a $\tau$-$D$-module.

From Proposition 1.9(2), we obtain that $M$ is a $D$-module if and only if $M$ is a $\chi(M)$-$D$-module.

Proposition 1.13. $M$ is a $D$-module if and only if $\chi(M) \in D$ and $\chi(M) \lor \xi(N) = \chi(M) \lor \xi(M)$ for all non-zero submodules $N$ of $M$.

Proof. ($\Rightarrow$) Since $M$ is a $D$-module, then $M$ is a $\chi(M)$-$D$-module. Therefore $\chi(M) \in D$ and $M$ is a $\chi(M)$-$A$-module. Hence for any non-zero submodule $N$ of $M$, we have that $\chi(M) \lor \xi(N) = \chi(M) \lor \xi(M)$.

($\Leftarrow$) It is enough to show that $\chi(M) \lor \xi(M)$ is an atom in $\text{gen}(\chi(M))$. Let $\tau \in R$-tors such that $\chi(M) < \sigma \leq \chi(M) \lor \xi(M)$. Hence $t_\tau(M) \neq 0$. So, $\chi(M) \lor \xi(t_\tau(M)) \leq \sigma \leq \chi(M) \lor \xi(M)$. We can conclude that $\sigma = \chi(M) \lor \xi(M)$ by hypothesis. Thus $M$ is a $\chi(M)$-$A$-module. \hfill \square
Proposition 1.14. Let $\tau \in R$-tors and let $M$ be a $\tau$-$D$-module. Denote $\mathcal{F} = \{N \subseteq M \mid N$ is decisive$\}$, then $D = \sum_{N \in \mathcal{F}} N$ is a decisive essential submodule of $M$.

Proof. Since $M$ is a $\tau$-$D$-module, there exists a decisive module $K$ such that $\chi(M) = \chi(K)$, hence $\text{Hom}_R(K, E(M)) \neq 0$. So there exist submodules $D' \subseteq D'' \subseteq K$ and a monomorphism $D''/D' \hookrightarrow M$. Thus $D''/D'$ is decisive by Proposition 1.3. So $\mathcal{F} \neq \emptyset$. Let $D = \sum_{N \in \mathcal{F}} N$. Assume that $D$ is not essential in $M$. Then there exists a non-zero pseudo-complement $T$ of $D$ in $M$, hence $T$ contains a decisive submodule as we have already proved, a contradiction.

Finally, let $D = \oplus_{N \in \mathcal{F}} N$. For all $N \in \mathcal{F}$, $\chi(N) = \chi(M)$ by Proposition 1.9(1). Hence $D$ is decisive by Proposition 1.2. Notice that $\chi(D) = \chi(M)$. Since $D \subseteq M$ and $D$ is a quotient of $D$, we have that $D$ is decisive by Proposition 1.3. □

Examples 1.15. (1) Let $R$ be the ring considered in Examples 1.5(4). Since $R$ is a prime ring, then $R$ is a $\chi(R)$-$A$-module by Propositions 2.20 and 1.21 of [4]. Now let $S$ be a simple submodule of $R$, hence $\chi(S) = \chi(R)$ by Corollary 2.17 of [4]. From the fact that $S$ is decisive, we have that $\chi(R) = \chi(S) \in D$. Thus $R$ is a $D$-module. This example shows that a $D$-module is not necessarily decisive.

(2) Let $R = \left(\begin{smallmatrix} 0 & 0 \\ R & 0 \end{smallmatrix}\right)$ and let $S = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then $S$ is a non-singular simple left $R$-module. The injective hull of $S$ can be described as $E(S) = \left(\begin{smallmatrix} \mathbb{R} & 0 \\ \mathbb{R} & 0 \end{smallmatrix}\right)$. Since $E(S)/S$ is a singular left $R$-module, then $\xi(S) < \xi(E(S))$. Hence $S$ is a $\xi$-$D$-module but $E(S)$ is not a $\xi$-$D$-module. This shows that the class of $\tau$-$D$-modules is not closed under injective hulls.

2. Decisive dimension

We will use the $D$-modules in order to define the decisive dimension in $R$-Mod.

Let $\tau \in R$-tors. The $D$-filtration of $\tau$ in $R$-tors is defined as follows:

1. $\delta_0 = \tau$.
2. If $i$ is not a limit ordinal, then
   \[ \delta_i = \delta_{i-1} \vee \xi(M \mid M$ is a $\delta_{i-1}$-$D$-module$)). \]
3. If $i$ is a limit ordinal, then $\delta_i = \bigvee_{j<i} \delta_j$.

Since $R$-tors is a set, there exists a minimal ordinal $k$ such that $\delta_k = \delta_{k+r}$ for all ordinals $r$.

Definition 2.1. A non-zero left $R$-module $M$ is said to have $\tau$-decisive dimension equal to an ordinal $h$ if $M$ is $\delta_h$-torsion, but it is not $\delta_i$-torsion for any $i < h$. The ring $R$ is said to have left $\tau$-decisive dimension if it has $\tau$-decisive dimension as left $R$-module. We will denote the $\tau$-decisive dimension (if it exists) of $M \in R$-Mod by $\tau$-$\text{dim}_D(M)$. The $\xi$-decisive dimension of an $R$-module is simply called the decisive dimension of the module. Notice that $\tau$-$\text{dim}_D(M) = k$ if and only if $\delta_k = \chi$.

We recall that left semiartinian rings are characterized by the fact that $\xi(R$-simp$) = \chi$.

As a direct consequence of the definition, we have the following result.

Proposition 2.2. Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence in $R$-mod. Then $\tau$-$\text{dim}_D(M) = \sup \{\tau$-$\text{dim}_D(M'), \tau$-$\text{dim}_D(M'')\}$ provided that either side exists.

In the next result we give equivalent conditions for $R$ to have left $\tau$-decisive dimension.

Theorem 2.3. Let $\tau \in R$-tors, $\tau \neq \chi$. The following conditions are equivalent.

1. $R$ has left $\tau$-decisive dimension.
2. For all $\sigma, \sigma' \in \text{gen}(\tau)$ with $\sigma < \sigma'$, there exists a $\sigma$-$A$-module $M$ such that $M \in T_{\sigma'}$ and $M$ is decisive.
3. For all $\sigma \in \text{gen}(\tau)$ with $\sigma \neq \chi$, $\sigma = \land(\chi(M) \mid M$ is decisive $\sigma$-$A$-module$).
4. For all $\sigma \in \text{gen}(\tau)$ with $\sigma \neq \chi$, there exists a $\sigma$-$A$-module $M$ such that $M$ is decisive.
Proof. (1) ⇒ (2) Let σ, σ′ ∈ gen(τ) such that σ < σ′ ≤ χ.

Since R has τ-decisive dimension, then there exists a minimal ordinal i such that σ′ ∧ δi ≤ σ. If i is a limit ordinal, then σ′ ∧ δi = σ′ ∧ (∨j<i δj) = ∨j<i (σ′ ∧ δj) ≤ σ. This contradicts the choice of i. Therefore i is not a limit ordinal. On the other hand, σ′ ∧ δ0 = σ′ ∧ τ = τ ≤ σ. So i ≥ 1. Now let N ∈ ℑσ′∧δi ∩ ℑσ. From the inequality σ′ ∧ δi−1 ≤ σ we get N ∈ ℑσ′∧δi−1. Since N ∈ ℑσ′∧δi−1, then N ∈ ℑδi ∧ δi−1. As δi = δi−1 ∨ ν {M | M is a δi−1-D-module}, there exists a δi−1-D-module M such that HomR(M, E(N)) ≠ 0. So there are submodules H, K of M with K ⊂ H ⊂ M and a monomorphism H/K ↪ N. Since M is a δi−1-D-module and N ∈ ℑδi−1, then H/K is a δi−1-D-module by Proposition 1.9(3).

On the other hand H/K ∈ ℑσ′, hence H/K is a δi−1 ∧ σ′-D-module by Proposition 1.9(4). As δi−1 ∧ σ′ ≤ σ and H/K ∈ ℑσ, H/K is a σ-D-module by Proposition 1.9(2). Since M is a δi−1-D-module there exists a decisive module D such that χ(M) = χ(D). From Proposition 1.9(3) we obtain χ(M) = χ(H) = χ(H/K) ≥ σ. Thus D ∈ ℑσ. Now denote U = H/K. Then there exist submodules V ⊂ W ⊂ U and a monomorphism W/V ↪ D. Since U is a σ-D-module and D ∈ ℑσ, then W/V is a σ-D-module by Proposition 1.9(3). So W/V is the decisive σ-A-module we were looking for.

(2) ⇒ (3) Let σ ∈ gen(τ) and let σ′ = ∨ {χ(M) | M is decisive σ-A-module}. Then σ ≤ σ′. If σ < σ′, then there exists a decisive module M such that M is a σ-A-module and M ∈ ℑσ′, which is a contradiction.

(3) ⇒ (4) It is clear.

(4) ⇒ (1) Suppose R does not have τ-decisive dimension and let k be the minimal ordinal such that δ_k = δ_{k+i} for all ordinal i. Hence δ_k < χ. By (4), there is a decisive module M such that M is a δ_k-A-module. Therefore M is a δ_k-D-module. So M ∈ ℑδ_k = ℑδ_{k+i}, which is a contradiction. □

Corollary 2.4. If R has τ-D-dimension and σ ∈ gen(τ), then R has σ-D-dimension.

A special case for R to have decisive dimension is considered in the next result.

Proposition 2.5. Suppose R-tors is an artinian lattice. Then for all τ ∈ R-tors, R has left τ-D-dimension.

Proof. Let τ ∈ R-tors and σ ∈ gen(τ). Since R-tors is artinian, then gen(τ) is artinian. So there exists σ′ ∈ R-tors such that σ′ is an atom of gen(σ). Now let M be a σ-A-module such that σ′ = σ ∨ ν(M). If M is decisive, then R has left τ-D dim by Theorem 2.3. If M is not decisive, then there exists τ_1 ∈ R-tors such that 0 ≠ τ_1(M) ⊂ M. If τ_1(M) is not decisive, then there exists τ_2 ∈ R-tors such that 0 ≠ τ_2(M) ⊂ τ_1(M) ⊂ M. Continuing this process we obtain a chain of submodules M ⊃ τ_1(M) ⊃ τ_2(M) ⊃ ··· ⊃ (t_n t_{n-1} ··· t_1(M)) ··· . Notice that for each k, (t_k t_{k-1} ··· t_1(M)) = t_1 ∧ ··· ∧ t_k(M). Hence we obtain a strictly descending chain in R-tors τ_1 > τ_2 > ··· > τ_1 ∧ ··· ∧ t_k ··· . So there exists n ∈ ℤ such that the chain stops at step n. Thus t_n ∧ ··· ∧ t_1(M) is a decisive submodule of M. This completes the proof. □

Corollary 2.6. Let R be a ring such that R-tors is a finite set. Then for all τ ∈ R-tors, τ ≠ χ, R has τ-D-dimension.

Note that the converse of Proposition 2.5 is not true in general. It is clear that ℤ has left decisive dimension, but ℤ-tors is not an artinian lattice.

Remark 2.7. In [4] we discussed a torsion theoretic dimension that we called atomic dimension. In order to define the τ-atomic dimension, we considered a filtration in R-tors as follows:

(i) α_0 = τ.
(ii) If i is not a limit ordinal, then

α_i = α_{i-1} ∨ ν {M | M is an α_{i-1}-A-module}.

(iii) If i is a limit ordinal, then α_i = ∨_{j<i} α_j.

The dimension associated to this filtration is called the τ-atomic dimension.

Among other results we proved in [4]:

(1) R has left τ-atomic dimension if and only if, for all σ, σ′ ∈ gen(τ) with σ < σ′, there exists a σ-A-module M such that M ∈ ℑσ′.
(2) If $R$ has left $\tau$-atomic dimension, then every element $\sigma \in \text{gen}(\tau)$, $\sigma \neq \chi$, uniquely decomposes as the meet of an irredundant family of irreducible elements of $\text{gen}(\tau)$.

**Theorem 2.8.** Let $\tau \in R$-tors and suppose $R$ has left $\tau$-decisive dimension. Then the following conditions hold.

1. Every $0 \neq M \in \mathbb{F}_r$ contains a decisive submodule $N$ such that $N$ is a $\chi(M)$-$A$-module.
2. For all $\sigma \in \text{gen}(\tau)$, $\sigma = \tau \lor \xi(M)$ is decisive and $N \in \mathbb{T}_\sigma$.
3. $R$ has left $\tau$-atomic dimension.
4. Let $\sigma \in \text{gen}(\tau)$, $\sigma \neq \chi$. Then $\sigma$ is an irreducible element of $R$-tors if and only if $\sigma$ is strongly irreducible.

**Proof.** (1) Let $0 \neq M \in \mathbb{F}_r$. Then $\chi(M) \in \text{gen}(\tau)$, hence there exists a $\chi(M)$-$A$-module $N$ such that $N$ is decisive by *Theorem 2.3*(4). So $M$ contains a non-zero submodule $K$ isomorphic to a subquotient of $N$. From [4, Corollary 2.16] and *Proposition 1.3*, we obtain that $K$ is a decisive $\chi(M)$-$A$-module.

(2) Let $\sigma \in \text{gen}(\tau)$. Denote $\sigma' = \tau \lor \xi(M)$ is decisive and $M \in \mathbb{T}_\sigma$). If $\sigma' < \sigma$, then there exists a decisive $\sigma'$-$A$-module $N$ such that $N \in \mathbb{F}_\sigma$ by *Theorem 2.3*(2), a contradiction. So $\sigma' = \sigma$.

(3) It follows from (1).

(4) It follows from (3), [4, *Proposition 3.6*], and (1). $\square$

In the following result we give a purely lattice theoretic characterization of rings with left $\tau$-decisive dimension.

**Theorem 2.9.** Let $\tau \in R$-tors. Then the following conditions are equivalent.

1. $R$ has left $\tau$-decisive dimension.
2. Every element $\sigma \in \text{gen}(\tau)$, $\sigma \neq \chi$, uniquely decomposes as the meet of an irredundant family of strongly irreducible elements of $\text{gen}(\tau)$.
3. Every $\sigma \in \text{gen}(\tau)$, $\sigma \neq \chi$, decomposes as the meet of an irredundant family of strongly irreducible elements of $\text{gen}(\tau)$.

**Proof.** (1) $\Rightarrow$ (2) By (1) and *Theorem 2.8*(3), we have that $R$ has left $\tau$-atomic dimension. Hence every $\sigma \in \text{gen}(\tau)$, $\sigma \neq \chi$, uniquely decomposes as the meet of an irredundant family of irreducible elements in $\text{gen}(\tau)$, by *Remark 2.7*, (2). From *Theorem 2.8*(4), we know that every irreducible element of $\text{gen}(\tau)$ is strongly irreducible. The uniqueness follows from the fact that every strongly irreducible element of $R$-tors is irreducible.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Let $\sigma \in \text{gen}(\tau)$, $\sigma \neq \chi$ and let $\{\chi(M_{\alpha})\}_{\alpha \in I}$ be a family of strongly irreducible elements of $\text{gen}(\tau)$ such that $\sigma = \wedge_{\alpha \in I} \{\chi(M_{\alpha})\}$. We can suppose that for all $\alpha \in I$, $M_\alpha$ is a decisive module by [6, *Proposition 32.7*, (3)]. Let $\beta \in I$; we claim that $M_{\beta}$ is a $\sigma$-$A$-module. Since $M_{\beta}$ is decisive and the family $\{\chi(M_{\alpha})\}_{\alpha \in I}$ is irredundant, then $M_{\beta} \in \mathbb{T}_{\chi(M_{\alpha})}$ for all $\alpha \neq \beta$. Hence $\sigma < \sigma \lor \xi(M_{\beta}) \leq \wedge_{\alpha \neq \beta} \chi(M_{\alpha})$. So we have $\sigma = \sigma \land \chi(M_{\beta}) \leq \left[\sigma \lor \xi(M_{\beta})\right] \land \chi(M_{\beta}) \leq \left[\wedge_{\alpha \neq \beta} \chi(M_{\beta})\right] \land \chi(M_{\beta}) = \sigma$. Therefore $\left[\sigma \lor \xi(M_{\beta})\right] \land \chi(M_{\beta}) = \sigma$. Since $R$-tors is a distributive lattice, we obtain that $\xi(M_{\beta}) \land \chi(M_{\beta}) \leq \sigma$. Since $M_{\beta}$ is decisive, then $M_{\beta}$ is a $\chi(M_{\beta})$-$A$-module. Therefore $M_{\beta}$ is a $\chi(M_{\beta})$-$A$-module by [4, *Proposition 2.8*, (2)]. Inasmuch as $M_{\beta} \in \mathbb{F}_{\sigma}$ and $\xi(M_{\beta}) \land \chi(M_{\beta}) \leq \sigma$, we obtain that $M_{\beta}$ is a $\sigma$-$A$-module by [4, *Proposition 2.8*, (1)]. Thus the claim is proved.

Finally, the result follows from *Theorem 2.3*(4). $\square$

An important concept of dimension in ring theory is the $\tau$-Gabriel dimension. See [6] for details about this dimension. In [4] we discussed some relations between $\tau$-atomic dimension and $\tau$-Gabriel dimension. Among other things we proved that if $R$ has left $\tau$-Gabriel dimension, then $R$ has left $\tau$-atomic dimension. In the following examples we will show that the condition “every proper element of $\text{gen}(\tau)$ uniquely decomposes as the meet of an irredundant family of irreducible elements of $\text{gen}(\tau)$” does not characterize rings with $\tau$-Gabriel dimension nor rings with $\tau$-atomic dimension.

**Example 2.10.** If $R$ is any non-discrete rank 1 valuation domain, then $R$-tors $= \{\xi, \chi(R), \chi\}$ and so $R$ has atomic dimension. We also observe that $R$ has decisive dimension by *Corollary 2.6*. However, as pointed out in [9, page 470], $R$ fails to have Gabriel dimension. Note that in this example every element of $R$-tors is irreducible. See also [4, Example 4.10].
**Example 2.11.** This example was provided to us by Professor Mark L. Teply in a personal communication.

Let \([0, 1]\) and \([0, 1]_\ast\) be the closed real interval and a set with two elements \(0 < 1\) respectively. Now let \(X = [0, 1] \times [0, 1]\). Define \((a, b) = (c, d)\) if \(a < c\) or \(a = c\) and \(b = d\).

Then \((X, \leq)\) satisfies the conditions of [10, Theorem 3.1]. Hence there exists a commutative Bezout domain \(R\) such that \(\text{Spec}(R) \cong X\) (as partially ordered sets).

The following facts are true.

1. \(R\) is a valuation domain.
2. \(R\) has a unique maximal ideal \(M\) that corresponds to the element \((1, 1)\) of \(X\).
3. Let \(P\) be a prime ideal. If \(0 \neq P \neq M\), then \(P\) is not finitely generated.
4. Let \(P \in \text{Spec}(R)\). Then \((\cap_{n \in N} P^n) \in \text{Spec}(R)\). If \(P \neq P^2\), then \(\cap_{n \in N} P^n\) is the maximal prime ideal properly contained in \(P\).
5. If \(P\) is the prime ideal associated to the element \((r, 0)\), then \(P\) is idempotent.

Let \(P \in \text{Spec}(R)\). We denote by \(\sigma_p\) the element of \(R\)-tors such that \(L_{\sigma_p} = \{I \mid I\) is an ideal of \(R\) and \(P \subseteq I\}\).

If \(P = P^2\), we denote \(\tau_p\) the element of \(R\)-tors such that \(L_{\tau_p} = \{I \mid I\) is an ideal of \(R\) and \(P \subseteq I\}\).

6. By [2, Theorem 3.3] we know that if \(\tau \in R\)-tors, then either
   i) there exists \(P \in \text{Spec}(R)\) such that \(\tau = \sigma_p\), or
   ii) there exists \(P \in \text{Spec}(R)\) with \(P = P^2\) and \(\tau = \tau_p\).

7. Let \(P \in \text{Spec}(R)\). Then
   i) \(R/P\) is a \(\sigma_p\)-cocritical module.
   ii) If \(P = P^2\), then \(\tau_p\) does not have cocritical modules. Hence \(R\) does not have \(\tau_p\)-G dim.

8. Let \(P\) be the prime ideal associated to \((r, 0)\). Then \(\text{gen}(\tau_p)\) does not have atoms and hence \(R\) does not have \(\tau_p\)-A dim. Also notice that every \(\rho \in R\)-tors is irreducible, but \(\tau_p\) is not strongly irreducible.

**Example 2.12.** In this example we show the existence of a ring \(R\) with the property that \(R\) has left Gabriel dimension and hence \(R\) has left atomic dimension but \(R\) does not have left decisive dimension. The ring involved was constructed by Goodearl [8] to answer a Goldie’s question.

Let \(K\) be a field of characteristic zero, and let \(S = K \llbracket t \rrbracket\) be the formal power series ring over \(K\) in an indeterminate \(t\). Define a \(K\)-linear derivation \(\delta\) on \(S\) as follows:

\[
\delta \left( \sum_{n=0}^{\infty} \alpha_n t^n \right) = \sum_{n=0}^{\infty} n \alpha_n t^n.
\]

Then let \(R = S \llbracket \theta \rrbracket\) be the formal linear differential operator ring with right-hand coefficients.

Goodearl shows that \(R\) is a left and right noetherian ring and \(S\) may be made into a left \(R\)-module that is isomorphic to the cyclic left \(R\)-module \(R/\theta R\). The lattice of \(R\)-submodules of \(S\) is \(S > St > St^2 > \ldots\)

The non-zero \(R\)-submodules of \(S\) are pairwise non-isomorphic and the simple subfactors of this module are pairwise non-isomorphic. See [8] for details.

Now we claim that \(S\) does not contain decisive \(R\)-submodules. To see this, we will show that \(\xi(S t^{n+1}) < \xi(S t^n)\) for all \(n = 0, 1, \ldots\). Assume that \(\xi(S t^{n+1}) = \xi(S t^n)\) for some \(n\). Then \(S t^n/\delta t^{n+1} \in \mathbb{Z}_{\xi(S t^{n+1})}\). Therefore \(\text{Hom}(S t^{n+1}, E(S t^n/\delta t^{n+1})) \neq 0\). Hence there exist submodules \(0 \neq C' \subseteq C \subseteq S t^{n+1}\) such that \(C' / C\) is isomorphic to \(S t^n/\delta t^{n+1}\). This is not possible. So \(\xi(S t^{n+1}) < \xi(S t^n)\) for all \(n = 0, 1, \ldots\). Thus \(S\) does not contain decisive \(R\)-submodules by Examples 1.5(6). Since \(R\) is a left noetherian ring, then \(R\) has left Gabriel dimension and hence \(R\) has left atomic dimension by [4, Proposition 4.2]. On the other hand, \(R\) does not have left decisive dimension by Theorem 2.8.

We also note the following. Let \(\tau = \chi(S) \in R\)-tors, then \(R\) has left \(\tau\)-Gabriel dimension and left \(\tau\)-atomic dimension, but \(R\) does not have \(\tau\)-decisive dimension. Since \(S\) is a \(\tau\)-cocritical module, then \(\tau\) is an irreducible element of \(R\)-tors, but \(\tau\) is not strongly irreducible.
3. \( \mathcal{D} \)-ass

In this section we associate to each module \( M \) a set of hereditary torsion theories; the elements in this set are called \( \mathcal{D} \)-associated to \( M \). As usual \( ass(M) \) will denote the set of prime ideals associated to \( M \).

**Definition 3.1.** Let \( M \in R\text{-Mod} \). We denote by \( \mathcal{D}\text{-}ass(M) = \{ \tau \in \mathcal{D} \mid \text{there is a submodule } N \text{ of } M \text{ such that } N \text{ is } \tau\text{-mod} \} \).

Notice that if \( M \) is a \( \mathcal{D} \)-module, then \( \mathcal{D}\text{-}ass(M) = \{ \chi(M) \} \), by [3, Proposition 1.6].

In the next proposition we will give some properties of \( \mathcal{D}\text{-}ass \) that are not difficult to prove.

**Proposition 3.2.** Let \( M \in R\text{-Mod} \). Then the following conditions hold.

1. For any submodule \( N \) of \( M \), \( \mathcal{D}\text{-}ass(N) \subseteq \mathcal{D}\text{-}ass(M) \subseteq \mathcal{D}\text{-}ass(N) \cup \mathcal{D}\text{-}ass(M/N) \).
2. If \( M = \bigoplus M_i \), then \( \mathcal{D}\text{-}ass(M) = \cup\mathcal{D}\text{-}ass(M_i) \).
3. If \( N \) is an essential submodule of \( M \), then \( \mathcal{D}\text{-}ass(N) = \mathcal{D}\text{-}ass(M) \).
4. If \( M \) is uniform and \( \mathcal{D}\text{-}ass(M) \neq \emptyset \), then \( \mathcal{D}\text{-}ass(M) \) is a singleton.

**Proposition 3.3.** Let \( \tau \in R\text{-}tors \). If \( R \) has left \( \tau\text{-dim} \), then \( \mathcal{D}\text{-}ass(M) \neq \emptyset \) for all \( 0 \neq M \in \mathbb{F}_\tau \).

**Proof.** Let \( 0 \neq M \in \mathbb{F}_\tau \). Since \( R \) has \( \tau\text{-dim} \), there exists a decisive module \( D \) such that \( D \) is a \( (\chi(M),\mathcal{A}) \)-module by Theorem 2.3(4). Hence \( D \in \mathbb{F}_{\mathcal{D}(M)} \). Therefore \( \text{Hom}_R(D, E(M)) \neq 0 \), so there are submodules \( D'' \), \( D' \) of \( D \) with \( D'' \subseteq D' \subseteq D \) and a monomorphism \( D'/D'' \hookrightarrow M \). Inasmuch as \( D'/D'' \in \mathbb{F}_{\mathcal{D}(M)} \), then \( \chi(D'/D'') = \chi(D) \) by Proposition 1.9(3). Thus \( \chi(D'/D'') \in \mathcal{D}\text{-}ass(M) \).

\( \mathcal{D}\text{-}ass(M) \) can be the empty set as the following examples show. \( \square \)

**Example 3.4.** (1) Let \( R = \mathbb{Z}_2 \mathbb{Z}_2 / \mathbb{Z}_2 (\mathbb{Z}_2) \) and let \( Q \) be the maximal ring of quotients of \( R \). Denote by \( \tau_R(R) \) and \( \tau_R(Q) \) the Goldie torsion theory in \( R \) and \( Q \) respectively. In [4, Example 4.11], we showed that \( gen(\tau_R(R)) \) and \( gen(\tau_R(Q)) \) do not contain atoms. Since \( gen(\tau_R(R)) \) and \( gen(\tau_R(Q)) \) are boolean lattices, then for each \( \sigma \in gen(\tau_R(R)) \) or \( \sigma \in gen(\tau_R(Q)) \), the lattice \( gen(\sigma) \) is boolean and atomless. So we can conclude that \( \mathcal{D}\text{-}ass(M) = \emptyset \) for every \( 0 \neq M \in \mathbb{F}_\sigma \).

(2) Let \( R \) and \( S \) be the ring and the module considered in Example 2.12. Since \( S \) does not contain decisive submodules, then \( \mathcal{D}\text{-}ass(S) = \emptyset \).

We denote by \( \mathcal{E}_\tau \) a complete set of representatives of isomorphism classes of indecomposable \( \tau\text{-}torsion \) free injective modules.

Let \( D_\tau = \{ \chi(D) \mid D \text{ is decisive and } D \in \mathbb{F}_\tau \} \).

Let \( E \in \mathcal{E}_\tau \), we denote by \( \mu_E \) the unique element (when it exists) of \( R\text{-tors} \) such that \( \mathcal{D}\text{-}ass(E) = \{ \mu_E \} \).

**Theorem 3.5.** Let \( \tau \in R\text{-}tors \) and suppose \( R \) has left \( \tau\text{-dimm} \). The following conditions are equivalent.

1. The assignment \( \varphi_\tau : \mathcal{E}_\tau \to \mathcal{D}_\tau \) defined by \( \varphi_\tau(E) = \mu_E \) is a bijective function.
2. \( \mathcal{D}\text{-}ass(M) \neq \emptyset \) for every non-zero \( \tau\text{-}torsion \) free module \( M \).
3. \( \mathcal{D}_\tau = \text{gen}(\tau) \cap R\text{-sp} \).
4. \( R \) has left \( \tau\text{-dim} \).
5. Every non-zero \( \tau\text{-torsion} \) free module \( M \) contains a decisive submodule.

**Proof.** (1) \( \Rightarrow \) (2). Let \( 0 \neq M \in \mathbb{F}_\tau \). Since \( R \) has left \( \tau\text{-dim} \), then \( M \) contains a cocritical submodule \( C \). Hence \( E(C) \in \mathcal{E}_\tau \). Therefore \( \varphi_\tau(E(C)) \) is defined by (1). Thus \( \varphi_\tau(E(C)) \in \mathcal{D}\text{-}ass(E(C)) = \mathcal{D}\text{-}ass(C) \subseteq \mathcal{D}\text{-}ass(M) \).

(2) \( \Rightarrow \) (3). Let \( \sigma \in \mathcal{D}_\tau \). Then there exists a decisive module \( D \) such that \( \sigma = \chi(D) \). By hypothesis, \( D \) contains a cocritical submodule \( C \). Therefore \( \chi(C) = \chi(D) \). So \( \sigma \in \text{gen}(\tau) \cap R\text{-sp} \). Now let \( \sigma \in \text{gen}(\tau) \cap R\text{-sp} \). Then there exists a cocritical module \( C \) such that \( \sigma = \chi(C) \). So \( \mathcal{D}\text{-}ass(C) \neq \emptyset \) by (2). Hence there exist \( \rho \in \mathcal{D}\text{-}ass(C) \) and a submodule \( C' \) of \( C \) such that \( C' \) is a \( \rho\text{-mod} \)-module. Since \( \rho \in \mathcal{D} \subseteq \mathcal{A}_C \), then \( \rho = \chi(C') \) by [3, Proposition 1.6]. Finally note that \( \rho = \chi(C') = \chi(C) = \sigma \). Thus \( \sigma \in \mathcal{D}_\tau \).

(3) \( \Rightarrow \) (4). Let \( \sigma \in \text{gen}(\tau) \). Since \( R \) has left \( \tau\text{-dim} \), then there exists a \( \sigma\text{-cocritical} \) module \( C \). So \( \chi(C) \in R\text{-sp} \cap \text{gen} \tau \). Therefore \( \chi(C) \in \mathcal{D}_\tau \) by (3). Hence there exists a decisive \( \tau\text{-torsion} \) free module \( D \) such that
Theorem 2.3

If \( \chi(C) = \chi(D) \). Since \( R \) has left \( \tau \)-G dim, there exists a cocritical submodule \( C' \) of \( D \). Therefore \( \chi(C) = \chi(C') \).

The cocriticality of \( C' \) implies that there exists a non-zero submodule \( C'' \) of \( C' \) that is isomorphic to a submodule of \( C \). Therefore \( R \) has left \( \tau \)-D dim by Theorem 2.3.(4).

(4) \( \Rightarrow \) (1) \( \varphi_\tau \) is a well defined function by (4) and Proposition 3.3. Now we let \( E, E' \in \mathcal{E}_\tau \) such that \( \mu_E = \mu_{E'} \).

Then there exist submodules \( N, N' \) of \( E \) and \( E' \) respectively with the property that \( N \) is a \( \mu_E \)-D-module and \( N' \) is a \( \mu_{E'} \)-D-module. We can assume that \( N \) and \( N' \) are cocritical by hypothesis. On the other hand, we know that \( \mu_E, \mu_{E'} \in \mathcal{D} \subseteq \mathcal{A} \). So \( \chi(N) = \mu_E \) and \( \chi(N') = \mu_{E'} \) by [3, Proposition 1.6]. Therefore \( \chi(N) = \chi(N') \).

Thus \( E = E(N) \cong E(N') = E' \). Now let \( \sigma \in D_\tau \), then there exists a \( \tau \)-torsion free decisive module \( D \) such that \( \sigma = \chi(D) \). We can assume \( D \) is cocritical. Hence \( E(D) \in \mathcal{E}_\tau \). Since \( D \)-ass\((D) = D\)-ass\((E(D)) \), we have that \( \varphi_\tau(E(D)) = \chi(D) = \sigma \).

(2) \( \Rightarrow \) (5) Let \( 0 \neq M \in \mathcal{F}_\tau \). We can assume that \( M \) is cocritical. \( D\)-ass\((M) \neq \emptyset \) by (2). Let \( \sigma \in D\)-ass\((M) \), then there exists a submodule \( N \subseteq M \) such that \( N \) is a \( \sigma \)-D-module. Since \( N \) is cocritical and \( \sigma \in \mathcal{D} \subseteq \mathcal{A} \), then \( \sigma = \chi(N) \) by [3, Proposition 1.6]. Since \( \sigma \in \mathcal{D} \), there exists a decisive module \( D \) such that \( \sigma = \chi(D) \). Then there is a non-zero submodule \( N' \) of \( N \) and a monomorphism \( N' \hookrightarrow D \). Thus \( N' \) is a decisive submodule of \( M \).

(5) \( \Rightarrow \) (2) Let \( 0 \neq M \in \mathcal{F}_\tau \), then \( M \) contains a decisive submodule \( D \) by (5). Therefore \( \chi(D) \in D\)-ass\((M) \).

Let us denote \( \mathcal{P} = \{ \chi(R/P) \mid P \text{ is a prime ideal of } R \} \).

Definition 3.6. Let \( \tau \in R\text{-tors} \) and \( M \in R\text{-Mod} \), \( M \) is called \( \tau \)-\( \mathcal{P} \)-module if \( M \) is a \( \tau \)-\( \mathcal{P} \)-module and \( \chi(M) \in \mathcal{P} \), [3].

The \( \mathcal{P} \)-filtration of \( \tau \in R\text{-tors} \) is defined to be a chain \( \pi_0 \leq \pi_1 \leq \cdots \leq \pi_i \leq \ldots \), satisfying the following conditions:

1. \( \pi_0 = \tau \).
2. If \( i \) is not a limit ordinal, then \( \pi_i = \pi_{i-1} \vee \xi(\{M \mid M \text{ is a } \pi_{i-1}\text{-}\mathcal{P}\text{-module}\}) \).
3. If \( i \) is a limit ordinal, then \( \pi_i = \bigvee_{j < i} \pi_j \).

The dimension associated to this filtration is called the \( \tau \)-\( \mathcal{P} \)-dimension. See [3] for details.

Proposition 3.7. Let \( \tau \in R\text{-tors} \) and suppose \( R \) has left \( \tau \)-G dim. If \( R \) has left \( \tau \)-\( \mathcal{P} \)-dim, then \( R \) has left \( \tau \)-D dim.

Proof. Let \( \sigma \in \text{gen}(\tau) \). Since \( R \) has left \( \tau \)-\( \mathcal{P} \)-dim, then there exists a \( \sigma \)-\( \mathcal{P} \)-module \( M \). So \( \chi(M) = \chi(R/P) \) for some prime ideal \( P \). By hypothesis, there are cocritical modules \( C \) and \( C' \) such that \( C \subseteq M \) and \( C' \subseteq R/P \). Therefore \( \chi(C) = \chi(C') \).

Then there is a non-zero submodule \( C'' \) of \( C \) and a monomorphism \( C'' \hookrightarrow C' \). So \( C'' \) is a decisive module by Proposition 1.4. Hence \( C'' \) is a decisive \( \sigma \)-\( \mathcal{P} \)-module. Thus \( R \) has left \( \tau \)-D dim by Theorem 2.3.

In the next example, we show that the converse of Proposition 3.7 is not in general true.

Example 3.8. Let \( C \) denote the ring of differential polynomials studied by Cozzens in [5] and let \( F \) be the left classical ring of quotients of \( C \). Then let \( R = \left( \begin{array}{cc} C & F \\ O & O \end{array} \right) \). Let \( M \) be a maximal left ideal of \( C \), then observe that up to isomorphism \( S_1 = R/\left( \begin{array}{cc} M \\ O \end{array} \right) \) and \( S_2 = R/\left( \begin{array}{cc} C \\ O \end{array} \right) \) are the only simple left \( R \)-modules.

\[
\text{Spec}(R) = \left\{ P_1 = \left( \begin{array}{cc} O & F \\ O & O \end{array} \right), P_2 = \left( \begin{array}{cc} C & F \\ O & O \end{array} \right) \right\}.
\]

Notice that \( R\text{-tors} = \{ \xi, \sigma_1, \sigma_2, \sigma_1 \vee \sigma_2, \tau, \chi \} \), where \( \sigma_1 = \xi(S_1), \sigma_2 = \xi(S_2) = \chi(S_1), \sigma_1 \vee \sigma_2 = \chi(R/P_1), \tau = \chi(R/P_2) = \chi(S_2) \).

The \( \mathcal{P} \)-filtration of \( \xi \) in \( R\text{-tors} \) is \( \{ \xi < \sigma_2 \} \). Therefore \( R \) does not have left \( \mathcal{P} \)-dimension.

The Gabriel filtration of \( \xi \) in \( R\text{-tors} \) is \( \{ \xi < (\sigma_1 \vee \sigma_2) < (\sigma_1 \vee \sigma_2) \vee \xi(R/P_1)) = \chi \} \). Therefore \( R \) has left Gabriel dimension. On the other hand, \( R\text{-tors} \) is a finite set, so by Corollary 2.6, \( R \) has left decisive dimension. Finally note that \( \mathcal{P} = \{ \chi(R/P_1), \chi(R/P_2) \} \) and \( D = \{ \chi(S_1), \chi(S_2), \chi(R/P_1) \} \).

In the next result, we show that the condition \( D_\tau = \mathcal{P}_\tau \) is necessary to obtain the equivalence in Proposition 3.7.

Proposition 3.9. Let \( \tau \in R\text{-tors} \) and suppose \( R \) has left \( \tau \)-G dim. Then the following conditions are equivalent.

1. \( R \) has left \( \tau \)-\( \mathcal{P} \)-dim.
2. \( R \) has left \( \tau \)-\( \mathcal{D} \)-dim and \( D_\tau = \mathcal{P}_\tau \).

Proposition 3.7

Theorem 3.10

3

R is a left fully bounded noetherian ring.

For all \( M \in R\text{-mod} \), let \( R \) be a \( \tau\)-G dim ring, then \( R/P \) contains a cocritical module \( C \) that is decisive by Proposition 1.4. So \( \chi(C) = \chi(R/P) \in \mathcal{D}_\tau \), hence \( \mathcal{P}_\tau \subseteq \mathcal{D}_\tau \).

Now let \( \chi(R/P) \in \mathcal{P}_\tau \) with \( P \) a prime ideal of \( R \). Since \( R \) has \( \tau\)-G dim, then \( R/P \) contains a cocritical module \( D \).

Proof. (1) \( \Rightarrow \) (2) It is enough to show that \( \mathcal{D}_\tau = \mathcal{P}_\tau \) by Proposition 3.7. Let \( \chi(D) \in \mathcal{D}_\tau \) with \( D \) decisive. Since \( R \) has left \( \tau\)-P dim, there exists \( M \in R\text{-Mod} \) such that \( M \) is a \( \chi(D)\text{-P-module} \). The fact that \( M \) and \( D \) are \( \mathcal{A}\text{-modules} \) implies \( \chi(M) = \chi(D) \) by [3, Proposition 1.6]. On the other hand we know that \( \chi(M) \in \mathcal{P}_\tau \), hence \( \chi(D) \in \mathcal{P}_\tau \). Thus \( \mathcal{D}_\tau \subseteq \mathcal{P}_\tau \).

(2) \( \Rightarrow \) (1) Let \( \sigma \in \text{gen}(\tau) \), \( \sigma \neq \chi \). As \( R \) has left \( \tau\)-D dim, there exists a decisive module \( D \) such that \( D \) is a \( \sigma\text{-A-module} \). Therefore \( D \in \mathcal{D}_\tau \subseteq \mathcal{P}_\tau \). So \( D \) is a \( \sigma\text{-P-module} \). Thus \( R \) has left \( \tau\)-P dim by [3, Proposition 2.5].

Let \( \text{Spec}_\tau(R) \) be the set of \( \tau\)-pure prime ideals of \( R \). In [1], for rings with \( \tau\)-Krull dimension and then in [3] for rings with \( \tau\)-Gabriel dimension, it was determined when \( \text{Spec}(R) \) is large enough for the assignment \( \phi_\tau : \mathcal{E}_\tau \rightarrow \text{Spec}_\tau(R) \) defined by \( \phi_\tau(E) = P_E \) (where \( P_E \) denotes the unique prime associated to \( E \)), to be a bijection. We will say that \( R \) has local bijective Gabriel correspondence with respect to \( \tau \) if \( \phi_\tau \) is a bijective function.

The following result characterizes rings with local bijective Gabriel correspondence with respect to \( \tau \) in terms of \( \tau\)-decisive dimension and the family \( \mathcal{D}_\tau \). □

Theorem 3.10. Let \( \tau \in R\text{-tors} \) and suppose \( R \) has left \( \tau\)-G dim. Then the following conditions are equivalent.

1. \( R \) has local bijective Gabriel correspondence with respect to \( \tau \).
2. \( R \) has left \( \tau\)-D dim and \( \mathcal{D}_\tau = \mathcal{P}_\tau \).
3. For all \( 0 \neq M \in \mathbb{F}_\tau \), \( \mathcal{D}\text{-ass}(M) = \{ \chi(R/P) \mid P \in \text{ass}(M) \} \neq \emptyset \).
4. Every \( 0 \neq M \in \mathbb{F}_\tau \) contains a decisive submodule and \( \mathcal{D}_\tau = \mathcal{P}_\tau \).

Proof. (1) \( \Rightarrow \) (2) By (1) and [3, Theorem 3.3] we have that \( R \) has left \( \tau\)-P dim. Therefore \( R \) has left \( \tau\)-D dim and \( \mathcal{D}_\tau = \mathcal{P}_\tau \) by Proposition 3.7.

(2) \( \Rightarrow \) (3) Let \( 0 \neq M \in \mathbb{F}_\tau \). Hence \( \mathcal{D}\text{-ass}(M) \neq \emptyset \) by (2) and Proposition 3.3. Let \( \sigma \in \mathcal{D}\text{-ass}(M) \); therefore, there exists a submodule \( N \) of \( M \) such that \( N \) is \( \sigma\text{-D-module} \). Since \( \sigma \in \mathcal{D}_\tau \), then \( \sigma = \chi(N) \in \mathcal{D}_\tau \). Therefore \( \sigma \in \mathcal{P}_\tau \) by (2). Hence \( N \) is a \( \sigma\text{-P-module} \) and \( \sigma \in \mathcal{P}\text{-ass}(M) \). Thus \( \mathcal{P}\text{-ass}(M) \neq \emptyset \) for all \( M \in \mathbb{F}_\tau \). Therefore \( \mathcal{D}\text{-ass}(M) = \{ \chi(R/P) \mid P \in \text{ass}(M) \} \neq \emptyset \) by [3, Theorem 3.3].

Now let \( \sigma \in \mathcal{P}\text{-ass}(M) \), then \( \sigma \in \mathcal{P}_\tau = \mathcal{D}_\tau \). So there exists a \( \sigma\text{-D-module} \) \( N \) and \( \sigma \in \chi(N) \) by [3, Proposition 1.6]. Therefore \( \sigma \in \mathcal{D}\text{-ass}(M) \) and thus \( \mathcal{D}\text{-ass}(M) = \mathcal{P}\text{-ass}(M) = \{ \chi(R/P) \mid P \in \text{ass}(M) \} \).

(3) \( \Rightarrow \) (1) Let \( M \in \mathbb{F}_\tau \) and \( \sigma \in \mathcal{D}\text{-ass}(M) \), then there exists a submodule \( N \) of \( M \) and a prime ideal \( P \) of \( R \) such that \( \chi(N) = \sigma = \chi(R/P) \). So \( N \) is a \( \sigma\text{-P-module} \).

Hence \( \sigma \in \mathcal{P}\text{-ass}(M) \). Thus \( R \) has local bijective Gabriel correspondence with respect to \( \tau \) by [3, Theorem 3.3].

(2) \( \Rightarrow \) (4) Clear.

(4) \( \Rightarrow \) (2) Let \( 0 \neq M \in \mathbb{F}_\tau \). By (2), \( M \) contains a decisive submodule \( D \). Hence \( \chi(D) \in \mathcal{D}_\tau = \mathcal{P}_\tau \). So \( \chi(D) \in \mathcal{P}\text{-ass}(M) \). Therefore \( R \) has left \( \tau\)-P dim by [3, Theorem 3.3]. Finally, \( R \) has left \( \tau\)-D dim by Proposition 3.7. □

As a consequence of Theorem 3.10 and [3, Theorem 3.3], we obtain a new characterization of left fully bounded noetherian rings.

Corollary 3.11. Let \( R \) be a left noetherian ring. Then the following conditions are equivalent.

1. \( R \) is a left fully bounded noetherian ring.
2. \( R \) has left \( \mathcal{D}\text{-dim} \) and \( \mathcal{D} = \mathcal{P} \).
3. For all \( 0 \neq M \in R\text{-mod} \), \( \mathcal{D}\text{-ass}(M) = \{ \chi(R/P) \mid P \in \text{ass}(M) \} \).
4. Every \( 0 \neq M \in R\text{-mod} \) contains a decisive submodule and \( \mathcal{D} = \mathcal{P} \).

4. Artinian and semiartinian rings

A well known result from the theory of commutative rings (Akizuki’s Theorem) asserts that a commutative noetherian ring \( R \) is artinian if and only if \( \text{Spec}(R) \) does not contain chains with more than one element. It is known that this result fails in the non-commutative case. In this section we will use the set \( \mathcal{D} \) instead of \( \text{Spec}(R) \) in order to obtain a result similar to Akizuki’s Theorem in the non-commutative context.
Proposition 4.1. The following conditions are equivalent.

(1) The family \( \mathcal{D} \) does not contain chains with more than one element.
(2) Every decisive module contains a simple submodule.

Proof. (1) \( \Rightarrow \) (2) Let \( D \) be a decisive module. We can assume that \( D \) is cyclic. Now, let \( H \) be a maximal submodule of \( D \). Since \( D \) is decisive, then \( D \in \mathbb{T}_{\chi(D/H)} \) or \( D \in \mathbb{F}_{\chi(D/H)} \). Notice that the first case is not possible. So \( D \in \mathbb{F}_{\chi(D/H)} \), hence \( \chi(D/H) \leq \chi(D) \). Since \( D/H \) is a simple module, then \( \chi(D/H) \leq \chi(D) \) is a chain in \( \mathcal{D} \). Therefore \( \chi(D/H) = \chi(D) \) by (1). So there exists a homomorphism from \( D/H \) to \( D \). This proves (2).

(2) \( \Rightarrow \) (1) Let \( D_1 \) and \( D_2 \) be decisive modules such that \( \chi(D_1) \leq \chi(D_2) \). By (1), \( D_1 \) contains a simple submodule \( S_1 \) and \( D_2 \) contains a simple submodule \( S_2 \). So \( \chi(S_1) = \chi(D_1) \) and \( \chi(S_2) = \chi(D_2) \). Therefore \( \chi(S_1) \leq \chi(S_2) \). From this we obtain \( S_1 \cong S_2 \). Thus \( \chi(D_1) = \chi(S_1) = \chi(S_2) = \chi(D_2) \). \( \square \)

As a consequence we obtain the following characterization of left semiartinian rings in terms of the family \( \mathcal{D} \).

Corollary 4.2. The following conditions are equivalent for a ring \( R \).

(1) \( R \) is a left semiartinian ring.
(2) (i) \( \mathcal{D} \) does not contain chains with more than one element.
(ii) Every \( 0 \neq M \in \text{R-mod} \) contains a decisive submodule.
(3) (i) \( \mathcal{D} \) does not contain chains with more than one element.
(ii) \( R \) has left \( \mathcal{D} \)-dimension.

Theorem 4.3. Let \( R \) be a left noetherian ring. Then the following conditions are equivalent.

(1) \( R \) is left artinian ring.
(2) \( \mathcal{D} \) is the set of coatoms in \( \text{R-tors} \).

Proof. (1) \( \Rightarrow \) (2) Since \( R \) is left artinian, then \( R \) is left semiartinian. Therefore the set of coatoms in \( \text{R-tors} \) is \( \{ \chi(S) \mid S \text{ is a simple left } R\text{-module} \} \), and every decisive module contains a simple submodule. Thus \( \mathcal{D} = \{ \chi(D) \mid D \text{ is decisive} \} = \{ \chi(S) \mid S \text{ is simple} \} \).

(2) \( \Rightarrow \) (1) Since \( \mathcal{D} \) is the set of coatoms in \( \text{R-tors} \), then \( \mathcal{D} \) does not contain chains with more than one element. So every decisive module contains a simple submodule by Proposition 4.1. Inasmuch as \( R \) is left noetherian, then every proper element of \( \text{R-tors} \) is a specialization of a coatom of \( \text{R-tors} \) by [6, Proposition 35.1]. Now, let \( 0 \neq M \in \text{R-Mod} \). Then there exists a simple left \( R \)-module \( S \) such that \( \chi(M) \leq \chi(S) \). So \( S \) is isomorphic to a submodule of \( M \). Therefore \( R \) is left semiartinian ring and hence artinian. \( \square \)

Other non-commutative versions of Akizuki’s Theorem can be seen in [11,12].

References