Three-equipped posets and their representations and corepresentations (inseparable case)☆

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We define three-equipped posets, i.e. partially ordered sets equipped with three kinds of binary relations, and their representations and corepresentations over an inseparable cubic field extension $F \subset G$, in characteristic 3. These representations and corepresentations lead to some matrix problems of mixed type over the pair of fields $(F, G)$. Through these problems of linear algebra, we completely describe in evident matrix form the indecomposables for some critical three-equipped posets, reducing the task for the rest of the critical sets to some matrix problems containing the pseudolinear pencil problem as a subproblem.

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1. Introduction

The aim of this paper is to begin the study of representations and corepresentations of three-equipped posets (i.e. of partially ordered sets equipped with three kinds of binary relations) over an inseparable cubic field extension $F \subset G$, in characteristic 3. This line of work is a natural extension of

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the representation theory of ordinary posets via continuing the previous researches on representations and corepresentations of equipped posets (introduced in [28] and [24], respectively).

Representation theory of posets is an important branch of the modern representation theory of algebraic structures. It has its roots in the representation theory of finite-dimensional algebras [1,8,10,23].

It starts at the beginning of the 70s with the work of Nazarova and Roiter [17] where poset representations are introduced. It was developed mainly by the Kiev representation theory school (see for instance [7,12,15,17,25,29,30,35]) using both the original matrix language of [17] and the language of vector spaces [9].

Those researches were extended soon by considering representations of posets with additional structures, in particular, with involution [4,16,19], bipartite [20,22], with an equivalence relation [4,31], dyadic [21], triadic [2], equipped [24,28,32,33].

An equipped poset is a partially ordered set with two kinds of relations. Its representations and corepresentations [24,28] determine some matrix problems of mixed type over a quadratic field extension, primarily investigated in the case of the classical extension $\mathbb{R} \subset \mathbb{C}$.

Developing these ideas, we introduce in the present paper three-equipped posets and their representations and corepresentations over an inseparable cubic field extension $\mathbb{F} \subset \mathbb{G}$, in characteristic 3. Analogously to the case of ordinary posets, we use both the matrix language and the invariant language of vector spaces. The choice of the given definitions is motivated by the real experience in handling and reducing the corresponding matrix problems of mixed type.

It should be mentioned that representations and corepresentations of the three-equipped posets of finite type were described in fact in [13], in the course of a research about schurian representation-finite vector space categories. So, our aim is to deal with the infinite representation case, beginning with a study of indecomposables of the critical three-equipped posets. For some of them, their indecomposables are completely classified and described in evident matrix form. Meanwhile for the others the task is reduced to some matrix problem containing the pseudolinear pencil problem (introduced and solved in [26]) as a subproblem.

The content of the paper is as follows, three-equipped posets are introduced in Section 2, their representations and corepresentations are defined in Sections 3 and 4. Section 5 contains some necessary information about the quadratic and bilinear forms attached to three-equipped posets. In Section 6, by using a ring theoretical interpretation of the problems we deal, the finite type criterion obtained in [13] is reformulated according to our objectives.

Section 7 contains the main classification results consisting in the descriptions of indecomposable representations of the three-equipped poset $K_{10}$ (Theorem 7) and indecomposable corepresentations of the critical three-equipped poset $K_{11}$ (Theorem 8). The proofs, based on linear algebraic methods, involve in particular the known classification of indecomposables for the classical Kronecker pencil problem solved originally in [14] (see also [10,34]). We also establish some properties of dimension vectors of indecomposables of three-equipped posets, in terms of the Tits quadratic form (Corollaries 9 and 11).

The problems on classification of indecomposable corepresentations for $K_{10}$ and representations for $K_{11}$ are reduced in Section 8 to some problem that contains the pseudolinear pencil problem as a subproblem (Theorems 12 and 13).

2. Three-equipped posets

A finite poset $(\mathcal{P}, \leq)$ is called three-equipped if to every pair of its comparable points $x \leq y$ it is assigned one and only one of the values 1, 2 or 3, with notations $x \leq^1 y, x \leq^2 y$ or $x \leq^3 y$, and the following condition holds:

$$\text{If } x \leq^1 y \leq^m z \text{ and } x \leq^n z, \text{ then } n \geq \min\{m, 3\}. \quad (1)$$

A relation $x \leq y$ is called weak, semi-strong or strong if $x \leq^1 y, x \leq^2 y$ or $x \leq^3 y$, respectively. It follows that the composition of a strong relation with any other relation is strong.

1 There exist precisely two non-trivially three-equipped critical posets $K_{10}$ and $K_{11}$ introduced in the text.
For every point \( x \in \mathcal{P} \) we have by (1) \( x \leq^1 x \), or \( x \leq^2 x \). In the first case the point \( x \) will be called weak, in the second one strong. Any relation between an arbitrary point and a strong point is strong.

We write \( x <^1 y \) if \( x \leq^1 y \) and \( x \neq y \). So, \( x \leq^2 y \) implies \( x <^2 y \).

Note that none of the three mentioned relations is a partial order relation in general.

If a three-equipped poset \((\mathcal{P}, \leq)\) is trivially equipped, i.e. it contains only strong points, then it is an ordinary poset.

Let \( X \) be a subset of some poset \( \mathcal{P} \) and \( a \in \mathcal{P} \), we write \( a < X (a \leq X) \) when \( a < x (a \leq x) \) for each \( x \in X \). Analogously, if \( Y \) is another subset of \( \mathcal{P} \) we write \( X < Y (X \leq Y) \) when \( x < y (x \leq y) \) for all \( x \in X \) and \( y \in Y \).

If \( x < y \), the open interval \((x, y)\) is the set of all points \( z \in \mathcal{P} \) such that \( x < z < y \). When \((x, y)\) is empty, we say that \( x < y \) is a short relation. Otherwise the relation is called long.

Let \( x <^n y \) be a long relation. For every \( z \in (x, y) \) such that \( x <^1 z <^m y \), denote \( \mu_z = \mu_z(x, y) = \min\{lm, 3\} \) and set

\[
\mu(x, y) = \max_{z \in (x, y)} \{\mu_z\}.
\]

Obviously \( n \geq \mu(x, y) \) by (1).

Each three-equipped poset is uniquely determined by its diagram constructed as follows.

1. Draw the ordinary Hasse diagram of \( \mathcal{P} \) denoting by \( \Box \) and \( \bigcirc \) the weak and strong points, respectively.
2. If \( x <^1 y \) is a short relation between weak points, join the points \( x \) and \( y \) with \( l - 1 \) additional lines.
3. If \( x <^1 y \) is a long relation between weak points, join \( x \) and \( y \) with \( l - \mu(x, y) \) additional lines.

**Example 1.** Let us consider a three-equipped poset \((\mathcal{P}, \leq)\) given by the diagram

Here \( z \) is the only strong point, so \( \{w, x, y\} <^2 z <^2 d \). Also \( a <^3 \{b, c, d\} \) because \( a <^3 b \). Note that \( b <^1 c <^1 d \) then \( \mu_c(b, d) = \mu(b, d) = 1 \) and the extra line between \( b \) and \( d \) represents \( b <^2 d \). In the same way \( w <^2 y \) although \( \mu(w, y) = 2 \) because \( w <^1 x <^2 y \). On the other hand \( \mu(w, b) = 1 \) due to \( w <^1 x <^1 b \), and \( w <^1 b \) since there are no additional lines between \( w \) and \( b \). As a consequence of \( x <^2 y <^2 c \) we have \( \mu(x, c) = 3 \) and \( x <^3 c \). For the open interval \((y, d) = \{c, z\} \), we get \( \mu_c(y, d) = 2 \) and \( \mu_z(y, d) = 3 \) hence \( \mu(y, d) = 3 \) and \( \{w, x, y\} <^3 d \).

A linearly ordered subset of a three-equipped poset \( \mathcal{P} \) is a chain. An antichain is a subset consisting of mutually incomparable points.

In the next two sections, we will define representations and corepresentations of three-equipped posets.
3. Representations

Throughout the paper, we deal over the pair of fields \((F, G)\) of characteristic 3, where \(G\) is an inseparable cubic extension of \(F\). So, \(G = F(\xi)\) for some fixed primitive element \(\xi\) with the minimal polynomial \(t^3 + p\) over \(F\).

Clearly, every element \(g \in G\) can be written in the form \(g = a + b\xi + c\xi^2\) for some \(a, b, c \in F\), where \(a = \text{Re} g\) is the real part of \(g\) and \(b = \text{Im}_1 g\) and \(c = \text{Im}_2 g\) are its first and second imaginary parts. Analogously, for a subset \(X \subseteq G\) set \(\text{Re} X = \{\text{Re} : g \in X\}\) and \(\text{Im} X = \{\text{Im} g : g \in X\}\) for \(i = 1, 2\).

Let \(\mathcal{M}_3(F)\) be the space of all \(3 \times 3\) matrices over \(F\) (with the identity matrix \(I_3\)) and \(e_{ij} \in \mathcal{M}_3(F)\) be the standard matrix identity having 1 at the place \((i, j)\) and 0 otherwise.

Since \(G\) is a three-dimensional \(F\)-linear space (say, the space \(F^3\) of row vectors \((a, b, c)\) over \(F\)), we can identify \(\mathcal{M}_3(F)\) with the endomorphism ring \(\text{End}_F(G)\) of this space. Thus, each element \(g \in G\), acting by multiplication as an \(F\)-endomorphism of \(G\), is identified with a matrix in \(\mathcal{M}_3(F)\). The obtained inclusion \(G \subseteq \mathcal{M}_3(F)\) induces on \(\mathcal{M}_3(F)\) the structure of a \((G, G)\)-bimodule with respect to ordinary matrix operations.

Denote by \(K(a_1, a_2, \ldots, a_n)\) the vector space spanned by the given vectors \(a_1, a_2, \ldots, a_n\) over a field \(K\). Under this notation the field \(G \subseteq \mathcal{M}_3(F)\) is identified with the matrix space

\[
\Delta_1 = F \left\langle I_3, \Sigma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p & 0 & 0 \end{bmatrix}, \Sigma^2 \right\rangle.
\]

One checks easily that the following three subsets \(\Delta_1 \subseteq \Delta_2 \subseteq \Delta_3\) of \(\mathcal{M}_3(F)\) have a \((G, G)\)-bimodule structure (i.e. \((\Delta_1, \Delta_1)\)-bimodule structure).

\[
\Delta_1, \quad \Delta_2 = \Delta_1 \left\langle I_3, e_{11} - e_{33} \right\rangle, \quad \Delta_3 = \mathcal{M}_3(F) = \Delta_1 \left\langle e_{11}, e_{22}, e_{33} \right\rangle.
\]

These bimodules appear in a natural way in the course of real handling with the matrix problem corresponding to some critical three-equipped posets.

For any finite-dimensional \(F\)-space \(U_0\), consider the induced \(G\) space \(\widetilde{U}_0 = U_0 \otimes_F G\). If \(X \subseteq \widetilde{U}_0\) is an \(F\)-subspace, then its real part \(\text{Re} X\) is the following subspace of \(U_0\)

\[
\text{Re} X = \{u \in U_0 : u \otimes 1 + u' \otimes \xi + u'' \otimes \xi^2 \in X, \text{ for some } u', u'' \in U_0\}.
\]

The imaginary parts \(\text{Im}_1 X\) and \(\text{Im}_2 X\) are defined analogously. Obviously, if \(X\) is a \(G\)-subspace of \(\widetilde{U}_0\), then \(\text{Re} X = \text{Im}_1 X = \text{Im}_2 X\).

A \(G\)-subspace \(X \subseteq \widetilde{U}_0\) is said to be strong if \(X = (\text{Re} X) \otimes_F G\) or, equivalently, if \((\text{Re} X) \otimes 1 \subseteq X\).

A representation \(U\) of a three-equipped poset \(\mathcal{P}\) over the pair \((F, G)\) is a collection

\[
U = (U_0; U_x : x \in \mathcal{P})
\]

where \(U_0\) is a finite-dimensional \(F\)-vector space, each \(U_x\) is a \(G\)-space of \(\widetilde{U}_0\), and for all points \(x, y \in \mathcal{P}\) the following condition holds

\[
x \preceq y \Rightarrow U_x \Delta_1 \subseteq U_y.
\]

For a strong point \(x \in \mathcal{P}\) we have \(U_x \Delta_3 \subseteq U_x\), hence \((\text{Re} U_x \otimes 1) \subseteq U_x\), that is \(U_x\) is a strong \(G\)-subspace.

Representations of a three-equipped poset \(\mathcal{P}\) are the objects of the category of representations \(\text{Rep} \mathcal{P}\).

A morphism \(U \to V\) is an \(F\)-linear map \(\varphi : U_0 \to V_0\) such that \((\varphi \otimes 1)(U_x) \subseteq V_x\) for all \(x \in \mathcal{P}\).

Two representations \(U\) and \(V\) are isomorphic, \(U \simeq V\), if there exists an \(F\)-space isomorphism \(\varphi : U_0 \to V_0\) such that \((\varphi \otimes 1)(U_x) = V_x\) for all \(x \in \mathcal{P}\).

A subrepresentation \(V = (V_0; V_x : x \in \mathcal{P})\) of a representation \(U\) is any representation of \(\mathcal{P}\) such that \(V_0 \subseteq U_0\) and \(V_x \subseteq U_x\) for each \(x \in \mathcal{P}\).

The radical of a representation \(U\) is a subrepresentation \(U = (U_0, U_x : x \in \mathcal{P})\) of \(U\), where \(U_x = \sum_{y \preceq x} U_y \Delta_1\).
The dimension of a representation \( U \) is a vector \( d = \text{dim} U = (d_0; d_x : x \in \mathcal{P}) \) where \( d_0 = \text{dim}_F U_0 \) and \( d_x = \text{dim}_G U_x/U_x \) for each \( x \in \mathcal{P} \).

A matrix representation of a three-equipped poset \( \mathcal{P} \) is a matrix \( M \) partitioned into vertical stripes \( M_x \) over \( G \) if \( x \in \mathcal{P} \) is a weak (strong) point. To this matrices we apply the following admissible transformations:

(a) \( F \)-elementary row transformations of the whole matrix \( M \);
(b) \( F \)-elementary \((G\text{-elementary}) \) column transformations of a stripe \( M_x \) for a strong (weak) point \( x \);
(c) If \( x <^1 y \), additions from columns of the stripe \( M_x \) to columns of the stripe \( M_y \) with coefficients in \( G \);
(d) If \( x <^2 y \), additions of columns from \( h(\text{Reg}M_x - \xi^2 \text{Im}gM_x) \) to columns of \( M_y \) with some coefficients \( h, g \in G \);
(e) If \( x <^3 y \), independent additions from the real or imaginary parts of columns of \( M_x \) to the real or imaginary parts of columns of \( M_y \), with coefficients in \( G \). If \( y \) is a strong point there are no additions to its imaginary parts, which in this case are equal to zero.

Two matrix representations are said to be equivalent or isomorphic if one of them can be obtained from the other one by a sequence of admissible transformations.

The dimension of a matrix representation \( M \) of a three-equipped poset \( \mathcal{P} \) is a vector \( d = \text{dim} M = (d_0; d_x : x \in \mathcal{P}) \), where \( d_0 \) is the number of rows of \( M \) and, for \( x \in \mathcal{P} \), \( d_x \) is the number of columns of the stripe \( M_x \).

Let \( U \) be a representation of a three-equipped poset \( \mathcal{P} \). By choosing a basis of the \( F \)-space \( U_0 \) (which is at the same time a \( G \)-basis of \( U_0 \)), we attach to \( U \) a matrix representation \( M \) such that for every weak (strong) point \( x \in \mathcal{P} \) the columns of the stripe \( M_x \) are formed by the coordinates of some system of generators of the \( G \)-space \( U_x \) \((F \text{-space } \text{Re}U_x) \) modulo \( U_x \) \((\text{Re}U_x) \) with respect to the chosen basis.

Notice that the admissible transformations of a matrix representation \( M \) correspond to those changes of the chosen basis of \( U_0 \) and of the systems of generators of \( U_x \) that do not change the number of generators. Therefore, we have chosen minimal systems of generators if and only if \( \text{dim} M = \text{dim} U \), and in this case the matrix \( M \) is reduced. Otherwise, \( \text{dim} M \geq \text{dim} U \).

So, a matrix isomorphism \( M \simeq M' \) implies an isomorphism of the corresponding representations \( U \simeq V \) (the converse is also true when \( M \) and \( M' \) are reduced).

To deal with posets with infinite non-isomorphic matrix representations we need the following definition of \( F \)-series of indecomposable representations of three-equipped posets.

Define matrix representations over the pair of polynomial rings \((F[t], G[t])\) analogously to the case over the pair \((F, G)\). That is, consider matrices partitioned into vertical stripes over the pair \((F[t], G[t])\) instead of \((F, G)\) (at the moment we do not introduce admissible transformations for these representations).

Each representation over \((F[t], G[t])\) generates an \( F \)-series of representations over \((F, G)\) by substituting a square matrix \( A \) with values in \( F \) for the variable \( t \) and scalar matrices \( gI \) of the same size, for each scalar \( g \in G \).

**Example 2.** For the poset \( K_{10} \)

\[
\begin{pmatrix}
3 & a & b \\
\end{pmatrix}
\]

the representation \( \begin{pmatrix} 1 & 1 \\ \xi & t \end{pmatrix} \) over \((F[t], G[t])\), generates the \( F \)-series \( I_n \) \( \xi I_n \) \( A \), where \( A \) is a matrix of order \( n \) in a standard canonical form with respect to ordinary similarity transformations \((X^{-1}AX)\) over \( F \).
4. Corepresentations

The three-equipped posets lead naturally to the notion of its corepresentations over the pair of fields \((F, G)\), in the following way.

Consider three \(F\)-subspaces \(\Omega_1 \subset \Omega_2 \subset \Omega_3\) of the field \(G\):

\[
\Omega_1 = F, \quad \Omega_2 = F \langle 1, \xi \rangle, \quad \Omega_3 = G = F \langle 1, \xi, \xi^2 \rangle.
\]

A corepresentation \(U\) of a three-equipped poset \(\mathcal{P}\) over the pair \((F, G)\) is any collection of the form

\[
U = (U_0; U_x : x \in \mathcal{P})
\]

where \(U_0\) is a \(G\)-vector space of finite dimension, each \(U_x\) is an \(F\)-subspace of \(U_0\), and the following condition is satisfied for all \(x, y \in \mathcal{P}\):

\[
x \leq y \Rightarrow U_x \Omega_1 \subseteq U_y.
\]

Note that if \(x\) is a strong point, \(U_x\) is a \(G\)-space.

The corepresentations of \(\mathcal{P}\) are the objects of the category of corepresentations \(\text{corep}\mathcal{P}\). A morphism \(U \rightarrow V\) is any \(G\)-linear map \(\varphi : U_0 \rightarrow V_0\) such that \(\varphi(U_x) \subseteq V_x\) for all \(x \in \mathcal{P}\).

We say that two corepresentations \(U\) and \(V\) are isomorphic \((U \simeq V)\) if there exists a \(G\)-isomorphism \(\varphi : U_0 \rightarrow V_0\) such that \(\varphi(U_x) = V_x\) for all \(x \in \mathcal{P}\).

The radical of a corepresentation \(U\) is a subcorepresentation \(U = (U_0, U_x : x \in \mathcal{P})\) in which \(U_x = \sum_{y \leq y} U_y \Omega_1\).

The dimension of a corepresentation \(U\) is a vector \(d = \dim U = (d_0; d_x : x \in \mathcal{P})\), where \(d_0 = \dim_G U_0\), \(d_x = \dim_G U_x / U_x\) (\(d_x = \dim_F U_x / U_x\) if \(x \in \mathcal{P}\) is a strong (weak) point).

A matrix corepresentation of a three-equipped poset \(\mathcal{P}\) is a matrix \(M\) over \(G\) partitioned into vertical stripes \(M_x (x \in \mathcal{P})\). To this matrices one can apply the following admissible transformations:

(a) \(G\)-elementary row transformations of the whole matrix \(M\);
(b) \(F\)-elementary (\(G\)-elementary) column transformations of the stripe \(M_x\) for a weak (strong) point \(x\);
(c) If \(x \leq y\), additions from columns of the stripe \(M_x\) to columns of the stripe \(M_y\) with coefficients in \(\Omega_1\).

Now, we consider some definitions and facts completely analogous to the case of representations of a three-equipped poset \(\mathcal{P}\).

Two matrix corepresentations are said to be equivalent or isomorphic if one of them can be obtained from the other one by a sequence of admissible transformations.

The dimension of a matrix corepresentation \(M\) of \(\mathcal{P}\) is a vector \(d = \dim M = (d_0; d_x : x \in \mathcal{P})\), where \(d_0\) is the number of rows of \(M\) and, for \(x \in \mathcal{P}\), \(d_x\) is the number of columns of the stripe \(M_x\).

A matrix corepresentation \(M\) corresponds to a corepresentation \(U \in \text{corep}\mathcal{P}\) if the columns of each stripe \(M_x\) are formed by coordinates (with respect to a chosen basis of \(U_0\)) of a system of generators of \(U_x\) modulo \(U_x\). Note that the admissible transformations of \(M\) correspond to those changes of the basis of \(U_0\) and of the systems of generators of \(U_x\) that do not change the number of generators.

Then, we have \(\dim M = \dim U\) if and only if the chosen systems of generators are minimal (i.e. the matrix \(M\) is reduced). Otherwise, \(\dim M \geq \dim U\).

So, a matrix isomorphism \(M \simeq M'\) implies an isomorphism of the corresponding corepresentations \(U \simeq V\) (the converse is also true when \(M\) and \(M'\) are reduced).

Every matrix corepresentation over the pair of polynomial rings \((F[t], G[t])\) generates an \(F\)-series of corepresentations over \((F, G)\) by substituting a square matrix \(A\) with values in \(F\) for the variable \(t\), and scalar matrices \(gl\) of the same size, for each number \(g \in G\).
5. The attached forms

Denote by $\mathcal{P}^0$ the enlargement $\mathcal{P}^0 = \mathcal{P} \cup \{0\}$ of the three-equipped poset $\mathcal{P}$ by a unique maximal strong point 0.

Let $\alpha = (\alpha_0; \alpha_x : x \in \mathcal{P})$ and $\beta = (\beta_0; \beta_x : x \in \mathcal{P})$ be vectors in $\mathbb{Z}^{|\mathcal{P}^0|}$, the non-symmetric bilinear form of $\mathcal{P}$ is defined as follows

$$B(\alpha, \beta) = \alpha_0 \beta_0 + \sum_{x \leq y} l_{xy} \alpha_x \beta_y - \alpha_0 \sum_{x \in \mathcal{P}} l_{xx} \beta_x$$

where $l_{xx} = 1$ if $x$ is strong, $l_{xx} = 3$ if $x$ is weak, and if $x < y$ then $l_{xy} = l_{yx}$. We also define the non-symmetric bilinear form of $\mathcal{P}$

$$\hat{B}(\alpha, \beta) = 3 \alpha_0 \beta_0 + \sum_{x \leq y} \hat{l}_{xy} \alpha_x \beta_y - 3 \alpha_0 \sum_{x \in \mathcal{P}} \beta_x$$

where $\hat{l}_{xy} = m$ if $x \leq^m y$.

The symmetric bilinear form $S(\alpha, \beta)$ and coform $\tilde{S}(\alpha, \beta)$ of $\mathcal{P}$ are obtained from the non-symmetric bilinear ones as follows

$$S(\alpha, \beta) = \frac{1}{2} [B(\alpha, \beta) + B(\beta, \alpha)]$$

$$\tilde{S}(\alpha, \beta) = \frac{1}{2} [\hat{B}(\alpha, \beta) + \hat{B}(\beta, \alpha)].$$

The Tits quadratic form $f$ of $\mathcal{P}$ is

$$f = f(\alpha) = S(\alpha, \alpha) = B(\alpha, \alpha) = \alpha_0^2 + \sum_{x \leq y} l_{xy} \alpha_x \alpha_y - \alpha_0 \sum_{x \in \mathcal{P}} l_{xx} \alpha_x.$$

Analogously, the Tits quadratic coform $\tilde{f}$ of $\mathcal{P}$ is

$$\tilde{f} = \tilde{S}(\alpha, \alpha) = \hat{B}(\alpha, \alpha) = \hat{f}(\alpha) = 3 \alpha_0^2 + \sum_{x \leq y} \hat{l}_{xy} \alpha_x \alpha_y - 3 \alpha_0 \sum_{x \in \mathcal{P}} \alpha_x.$$

A simple root $d_i$ of the Tits quadratic form $f$ (coform $\tilde{f}$) associated to a three-equipped poset $\mathcal{P}$, is a vector in $\mathbb{Z}^{|\mathcal{P}^0|}$ with the unique non-zero coordinate $i$ equal to 1.

The reflections $w_x(\alpha)$ and $\hat{w}_x(\alpha)$ of a vector $\alpha$ at some point $x \in \mathcal{P}^0$ are

$$w_x(\alpha) = \alpha - \frac{2}{l_{xx}} S(\alpha, d_x) d_x, \quad \hat{w}_x(\alpha) = \alpha - \frac{2}{\hat{l}_{xx}} \tilde{S}(\alpha, d_x) d_x.$$

The positive vectors obtained by reflections $w_x$ ($\hat{w}_x$) of the simple roots of $f$ ($\tilde{f}$) are the admissible roots of the Tits quadratic form $f$ (coform $\tilde{f}$). A vector $d \in \mathbb{Z}^{|\mathcal{P}^0|}$ is an imaginary root of $f$ or $\tilde{f}$ if $f(d) = 0$ or $\tilde{f}(d) = 0$, respectively. For the three-equipped posets case, the value of the Tits form $f$ (coform $\tilde{f}$) evaluated on the admissible roots is 1 or 3.

In the following examples we have calculated some admissible roots.

**Example 3.** The admissible roots of the Tits quadratic form

$$f(d) = d_0^2 + 3d_a^2 + d_b^2 - d_0(3d_a + d_b)$$

of the three-equipped poset $K_{10}$, of the form (2), are (1, 1, 1); (2, 1, 1); (2 + 1, n + 1, 1 + 1); (2 + 1, n + 1, n + 1); (2 + 1, n + 1, n + 1); (2 + 2, n + 1, n + 1); (2 + 3, n + 2, n + 1); (3, 2, 3); (6, 4, 3); (3, 1, 3); (6, 2, 3); (9, 4, 3); (12, 5, 6); (6m + 3, 3m + 2, 3m); (6m + 3, 3m + 1, 3m + 3); (6n + 3, 3n + 2, 3n + 3); (6n + 9, 3n + 4, 3n + 3); (6n + 6, 3n + 4, 3n + 3); (6n + 12, 3n + 5, 3n + 6).

The value of the Tits form evaluated on the reflections of the simple root (0, 1, 0) is equal to 3, the value of the Tits form for the other admissible roots is 1.
Example 4. The admissible roots of the Tits quadratic coform
\[ \tilde{f}(d) = 3d_0^2 + d_1^2 + d_2^2 + d_1d_2 - 3d_0(d_1 + d_2) \]
of the three-equipped poset \( K_{11} \)
\[
\begin{array}{cc}
3 & b \\
3 & a \\
\end{array}
\]
are the following (2, 3, 3); (4, 6, 3); (4, 3, 4); (4, 3, 3); (5, 3, 6); (n + 1, n + 1, n); (n, n, n + 1); (n + 1, n, n + 2); (n + 1, n + 2, n); (n, n + 1, n); (n + 1, n, n + 1); (3m + 2, 3m + 3, 3m); (3m + 1, 3m, 3m + 3); (3n + 2, 3n + 3, 3n + 3); (3n + 4, 3n + 3, 3n + 3); (3n + 4, 3n + 6, 3n + 3); (3n + 5, 3n + 3, 3n + 6).

In this case, the value of the Tits coform evaluated on the reflections of the simple root (1,0,0) is 3, and the value of the Tits coform for the other admissible roots is 1.

In the next sections we will show how the roots of the Tits form and coform are related to the representations and corepresentations.

6. Three-equipped posets of finite type

In this section, the known criterion of finite type for representations and corepresentations of three-equipped posets (obtained by Klemp and Simson, in fact in [13]) is reformulated in the language introduced above. First we need to give some definitions.

A three-equipped poset \( Q \) is an implicit subposet of a three-equipped poset \( P \) if every point \( x \in Q \) is a point of \( P \), and, for \( x, y \in Q \), if \( x \leq^l y \) in \( P \) then \( x \leq^m y \) in \( Q \) with \( l \leq m \).

In other words, an implicit subposet of a three-equipped poset \( P \) can be obtained by eliminating points of \( P \), by adding relations, or strengthening some of its existing relations.

Example 5

(i) Any subposet of a three-equipped poset, in the usual sense, is an implicit subposet.

(ii) The poset \( \begin{array}{cc}
3 & b \\
3 & a \\
\end{array} \)
contains the implicit subposets \( \begin{array}{cc}
3 & \text{b} \\
\text{a} \\
\end{array} \) and \( \begin{array}{cc}
3 & \text{b} \\
\text{a} \\
\end{array} \).

(iii) The three-equipped poset \( \begin{array}{cc}
3 & \text{a} \\
\text{a} \\
\end{array} \)
is an implicit subposet of \( \begin{array}{cc}
3 & \text{a} \\
\text{a} \\
\end{array} \).

The decomposable and indecomposable representations and corepresentations, and their direct sums, are defined in a usual way.

A three-equipped poset is representation-finite (corepresentation-finite) if it has a finite number of pairwise non-isomorphic indecomposable representations (corepresentations).

We associate to each three-equipped poset \( P \) two incidence rings:
\[ \Lambda' = \Lambda'(P) = \bigoplus_{x,y \in P} \Lambda'_{xy}, \quad \widehat{\Lambda} = \widehat{\Lambda}(P) = \bigoplus_{x,y \in P} \widehat{\Lambda}_{xy}, \]
where
\[ \Lambda'_{xy} = \begin{cases} \Delta_l, & \text{if } x \leq^l y, \\ 0, & \text{if } x \not\leq^l y, \end{cases} \quad \text{and} \quad \widehat{\Lambda}_{xy} = \begin{cases} \Omega_l, & \text{if } x \leq^l y, \\ 0, & \text{if } x \not\leq^l y. \end{cases} \]

Obviously \( \widehat{\Lambda}(P^0) \) is an artinian schurian right peak ring in the sense of [13]. From the characterization given there, of artinian schurian right peak rings with a finite number of isomorphism classes of indecomposable socle projective right modules, we can obtain a finite type criterion in our situation.
To apply that result, instead of the ring \( \Lambda'(P^0) \), we have to deal with its Morita equivalent artinian schurian right peak ring \( \Lambda(P^0) \) obtained as follows.

For a three-equipped poset \( P \), assign to each point \( x \in P \) the field \( \Lambda_{xx} = G(\Lambda_{xx} = F) \) if \( x \) is weak (strong). And to every pair of distinct points \( x \) and \( y \) it is assigned the following \( (\Lambda_{xx}, \Lambda_{yy}) \)-bimodule \( \Lambda_{xy} \)

\[
\Lambda_{xy} = \begin{cases} 
\Delta_1, & \text{if } x <^1 y, \\
\Delta_2, & \text{if } x <^2 y, \\
\text{Hom}_F(\Lambda_{xx}, \Lambda_{yy}), & \text{if } x <^3 y, \\
0, & \text{if } x \not< y.
\end{cases}
\]

Then, the incidence ring \( \Lambda \) of \( P \) is

\[
\Lambda = \Lambda(P) = \bigoplus_{x,y \in P} \Lambda_{xy}.
\]

The representations and corepresentations of \( P \) are the socle projective right modules over the incidence rings \( \Lambda(P^0) \) and \( \Lambda'(P^0) \), respectively.

Notice that the coefficients \( l_{xy}, \hat{l}_{xy} \) of the Tits quadratic form and coform (see Section 5) of a three-equipped poset \( P \) are \( l_{xy} = \dim_F \Lambda_{xy} \) and \( \hat{l}_{xy} = \dim_F \hat{\Lambda}_{xy} \).

As a consequence of [13, Theorem A], we get the following result.

**Theorem 6.** For any three-equipped poset \( P \), the following conditions are equivalent.

1. \( P \) is representation-finite.
2. \( P \) is corepresentation-finite.
3. The Tits quadratic form \( f \) of \( P \) is weakly positive, i.e. \( f(d) > 0 \) for every vector \( d > 0 \).
4. The Tits quadratic coform \( \hat{f} \) of \( P \) is weakly positive.
5. \( P \) does not contain any of the critical three-equipped posets \( K_1, \ldots, K_5, K_{10} \) and \( K_{11} \) as implicit subposets, where \( K_1 = (1, 1, 1, 1), K_2 = (2, 2, 2), K_3 = (1, 3, 3), K_4 = (N, 4), K_5 = (1, 2, 5) \) are the well known ordinary Kleiner’s critical posets and \( K_{10}, K_{11} \) are the non-trivially equipped ones of the form \( (2) \) and \( (3) \), respectively.

Our notations \( K_{10}, K_{11} \) continue the notations of [32] where the critical non-trivially equipped posets \( K_6, \ldots, K_9 \), were considered.

The indecomposable representations and corepresentations whose dimensions do not have zero coordinates are called sincere. A three-equipped poset is sincere with respect to representations or corepresentations if it has at least one sincere indecomposable representation or corepresentation, respectively.

By [13, Theorem B], the only sincere non-trivial three-equipped posets of finite type (no difference, with respect to representations or corepresentations) are the following two

\[
F_{19} \quad 3 \quad F_{20} \quad 3.
\]

Their sincere indecomposables representations and corepresentations can be calculated in a simple way. We have obtained their evident matrix forms presented in Appendix A (together with the values of the Tits quadratic form \( f \) and coform \( \hat{f} \) on their dimensions).

7. The main classification results

In this section we prove the following two main theorems.
Theorem 7. The sincere indecomposable representations of the critical three-equipped poset \( K_{10} \) of the form (2) are exhausted, up to isomorphism, by the pairwise non-isomorphic matrix representations listed in Appendix B.

Theorem 8. The sincere indecomposable corepresentations of the critical three-equipped poset \( K_{11} \) of the form (3) are exhausted, up to isomorphism, by the pairwise non-isomorphic matrix representations listed in Appendix C.

In the proof we apply the following conventions and observations.

(\( \alpha \)) The \( n \)th indecomposable representation (corepresentation) of the poset \( K_{10} (K_{11}) \) presented in Appendix B (Appendix C) will be denoted by \( K_{10} - n (K_{11} - n) \).

In the matrices the empty blocks represent zero-blocks. A block marked by \( F(G) \) is an arbitrary matrix over \( F(G) \). The symbol \( I_n \uparrow (I_n \downarrow) \) means the identity matrix of size \( n \), with an additional zero row above (below). In the same way, the symbol \( T_n \uparrow (T_n \downarrow) \) represents the identity matrix with an additional zero column on its right (left). We denote by \( e^i_j \) the standard matrix identities, by \( e^i \) a column vector with 1 in its \( i \)th coordinate and 0 in the others, and by \( f^j \) a row vector with its unique non null \( j \)th coordinate equal to 1. A block with the symbol \( J_n \uparrow (0) (J_n \downarrow (0)) \), is a Jordan block of order \( n \) with eigenvalue 0 and every element in its superdiagonal (subdiagonal) equal to 1. If there is an arrow in the upper part of the symbol \( J_n \uparrow (0) (J_n \downarrow (0)) \) the corresponding Jordan block has an additional null row (column) in the upper or lower (left or right) part.

In the proofs, we draw in front of each matrix, the diagram determined by those admissible transformations which do not change the corresponding matrix form. If we have a weak point diagram in front of a matrix consisting in three blocks over \( F \) in the following way

\[
\begin{bmatrix}
F \\
F \\
F
\end{bmatrix}
\]

that means that the corresponding horizontal blocks have the same number of rows and one can multiply them (i.e. the three blocked matrix) from the left to arbitrary non-singular matrices over \( F \), of the form

\[
\begin{bmatrix}
X & Y & Z \\
-pZ & X & Y \\
-pY & -pZ & X
\end{bmatrix}
\]

where \( X, Y \) and \( Z \) are square matrices over \( F \), too. The effect of this matrix multiplication is that the three \( F \)-blocks to behave like the real and imaginary parts of a matrix over \( G \). In fact, remember that \( G = \Delta_1 \) is the subset \( F \left( I_3, \Sigma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p & 0 & 0 \end{bmatrix}, \Sigma^2 \right) \) of \( M_3(F) \). Thus, we reduce these three blocks together as a weak point: \( F \)-column transformations and, in the matrix sense, \( G \)-row transformations.

(\( \beta \)) By Kronecker rhombus we mean the next diagram.

\[
\begin{tikzpicture}
  \node (x) at (0,0) {$x$};
  \node (y) at (1,1) {$y$};
  \node (z) at (1,2) {$z$};
  \node (w) at (0,-1) {$w$};
  \draw (x) -- (y) -- (z) -- (w) -- (x);
  \draw (y) -- (w);
\end{tikzpicture}
\]
Its matrix problem has infinite representation type and it can be described in the following way.

\[
M = \begin{bmatrix}
G & F & F & G \\
G & F \oplus F & G & G
\end{bmatrix}
\]

It consists in a four striped matrix \( M \), to which we apply \( F \)-elementary row transformations (represented by \( F \) on the right of the matrix). The stripes corresponding to the weak points \( M_w \) and \( M_z \) are over \( G \) and they admit \( G \)-elementary column transformations (indicated by \( G \) below). The stripes \( M_x \) and \( M_y \) are over \( F \), they correspond to the Kronecker problem (see (34)) so they have the same size, and their columns are transformed by simultaneous \( F \)-elementary transformations (indicated by \( F \oplus F \) below). We can add columns from \( M_w \) simultaneously to a column of \( M_x \) and a column of \( M_y \) numbered equally, with coefficients \( * \) and \( * \xi \), respectively, where \( * \in G \) is an arbitrary element. Also, we can simultaneously add any column from \( M_x \) and the column of \( M_y \) numbered equally, to any column of \( M_x \) with coefficients \(- * \xi \) and \(* \), respectively.

The two points \( x \) and \( y \), in the Kronecker problem, are strong points so we have not additions to their imaginary parts (equal to zero in this case).

(\( \gamma \)) Let \( \mathcal{P} \) be a three-equipped poset with a maximum strong point \( z \) and \( U \) be a corepresentation of \( \mathcal{P} \). Notice that \( U_z = \sum_{x \in \mathcal{P} \setminus \{ z \}} U_x \Omega_3 \) is a \( G \)-space and if \( U_z \) and \( U_x \setminus U_z \) are not nulls, \( U \) can be decomposed into a direct sum of corepresentations \( U = V \oplus W \) where \( V_0 = U_v, V_x = U_x \) for every \( x \in \mathcal{P} \setminus \{ z \} \), \( V_z = U_z, W_0 = U_0 \setminus U_z, W_x = 0 \) for every \( x \in \mathcal{P} \setminus \{ z \} \), and \( W_z = U_z \setminus U_z \).

If \( U \) is a representation of \( \mathcal{P} \) we have that \( U_z = \sum_{x \in \mathcal{P} \setminus \{ z \}} U_x \Delta_3 \), so it is a strong \( G \)-subspace of the strong \( G \)-subspace \( U_z \) and it has the form \( \overline{U}_z = \sum_{x \in \mathcal{P} \setminus \{ z \}} \text{Re} U_x \otimes F \ G = \text{Re} U_z \otimes F \ G \). Again, if \( U_z \) and \( U_x \setminus U_z \) are non-zero we can decompose \( \overline{U} \) into a direct sum of representations \( U = V \oplus W \). These representations \( V \) and \( W \) are formed by the vector spaces \( V_0 = \text{Re} U_z, V_x = U_x \) for every \( x \in \mathcal{P} \setminus \{ z \} \), \( V_z = U_z, W_0 = U_0 \setminus \text{Re} U_z, W_x = 0 \) for every \( x \in \mathcal{P} \setminus \{ z \} \), and \( W_z = \overline{U}_z \setminus U_z \).

Then each indecomposable corepresentation (representation) of \( \mathcal{P} \) is completely determined by an indecomposable corepresentation (representation) of \( \mathcal{P} \setminus \{ z \} \) or by the indecomposable corepresentation (representation) of \( \{ z \} \) (it is an ordinary poset consisting in a point, so its indecomposables are of the form \( Q(z) \) or \( T(z) \) only). Dually, we have the same situation if the strong point \( z \) is the minimum of \( \mathcal{P} \).

In matrix terms we can make null any element in the stripe corresponding to the maximum point \( z \) with additions of type (e) or (c) (see pages 1831 and 1832) from any no null element in the same row of a stripe corresponding to a point \( x \in \mathcal{P} \setminus \{ z \} \).

For the dual case when \( z \) is not the maximum but the minimum, with additions from the no null elements of the stripe corresponding to \( z \) the elements in other stripes get turned into 0.

**Proof of Theorem 8.** Let \( M \) be a matrix corepresentation of \( K_{11} \), we reduce the stripe \( M_0 \) corresponding to the minimal point into direct sums of indecomposable corepresentations of a weak point (poset \( F_{19} \)). Then, by using admissible additions from \( M_0 \) we make null some blocks of the stripe \( M_0 \). Recall that the column transformations of \( M_0 \) are over \( F \). The following problem is obtained in the stripe \( M_0 \).

\[
\begin{array}{c|cc|c|c|}
\hline
l & \xi l & \xi^2 l & \xi F \\
\hline
l & \xi l & \xi^2 l & 0 \\
0 & l & -\xi l & G \\
1 & \xi F + \xi^2 F & G \\
\hline
\end{array}
\]
By \((\gamma)\), the corepresentation of \(K_{11}\) that can not be restored from the rhombus indecomposables is \(\hat{K}_{11} - 4\) which is obtained by the only indecomposable representation of the maximum strong point in the problem of \(M_b\).

Now, we are going to restore the rest of indecomposable corepresentations of \(K_{11}\) from the indecomposable representations of the Kronecker rhombus \((4)\).

In the case \(M_w = M_x = M_y = 0\), the indecomposable representations of the weak point \(z\) (maximum of the Kronecker rhombus) produces \(\hat{K}_{11} = 5, \hat{K}_{11} - 6, \hat{K}_{11} - 18\) and \(\hat{K}_{11} - 19\).

Let us consider the central part of the Kronecker rhombus when \(M_w = M_z = 0\)

\[
\begin{pmatrix}
\xi F + \xi^2 F
\end{pmatrix}
\begin{array}{c}
\circ
\
\circ
\end{array}
\]

That represents precisely the classical Kronecker problem. So the pair of matrices over \(F\) are transformed by the rule \((A, B) \mapsto (X^{-1}AY, X^{-1}BY)\), where \(X\) and \(Y\) are invertible square matrices over \(F\).

To write the corresponding matrix representations we use the formulation of a more general result given in [34, Theorem 1]. According to it, the indecomposable pair of matrices in which \((A, B)\) could be transformed are exhausted, up to isomorphism, by pairs of the following types

0: \((I_n, \Phi(f))\), where \(\Phi(f)\) is the companion matrix of a power \(f\) of a prime polynomial.
1: \((J_n^+, J_n^- (0))\).
2: \((I_n^+, I_n^- )\).
3: \((J_n^+, J_n^- )\).

The indecomposable pair of matrices of 0 and 1 type give us the \(F\)-series \(\hat{K}_{11} - 1\). The matrix corepresentations given by the 2 and 3 types of indecomposables of the Kronecker problem are \(\hat{K}_{11} - 10\) and \(\hat{K}_{11} - 11\), respectively.

The points \(x\) and \(y\) in the central part of the rhombus \((4)\) are strong, so we have not additions to its imaginary parts. That is, when we add a column of the stripe \(M_w\) to the \(i\)th-column of \(M_x\) with an arbitrary coefficient \(g \in G\), we add only the real part of \(g M_w\), and we simultaneously add the same column of \(M_w\) multiplied by \(g\xi\) to the corresponding \(i\)th-column of \(M_y\), but we add only the real part of \(\xi g M_w\), i.e. \(-p \text{Im}_2 g M_w\). Then we simultaneously add the \(i\)th-column of \(M_x\) with coefficient \(-h\xi\) \((h \in G)\) and the \(i\)th-column of \(M_y\) with coefficient \(h\) to a column of \(M_z\). The resulting additions from \(M_w\) to \(M_z\) are \(h(-\xi \text{Reg}\_M_w - p \text{Im}_2 g M_w)\).

Notice that

\[
\begin{align*}
    h(-\xi \text{Reg}\_M_w - p \text{Im}_2 g M_w) &= h(-\xi)(-\xi^{-1})(-\xi \text{Reg}\_M_w - p \text{Im}_2 g M_w) \\
    &= h'(p^{-1}\xi^2)(-\xi \text{Reg}\_M_w - p \text{Im}_2 g M_w) \\
    &= h'(\text{Reg}\_M_w - \xi^2 \text{Im}_2 g M_w)
\end{align*}
\]

then, the additions from \(M_w\) to \(M_z\) correspond to additions of the case \(w <^2 z\). That is, in the matrix corepresentation problem of \(K_{11}\),

\[
\begin{array}{ccc}
M_a & M_b \\
I & \xi I & \xi^2 I & 0 \\
0 & I & -\xi I & G \\
& & & \mathbb{G} \\
\end{array}
\]

which produces \(\hat{K}_{11} - 2\) (by using the sincere indecomposable representation of the poset \(F_{20}\)).
For the lower part of the rhombus

we reduce first the minimal weak point and using suitable columns additions to the stripes $M_x$ and $M_y$, we annul some blocks to obtain the following problem:

To draw the problem in the last way, notice that the row transformations are simultaneous for the blocks marked by $A$ and $B$ (they are over $F$), and every column transformation of the stripe $M_y$ must be apply simultaneously to the stripe $M_x$. Then by transposing matrices, we have in matrix diagram language

The maximum strong point problem in the stripe $M_y$ produces $\tilde{K}_{11} - 7$.

In the case when $A = B = 0$, by reducing the weak point problem in the stripe $M_y$ we have $\tilde{K}_{11} - 8, \tilde{K}_{11} - 9, \tilde{K}_{11} - 20$ and $\tilde{K}_{11} - 21$.

To find the sincere indecomposables of the upper part of the rhombus
we reduce first the pencil into the matrix representations shown in [34, Theorem 1] (see page 1838). When the stripes corresponding to the pencil are 0 type, the admissible additions from \(M_x\) and \(M_y\) to \(M_z\) are given by

\[
(-\xi I_n + J(0))X
\]

where \(X\) is any column vector of size \(n\) with values in \(G\). The determinant of \(-\xi I_n + J(0)\) is equal to \(\xi^n\), then we can make always null the stripe \(M_z\).

When the pair \((M_x, M_y)\) is reduced as I type, that is \(M_x = I_n\), and \(M_y = \Phi(f)\) the companion matrix of a power \(f\) of a prime polynomial. Their admissible additions to \(M_z\) are given by

\[
(-\xi I_n + \Phi(f))X
\]

\[X\] is any column vector of size \(n\) with values in \(G\). The determinant of \(-\xi I_n + \Phi(f)\) is equal to \((-1)^n f(\xi)\), then when \(f(\xi) \neq 0\) we turn to 0 all the stripe \(M_z\).

When the stripes corresponding to the pencil are 0 type, the admissible additions from \(M_x\) and \(M_y\) to \(M_z\) are given by

\[
(-\xi I_n + J(0))X
\]

where \(X\) is any column vector of size \(n\) with values in \(G\). The determinant of \(-\xi I_n + J(0)\) is equal to \(\xi^n\), then we can make always null the stripe \(M_z\).

When the pair \((M_x, M_y)\) is reduced as I type, that is \(M_x = I_n\), and \(M_y = \Phi(f)\) the companion matrix of a power \(f\) of a prime polynomial. Their admissible additions to \(M_z\) are given by

\[
(-\xi I_n + \Phi(f))X
\]

\[X\] is any column vector of size \(n\) with values in \(G\). The determinant of \(-\xi I_n + \Phi(f)\) is equal to \((-1)^n f(\xi)\), then when \(f(\xi) \neq 0\) we turn to 0 all the stripe \(M_z\).

When the pair \((M_x, M_y)\) is reduced as I type, that is \(M_x = I_n\), and \(M_y = \Phi(f)\) the companion matrix of a power \(f\) of a prime polynomial. Their admissible additions to \(M_z\) are given by

\[
(-\xi I_n + \Phi(f))X
\]

\[X\] is any column vector of size \(n\) with values in \(G\). The determinant of \(-\xi I_n + \Phi(f)\) is equal to \((-1)^n f(\xi)\), then when \(f(\xi) \neq 0\) we turn to 0 all the stripe \(M_z\).

When the pair \((M_x, M_y)\) is reduced as I type, that is \(M_x = I_n\), and \(M_y = \Phi(f)\) the companion matrix of a power \(f\) of a prime polynomial. Their admissible additions to \(M_z\) are given by

\[
(-\xi I_n + \Phi(f))X
\]

\[X\] is any column vector of size \(n\) with values in \(G\). The determinant of \(-\xi I_n + \Phi(f)\) is equal to \((-1)^n f(\xi)\), then when \(f(\xi) \neq 0\) we turn to 0 all the stripe \(M_z\).

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When the pair \((M_x, M_y)\) is reduced as I type, that is \(M_x = I_n\), and \(M_y = \Phi(f)\) the companion matrix of a power \(f\) of a prime polynomial. Their admissible additions to \(M_z\) are given by

\[
(-\xi I_n + \Phi(f))X
\]

\[X\] is any column vector of size \(n\) with values in \(G\). The determinant of \(-\xi I_n + \Phi(f)\) is equal to \((-1)^n f(\xi)\), then when \(f(\xi) \neq 0\) we turn to 0 all the stripe \(M_z\).

When the pair \((M_x, M_y)\) is reduced as I type, that is \(M_x = I_n\), and \(M_y = \Phi(f)\) the companion matrix of a power \(f\) of a prime polynomial. Their admissible additions to \(M_z\) are given by

\[
(-\xi I_n + \Phi(f))X
\]

\[X\] is any column vector of size \(n\) with values in \(G\). The determinant of \(-\xi I_n + \Phi(f)\) is equal to \((-1)^n f(\xi)\), then when \(f(\xi) \neq 0\) we turn to 0 all the stripe \(M_z\).

When the pair \((M_x, M_y)\) is reduced as I type, that is \(M_x = I_n\), and \(M_y = \Phi(f)\) the companion matrix of a power \(f\) of a prime polynomial. Their admissible additions to \(M_z\) are given by

\[
(-\xi I_n + \Phi(f))X
\]

\[X\] is any column vector of size \(n\) with values in \(G\). The determinant of \(-\xi I_n + \Phi(f)\) is equal to \((-1)^n f(\xi)\), then when \(f(\xi) \neq 0\) we turn to 0 all the stripe \(M_z\).

When the pair \((M_x, M_y)\) is reduced as I type, that is \(M_x = I_n\), and \(M_y = \Phi(f)\) the companion matrix of a power \(f\) of a prime polynomial. Their admissible additions to \(M_z\) are given by

\[
(-\xi I_n + \Phi(f))X
\]

\[X\] is any column vector of size \(n\) with values in \(G\). The determinant of \(-\xi I_n + \Phi(f)\) is equal to \((-1)^n f(\xi)\), then when \(f(\xi) \neq 0\) we turn to 0 all the stripe \(M_z\).
The indecomposables of each weak point of the chain lead to four indecomposables of the upper part of the rhombus, from them we have $K_{11} - 14$, $K_{11} - 16$, $K_{11} - 24$ and $K_{11} - 26$. There are also the four corresponding indecomposables of the lower part of the rhombus which produce $K_{11} - 13$, $K_{11} - 17$, $K_{11} - 25$ and $K_{11} - 27$.

Every $n \in \mathbb{N}$ has a corresponding subposet of the chain of the form $F_{20}$. Its sincere indecomposable matrix corepresentation produces the following indecomposable of the upper part of the rhombus:

$$
\begin{align*}
&M_x & & M_y & & M_z \\
0 & 0 & & 0 & & 0 \\
I_m & 0 & & 0 & & 0 \\
0 & I_m & & 0 & & 0 \\
0 & 0 & & I_k & & 0 \\
I_{k(n-1)} & 0 & & 0 & & 0 \\
0 & I_k & & 0 & & 0 \\
0 & 0 & & 0 & & 0
\end{align*}
$$

From it we obtain $K_{11} - 28$ and an indecomposable of the lower part of the rhombus which produces $K_{11} - 29$.

If $n = 0$ the indecomposable of the upper part of the rhombus has the form

$$
\begin{align*}
&M_x & & M_y & & M_z \\
0 & 1 & & 0 & & 0 \\
1 & 0 & & 0 & & 0 \\
0 & 0 & & 0 & & 0 \\
0 & 0 & & 0 & & 0 \\
0 & 0 & & 0 & & 0 \\
0 & 0 & & 0 & & 0
\end{align*}
$$

and it gives us $K_{11} - 15$. Its corresponding indecomposable of the lower part of the rhombus gives us $K_{11} - 12$.

The complete rhombus has the sincere matrix representation

$$
\begin{align*}
&M_w & & M_x & & M_y \\
1 & 0 & & 0 & & 0 \\
\xi & 0 & & 0 & & 0 \\
\xi^{-2} & 0 & & 0 & & 0 \\
0 & I_{3m} & & \Phi(t^2 + p)^m & & 0
\end{align*}
$$

from it we obtain $K_{11} - 3$, and the proof of Theorem 8 is complete.

The following result is a direct consequence of Theorem 8 and Example 4.

**Corollary 9.** A vector $d \in \mathbb{N}^3$ is a dimension of an indecomposable corepresentation of the poset $K_{11}$ if an only if $d$ is an admissible or imaginary root of the corresponding Tits quadratic coform. Moreover:

- If $d$ is an admissible root, then there exists precisely one (up to isomorphism) indecomposable corepresentation of dimension $d$.
- If $d$ is an imaginary root, it has the form $(n, n, n)$ for $n \geq 1$, and in case of infinite fields $F, G$, there exist infinitely many non-isomorphic indecomposable corepresentations of dimension $d$. 

The following proposition and its proof explain how Theorem 7 follows from Theorem 8.

**Proposition 10.** The matrix problem on classification of representations (corepresentations) of the three-equipped poset $K_{10}$ is reduced to the matrix problem on classification of corepresentations (representations) of the three-equipped poset $K_{11}$.

**Proof.** Consider a matrix representation (corepresentation) of the critical three-equipped poset $K_{10}$. Reducing the stripe of the strong point by $F(G)$-elementary row and column transformations, we get a matrix problem of the form.

\[
\begin{pmatrix} 3 & \circ \\ G & 1 & 0 \\ G & 0 & 0 \end{pmatrix}
\]

where it is supposed that one can apply those admissible transformations which do not change the form of the reduced strong point stripe. Then, we apply $G(F)$-elementary column transformations to the stripe corresponding to the weak point. It is now divided in two blocks, each one of them can be reduced by $F(G)$-elementary row transformations and we also have row additions from the lower block to the upper one with coefficients in $F(G)$. Then in the stripe corresponding to the weak point we have precisely the transpose problem to the problem on classification of corepresentations (representations) of the three-equipped poset $K_{11}$. □

The sincere indecomposable matrix representations of $K_{10}$ are obtained (following the proposition) from the sincere indecomposable matrix corepresentations presented in Appendix C and those indecomposable non-sincere matrix corepresentations of $K_{11}$ of the form $Ma = 0, M b = \hat{P}(b)$, with $\hat{P} \in \{\hat{Q}, \hat{R}, \hat{S}, \hat{T}\}$ (see Appendix A). The last ones produce $K_{10} - 4, K_{10} - 12$ for $n = 1, K_{10} - 22$ and $K_{10} - 24$. To list all the representations of $K_{10}$ in Appendix B we have reduced the first stripe (corresponding to the weak point) into direct sums of the indecomposable representations of a weak point (poset $F_{19}$).

The following result clearly expresses a relation between the roots of the Tits form and the dimensions of the indecomposable representations of the three-equipped poset $K_{10}$.

**Corollary 11.** A vector $d \in \mathbb{N}^3$ is a dimension of an indecomposable representation of the poset $K_{10}$ if and only if $d$ is an admissible or imaginary root of the corresponding Tits quadratic form. Moreover:

- If $d$ is an admissible root, then there exists precisely one (up to isomorphism) indecomposable representation of dimension $d$.
- If $d$ is an imaginary root, it has the form $(2n, n, n)$ for $n \geq 1$, and in case of infinite fields $F, G$, there exist infinitely many non-isomorphic indecomposable representations of dimension $d$.

8. **Reduction to some pseudolinear matrix problem**

In this section, the problem on classifying indecomposable matrix representations of the three-equipped poset $K_{11}$ is reduced to some matrix problem of corepresentation type, that involves the pseudolinear pencil problem in the sense of [26]. Recall that the last problem is determined in [26], over a field $K$, by a pair $(\tau, \delta)$ where $\tau$ is an automorphism of $K$, and $\delta$ is a right derivation of $K$ such that $(gh)\delta = gh\delta + g\delta(h)$, for every $g, h \in K$.

In our case, there appears naturally some matrix problem which contains, as a subproblem, a pseudolinear pencil problem over $G$ determined by the pair $(1, \delta)$ where $1$ is the identity automorphism of $G$ and $\delta$ is a natural right derivation of $G$ such that

\[
\delta(F) = 0; \quad \delta(\xi) = 1; \quad \delta(\xi^2) = -\xi.
\]
This pseudolinear pencil subproblem is depicted by the diagram

\[
\begin{array}{c}
\bullet \\
(1, \delta) \\
\bullet \\
\end{array}
\begin{array}{c}
d \\
3 \\
\ast \delta
\end{array}
\begin{array}{c}
a \\
(1, \delta) \\
b \\
\ast \\
3 \\
\ast \delta \\
c
\end{array}
\begin{array}{c}
d \\
\end{array}
\]

(5)

Now, by pseudolinear rhombus we mean the following diagram containing 5.

\[
\begin{array}{c}
d \\
3 \\
\ast \delta \ast \\
(1, \delta) \\
\ast \\
3 \\
\ast \delta \\
a \\
b \\
\ast
\end{array}
\begin{array}{c}
F \\
\Gamma \\
G \\
F
\end{array}
\begin{array}{c}
a \\
\end{array}
\begin{array}{c}
b \\
\end{array}
\begin{array}{c}
d \\
\end{array}
\begin{array}{c}
c
\end{array}
\]

(6)

This pseudolinear rhombus determine a matrix problem that can be described by the next diagram.

\[
M = \begin{bmatrix}
G & G & G & G & G \\
\end{bmatrix}
\begin{array}{c}
F \\
\Gamma \\
F
\end{array}
\]

It consists of a rectangular matrix \( M \) over \( G \) separated into four vertical stripes \( M_a = A, M_b = B \), which have the same size, and \( M_c, M_d \) (any of them may be empty). The following admissible transformations of such matrices are accepted:

(a) \( G \)-elementary row transformations of the whole matrix (indicated by \( G \) on the right).
(b) \( F \)-elementary column transformations of the stripes \( M_w \) and \( M_z \) (indicated by \( F \) below).
(c) The ring \( \Gamma = \left\{ \begin{pmatrix} Y & Y^\delta \\ 0 & Y \end{pmatrix} : Y \text{ is an invertible square matrix over } G \right\} \) represents the column transformations of the stripes \( A \) and \( B \), so they are simultaneously transformed by the rule \((A, B) \mapsto (AY, BY + AY^\delta)\).
(d) Additions from any column of \( M_c \) simultaneously to a column of \( A \) and a column of \( B \) numbered equally, with coefficients \( \ast \delta \) and \( \ast \), respectively, where \( \ast \in G \) is an arbitrary element.
(e) Simultaneous additions from a column of \( A \) and a column of \( B \) numbered equally, with coefficients \( \ast \) and \( \ast \delta \), respectively, to any column of \( M_d \).

**Theorem 12.** All but one sincere indecomposable representations of the three-equipped poset \( K_{11} \) of the form (3), considered up to isomorphism, can be restored from the indecomposable, not necessarily sincere, matrix corepresentations of the pseudolinear rhombus (6).

**Proof.** We reduce first the stripe corresponding to the minimal point \( a \) into direct sums of indecomposable matrix representations of a weak point (poset \( F_{1g} \)), then with the possible additions to the maximal point stripe we annul some blocks, obtaining the following problem in the stripe \( M_b \).
The notation $\begin{bmatrix} X - \xi Y \\ -\xi^2 Y - \xi Z \end{bmatrix}$ means that the two blocks can be multiplied by the left for matrices of the form $X, Y, Z$ are square invertible matrices over $F$.

The solution for the last linear algebra problem expressed in this way is unknown. However, note that the transformations are spanned by the vector space $F\langle I, \Upsilon = \begin{bmatrix} -\xi \xi^2 \end{bmatrix} \rangle$.

This $F$-space is isomorphic to $G$ by identifying $\Upsilon$ with $\xi$. So we know intuitively, that we have some kind of simultaneous $G$-transformations of the pair matrices $(A, B)$ in the corresponding blocks.

The characteristic polynomial of $\Upsilon$ has the form $t^2 + \xi t + \xi^2 = (t - \xi)^2$ therefore its eigenvalues are equals to $\xi$ and we have that $\Upsilon$ is similar to the matrix $\begin{bmatrix} \xi & 1 \\ 0 & \xi \end{bmatrix}$. In fact

$$\begin{bmatrix} \xi \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ \xi & 1 \end{bmatrix} - \begin{bmatrix} -\xi & 1 \\ -\xi^2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\xi & 1 \end{bmatrix}.$$

So, with a base change given by the matrix $\Psi = \begin{bmatrix} 1 & 0 \\ \xi & 1 \end{bmatrix}$ the pair $(A, B)$ can be reduced as follows

$$(A, B) \mapsto (XAY + X^\delta BY, XBY)$$

where $X$ and $Y$ are square invertible matrices over $G$ and $\delta$ is the natural derivation of the field $G$, which turn this problem into the $(1, \delta)$-pseudolinear pencil problem (5).

Also, by using the base change given by $\Psi$ the additions to and from the points in the pseudolinear pencil get turned into the additions (d) and (e) of the matrix problem of the pseudolinear rhombus (6).

Then, the problem obtained in the stripe $M_b$ without the maximal strong point became the pseudolinear rhombus. By $\gamma$ (see page 1837) every indecomposable of the pseudolinear rhombus produces a representation of the three-equipped poset $K_{11}$ and the only representation of $K_{11}$ which is not obtained from an indecomposable of the pseudolinear rhombus is $\begin{bmatrix} 1 & 0 \\ \xi & 1 \\ \xi^2 & -\xi \end{bmatrix}$ produced by the maximal strong point problem in $M_b$. So, Theorem 12 is proved. $\square$
Notice that at the moment we do not have at our disposal a complete solution to the pseudolinear rhombus problem (6).

Although corepresentations of $K_{10}$ may be restored from representations of $K_{11}$ (see Proposition 10) we are able to get a direct reduction to the pseudolinear rhombus.

**Theorem 13.** All but one sincere indecomposable corepresentations of the three-equipped poset $K_{10}$ of the form (2), considered up to isomorphism, are in one-to-one correspondence with the indecomposables (not necessarily sincere) of the pseudolinear rhombus (6).

**Proof.** First, the weak point is reduced into direct sums of its indecomposables (poset $F_{19}$), obtaining the following problem in the strong point stripe.

The problem obtained in the stripe $M_b$ (with exception of its maximal and minimal strong points) coincides with the pseudolinear rhombus (6). By $\gamma'$ (see page 1837), every indecomposable of this pseudolinear rhombus produces a matrix corepresentation of the three-equipped poset $K_{10}$. The only corepresentation of $K_{10}$ which is not obtained from indecomposables of the rhombus, is \[ \begin{pmatrix} 1 & \xi & \xi^2 & 1 \\ \end{pmatrix} \]
produced by the maximal strong point problem in $M_b$ (note that the minimal strong point problem does not produce a sincere corepresentation). □

We finish this work by the conclusion that the research on representations and corepresentations of three-equipped posets is an interesting direction of the theory of representations of posets with additional structures. It confirms the importance of the matrix technique widely used during several decades in the modern representation theory (see for example [21,22,18,25,11,3,5,27,36,6]).

**Appendix A. Sincere indecomposables of the three-equipped posets of finite type**

<table>
<thead>
<tr>
<th>Poset $F_{19}$</th>
<th>Representations $Q(a)$</th>
<th>Corepresentations $\hat{Q}(a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{Q}(a) = \begin{pmatrix} a \ 1 \ \xi \ \xi^2 \end{pmatrix}$ $f = 1$</td>
<td>$\hat{Q}(a) = \begin{pmatrix} a \ 1 &amp; \xi &amp; \xi^2 \end{pmatrix}$ $\hat{f} = 3$</td>
<td></td>
</tr>
<tr>
<td>$\hat{R}(a) = \begin{pmatrix} a \ 0 \ \xi \ \xi^2 \end{pmatrix}$ $f = 3$</td>
<td>$\hat{R}(a) = \begin{pmatrix} a \ 1 &amp; \xi \end{pmatrix}$ $\hat{f} = 1$</td>
<td></td>
</tr>
</tbody>
</table>
\[ S(a) = \begin{vmatrix} a \\ 1 \\ \xi \end{vmatrix}, \quad f = 1 \]
\[ T(a) = \begin{vmatrix} 1 \\ \xi \\ \xi^2 \end{vmatrix}, \quad f = 3 \]
\[ \tilde{S}(a) = \begin{vmatrix} a \\ 1 \\ \xi \xi^2 \end{vmatrix}, \quad \tilde{f} = 3 \]
\[ \tilde{T}(a) = \begin{vmatrix} a \\ 1 \\ \xi \end{vmatrix}, \quad \tilde{f} = 1 \]

<table>
<thead>
<tr>
<th>( F_{20} )</th>
<th>( H(a, b) = \begin{vmatrix} a &amp; b \ 1 &amp; \xi \end{vmatrix}, \quad f = 3 )</th>
<th>( \tilde{H}(a, b) = \begin{vmatrix} a &amp; b \ 1 &amp; \xi^2 \end{vmatrix}, \quad \tilde{f} = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3 )</td>
<td>( b )</td>
<td></td>
</tr>
<tr>
<td>( 3 )</td>
<td>( a )</td>
<td></td>
</tr>
</tbody>
</table>

### Appendix B. Representations of the three-equipped poset \( K_{10} \)

The table below contains the matrix forms of all the sincere indecomposable representations of the poset \( K_{10} \), up to isomorphism. In the matrix representations the left stripe corresponds to the weak point \( a \) and the right one to the strong point \( b \). We follow the convention (\( \alpha \)) of Section 7 and the notations of Appendix A.

#### Representations with Tits form 0

1) \[ \begin{vmatrix} I_n & I_n \\ \xi I_n & A \end{vmatrix} \]

where \( A \) is a matrix in a standard canonical form with respect to ordinary similarity transformations \((X^{-1}AX)\) over \( F \).

2) \[ \begin{vmatrix} R(a) & 0 & e_{11} \\
0 & T(a) & l_3 \end{vmatrix} \]

3) \[ \begin{vmatrix} R(a) & 0 & 0 & e_3 & 0 & 0 \\
0 & \xi^{3m} & 0 & 0 & 0 & l^{3m} \\
0 & 0 & e_{3m} & 0 & \Phi((t^3 + p)^m) \\
0 & 0 & T(a) & e_1 & l_2 & 0 \end{vmatrix} \]

\( \Phi((t^3 + p)^m) \) is the companion matrix of the \( m \)th power of the minimal polynomial \( t^3 + p \) of \( \xi \) over \( F \).

#### Representations with Tits form 1

4) \[ \begin{vmatrix} 1 & 1 \\
T(a) & I^\uparrow_2 \end{vmatrix} \]

5) \[ \begin{vmatrix} R(a) & e_1 \\
R(a) & I^\uparrow_2 \end{vmatrix} \]

6) \[ \begin{vmatrix} R(a) & e_i \\
R(a) & I^\uparrow_2 \end{vmatrix} \]

7) \[ \begin{vmatrix} T(a) & e_1 \\
T(a) & I^\uparrow_2 \end{vmatrix} \]

8) \[ \begin{vmatrix} S(a) & 0 & e_{22} \\
0 & T(a) & l_2 \end{vmatrix} \]

9) \[ \begin{vmatrix} S(a) & 0 & e_{21} \\
0 & T(a) & l_3 \end{vmatrix} \]

10) \[ \begin{vmatrix} S(a) & 0 & e_{21} \\
0 & T(a) & l_3 \end{vmatrix} \]

11) \[ \begin{vmatrix} T(a) & 0 & 0 \\
0 & l_n & I_n \end{vmatrix} \]

12) \[ \begin{vmatrix} T(a) & 0 & 0 \\
0 & l_n & I_n \end{vmatrix} \]

13) \[ \begin{vmatrix} I_{n+1} & 0 \\
\xi I_{n+1} & l_n \end{vmatrix} \]

14) \[ \begin{vmatrix} I_{n+1} & 0 \\
\xi I_{n+1} & l_n \end{vmatrix} \]

15) \[ \begin{vmatrix} T(a) & 0 & e_{12} + e_{33} \\
0 & S(a) & l_2 \end{vmatrix} \]

16) \[ \begin{vmatrix} I_{n+1} & 0 \\
\xi I_{n+1} & l_n \end{vmatrix} \]

17) \[ \begin{vmatrix} I_3 & 0 \\
\xi I_3 & 0 \end{vmatrix} \]

18) \[ \begin{vmatrix} e_{11} \end{vmatrix} \]

19) \[ \begin{vmatrix} e_{24} \\
e_{23} + e_{34} \end{vmatrix} \]
Representations with Tits form 3

\begin{align*}
18) & \begin{bmatrix}
T(a) & 0 & 0 & 0 \\
0 & I_{2n} & 0 & 0 \\
0 & \xi I_{2n} & I_2 & 0 \\
0 & l_2n & 0 & I_{2n-2}
\end{bmatrix} & \begin{bmatrix}
l_{2n+2} & 0 & 0 & 0 \\
l_2n & 0 & 0 & 0 \\
\xi l_{2n+2} & 0 & T(a) & 0 \\
0 & E_{2n} & I_{2n} & 0
\end{bmatrix} & \begin{bmatrix}
l_{2n} & 0 & 0 & 0 \\
l_{2n} & 0 & 0 & 0 \\
\xi l_{2n} & 0 & T(a) & 0 \\
0 & E_{2n} & I_{2n} & 0
\end{bmatrix}
\end{align*}

19) 

20) 

21) 

22) 

23) 

24) 

25) 

26) 

27) 

28) 

29) 

30) 

31) 

32)
Appendix C. Corepresentations of the three-equipped poset $K_{11}$

The matrix forms of all the sincere indecomposable corepresentations of the poset $K_{11}$, up to isomorphism, are presented in the table below.

The left stripe of the matrix corepresentations corresponds to the minimal weak point $a$ and the right stripe to the maximal one $b$. We follow the convention ($\alpha$) of Section 7 and the notations of Appendix A.

<table>
<thead>
<tr>
<th>Corepresentations with Tits coform 0</th>
<th>Corepresentations with Tits coform 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) $I_n \xi I_n + \xi^2 A$ where $A$ is a matrix in a standard canonical form with respect to ordinary similarity transformations $(X^{-1}AX)$ over $F$.</td>
<td>2) $\tilde{S}(a) \xi 0$ $e_{23} e_{23}$ $\tilde{S}(a) 0 \hat{R}(b)$</td>
</tr>
<tr>
<td>3) $\tilde{S}(a) 0 0 e_{23} \xi I_{3n} + \xi^2 \Phi((t^3 + p)^m)$ where $\Phi((t^3 + p)^m)$ is the companion matrix of the $m$th power of the minimal polynomial $t^3 + p$ of $\xi$ over $F$.</td>
<td>4) $\hat{R}(a) \xi 0$ $\xi \xi I_{n+1}$ $\xi I_n + \xi^2 I_n$</td>
</tr>
<tr>
<td>5) $\tilde{S}(a) \xi 0$ $\xi 0 \hat{R}(b)$</td>
<td>6) $\tilde{S}(a) \xi 0$ $\hat{R}(b)$</td>
</tr>
<tr>
<td>7) $\xi 0 1 0$</td>
<td>8) $\xi f_{1n} \xi I_n + \xi^2 I_n$</td>
</tr>
<tr>
<td>9) $l_2 0$ $\xi I_n + \xi^2 I_n$ $\xi I_n + \xi^2 I_n$</td>
<td>10) $l_{n-1} \xi I_n + \xi^2 I_n$</td>
</tr>
<tr>
<td>11) $I_n \xi I_n + \xi^2 I_n$</td>
<td>12) $l_3 0$ $\xi I_n + \xi^2 I_n$</td>
</tr>
<tr>
<td>13) $e_{2n+1,3} \xi I_n + \xi^2 I_n$</td>
<td>14) $\tilde{S}(a) 0 e_{2,n+1} e_{2,n+1}$</td>
</tr>
<tr>
<td>15) $\xi f_{1n} \xi I_n + \xi^2 I_n$</td>
<td>16) $\xi e_2 e_{2n}$ $\xi e_2$ $\xi e_2$</td>
</tr>
<tr>
<td>17) $l_{2n+2} 0 0 0$ $\xi f_{1n} \xi I_n + \xi^2 I_n$</td>
<td></td>
</tr>
</tbody>
</table>

The table above shows the corepresentations with different coforms for the poset $K_{11}$. Each row corresponds to a specific corepresentation, with the corresponding matrix entry presented in the designated spot. The table is structured to reflect the minimal and maximal points of the poset, with the left and right stripes respectively.
### Corepresentations with Tits coform 3

<table>
<thead>
<tr>
<th>18) $\widehat{S}(a)$</th>
<th>0</th>
<th>$\widehat{Q}(b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_2$</td>
<td>$\xi I_2$</td>
<td>$\xi^2 I_2$</td>
</tr>
<tr>
<td>$\xi I_2$</td>
<td>$I_2$</td>
<td>$-\xi I_2$</td>
</tr>
<tr>
<td>0</td>
<td>$\xi I_2$</td>
<td>0</td>
</tr>
<tr>
<td>19) $I_3$</td>
<td>$\xi I_2$</td>
<td>$\xi^2 I_2$</td>
</tr>
<tr>
<td>$\xi I_2$</td>
<td>$e_{12}$</td>
<td>$e_{12}$</td>
</tr>
<tr>
<td>$\xi^2 I_2$</td>
<td>$I_3$</td>
<td>$0$</td>
</tr>
<tr>
<td>0</td>
<td>$\xi^2 I_2$</td>
<td>$\xi I_2$</td>
</tr>
<tr>
<td>20) $I_3$</td>
<td>0</td>
<td>$\xi I_3$</td>
</tr>
<tr>
<td>0</td>
<td>$\xi I_3$</td>
<td>$\widehat{S}(b)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>21) $I_{3n}$</th>
<th>$e_{3n,3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi I_{3n}$</td>
<td>$\xi^2 I_{3n} + \xi^2 \Phi ((t^3 + p)^m)$</td>
</tr>
<tr>
<td>$\xi I_{3n}$</td>
<td>$\xi^2 I_{3n}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>22) $S(a)$</th>
<th>0</th>
<th>$I_{3n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_{2,3n}$</td>
<td>$\xi^2 e_1$</td>
<td>$\xi^2 f_1$</td>
</tr>
<tr>
<td>$e_{2,3n}$</td>
<td>$\xi f_{3n}$</td>
<td>$\xi^2 f_{3n}$</td>
</tr>
<tr>
<td>0</td>
<td>$\xi^2 f_1$</td>
<td>$\xi^2 f_{3n}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>23) $I_{3n+3}$</th>
<th>$e_{3n}$</th>
<th>$e_{3n+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$\xi$</td>
<td>$\xi^2$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>$\xi$</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>$\xi^2$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>24) $I_{3n+3}$</th>
<th>$e_{3n}$</th>
<th>$e_{3n+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$\xi$</td>
<td>$\xi^2$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>$\xi$</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>$\xi^2$</td>
</tr>
<tr>
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</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>25) $I_{3n+3}$</th>
<th>$e_{3n}$</th>
<th>$e_{3n+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$\xi$</td>
<td>$\xi^2$</td>
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### References


