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Note

Polychromatic cliques

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Abstract

The sub-Ramsey number $\mathrm{sr}(K_n,k)$ is the smallest integer m such that in any edge-colouring of K_m which uses every colour at most k times some subgraph K_n has all edges of different colours. It was known that, for a fixed k, the function $\mathrm{sr}(K_n,k)$ is $O(n^3)$ and $\Omega(n)$. We improve these bounds to $O(n^2)$ and $\Omega(n^{3/2})$ (slightly less for small values of k).

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1. Background and notation

An edge-colouring of a graph G (i.e., a mapping of $E(G)$ to a set of 'colours') is *k-bounded* if each colour is used at most k times. If $k=1$, i.e., if all edges of G have different colours, we say that G is *polychromatic*. The *sub-Ramsey number* $\mathrm{sr}(G,k)$ is the smallest integer m such that each k-bounded edge-colouring of K_m contains a polychromatic subgraph isomorphic to G. Recall that the *Ramsey number* $r(G,k)$ is the smallest integer m such that each edge-colouring of K_m with k colours contains a monochromatic (i.e., all edges coloured with the same colour) subgraph isomorphic to G. Thus sub-Ramsey numbers are in this sense dual to Ramsey numbers, and it is easy to see that each $\mathrm{sr}(G,k) \leqslant r(G,k)$, and hence each $\mathrm{sr}(G,k)$ is guaranteed to be finite [2].

Galvin appears to be the first person to suggest investigating $\mathrm{sr}(G,k)$ [9]. In [2] it is shown that $\mathrm{sr}(K_n,k)$ is $O(kn^3)$ and $\Omega(kn)$. In [12] the authors show that $\mathrm{sr}(P_n,k)=\mathrm{sr}(C_n,k)=n$ when n is large enough with respect to k. (P_n and C_n denote respectively the path and the cycle with n vertices.) Results on sub-Ramsey numbers of stars can be found in [5,10,11]. Related questions on k-bounded edge-colourings without polychromatic subgraphs which are not necessarily of fixed size (such as polychromatic Hamilton paths or cycles) are investigated in [1,4,8,12,14].

In this note we improve both the upper bound and the lower bound for $\mathrm{sr}(K_n,k)$. In particular we prove that $\mathrm{sr}(K_n,k)$ is $O(kn^2)$ (better than $O(kn^3)$) and, for k greater than or equal to 15, $\mathrm{sr}(K_n,k)$ is $\Omega(n^{3/2})$ (better than $\Omega(kn)$ for any fixed k). The lower bound for k between 3 and 15 is $\Omega(n^{4/3})$. (We have no improvement of the lower bound when $k=2$.)

2. The upper bound

Theorem 1. *Let $n \geqslant 3$ and $k \geqslant 2$ be positive integers. Then*

$$\mathrm{sr}(K_n,k) \leqslant (2n-3)(n-2)(k-1)+3.$$

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Proof. Assume that K_m admits a *good* edge-colouring, i.e., a k-bounded edge-colouring in which no subgraph isomorphic to K_n is polychromatic. We shall show that $m \leqslant (2n-3)(n-2)(k-1)+2$.

Fix a good edge-colouring of K_m, with colours $1, 2, \ldots, r$, and denote by d_i the number of edges coloured i. Note that each $d_i \leqslant k$. Let W denote the number of unordered pairs of edges of K_m coloured by the same colour. We claim that

$$W \leqslant \binom{m}{2} \frac{k-1}{2}. \tag{1}$$

Indeed, $\binom{m}{2} = |E(K_m)| = \sum_{i=1}^{r} d_i$ and $W = \sum_{i=1}^{r} \binom{d_i}{2}$. Thus

$$W = \sum_{i=1}^{r} d_i \frac{d_i - 1}{2} \leqslant \sum_{i=1}^{r} d_i \frac{k-1}{2} = \binom{m}{2} \frac{k-1}{2}.$$

Consider now a subgraph Q of K_m isomorphic to K_q, for some integer $q > n$. We claim that Q must have many pairs of edges of the same colour, otherwise some subgraph of Q isomorphic to K_n would be polychromatic. Specifically, denote by $T(q,n)$ the maximum number of edges of a graph with q vertices which does not have a complete subgraph of n vertices. (The value of $T(q,n)$ is known by Turan's theorem [3].) If Q obtained more than $T(q,n)$ colours then taking one edge of each colour would result in a graph G with q vertices and more than $T(q,n)$ edges, which therefore would have to contain a subgraph isomorphic to K_n. This contradicts the fact that we have a good edge-colouring. Therefore, Q is coloured by at most $T(q,n)$ colours, say, the colours $1, 2, \ldots, t$, where $t \leqslant T(q,n)$. Denote by $w(Q)$ the number of unordered pairs of edges of Q which obtain the same colour, and by $d_i(Q)$ the number of edges of Q of colour i, $i \leqslant t$. We claim that

$$w(Q) \geqslant \binom{q}{2} - T(q,n). \tag{2}$$

Indeed,

$$w(Q) = \sum_{i=1}^{t} \binom{d_i(Q)}{2} = \sum_{i=1}^{t} (d_i(Q) - 1)\frac{d_i(Q)}{2} \geqslant \sum_{i=1}^{t} (d_i(Q) - 1)$$

$$= \left(\sum_{i=1}^{t} d_i(Q) \right) - t = \binom{q}{2} - t \geqslant \binom{q}{2} - T(q,n).$$

Let $\mathcal{Q}$ denote the set of all subgraphs of K_m isomorphic to K_q. Since each pair of edges of K_m belongs to $\binom{m-3}{q-3}$ or $\binom{m-4}{q-4}$ graphs in $\mathcal{Q}$, depending on whether the two edges are adjacent or not, we see that $MW \geqslant \sum_{Q \in \mathcal{Q}} w(Q)$, where $M = \max \left\{ \binom{m-3}{q-3}, \binom{m-4}{q-4} \right\}$.

Suppose first that $\binom{m-3}{q-3} \leqslant \binom{m-4}{q-4}$. This inequality implies that $m-3 \leqslant q-3$ and thus $m \leqslant q$. Therefore, our desired conclusion $m \leqslant (2n-3)(n-2)(k-1)+2$ will follow as long as we choose q not too big. Let us in fact choose $q = 2n-2$.

In the remaining case, $\binom{m-3}{q-3} > \binom{m-4}{q-4}$, we have $\binom{m-3}{q-3} W \geqslant \sum_{Q \in \mathcal{Q}} w(Q)$ which by (1) and (2) implies that

$$\binom{m-3}{q-3} \binom{m}{2} \frac{k-1}{2} \geqslant \binom{m-3}{q-3} W \geqslant \binom{m}{q} \left(\binom{q}{2} - T(q,n) \right). \tag{3}$$

Since $(n-1)$ divides q,

$$T(q,n) = \frac{n-2}{2n-2} q^2$$

[3], and then, using $q = 2n-2$, we have

$$\binom{q}{2} - T(q,n) = \binom{q}{2} - \frac{n-2}{2n-2} q^2 = \frac{q}{2}.$$

Therefore, we obtain from (3) that

$$\frac{(m-3)!}{(q-3)!(m-q)!}\cdot\frac{m(m-1)}{2}\cdot\frac{(k-1)}{2}\geqslant\frac{m!}{(m-q)!q!}\cdot\frac{q}{2}.$$

Simplifying the last inequality we reach the conclusion that

$$m-2\leqslant(k-1)(q-1)(q-2)/2$$

which is equivalent to $m\leqslant(2n-3)(n-2)(k-1)+2$. □

3. The lower bound

Let m and t be positive integers.

Let $U(t)$ denote the maximum number of ways a set can be written as a union of two t-element subsets. For instance, $U(2)=3$ because the set $\{1,2,3,4\}$ can be written as $\{1,2\}\cup\{3,4\}$, or $\{1,3\}\cup\{2,4\}$, or $\{1,4\}\cup\{2,3\}$, and no set can be written as the union of two 2-element subsets in four different ways. It is easy to check also that $U(3)=15$.

Let $F(m,t)$ be the maximum size of a union free family of t-element subsets of $\{1,2,\dots,m\}$. (A family of sets S_i, $i\in I$ is *union free* if all the $\binom{|I|}{2}$ unions $S_i\cup S_j$, $S_i,S_j\in F$, are distinct.)

The following result of Frankl and Füredi gives the asymptotic behaviour of $F(m,t)$:

Theorem 2 (Frankl and Füredi [7]). *Let t be a positive integer.*
There exist positive constants c_t, c_t' so that

$$c_t' m^{\lceil\frac{4t}{3}\rceil/2}\leqslant F(m,t)\leqslant c_t m^{\lceil\frac{4t}{3}\rceil/2}.$$

The relevance of the functions $U(t)$ and $F(m,t)$ to our problem arises from the following fact:

Theorem 3. *If $n>F(m,t)$ and $k\geqslant U(t)$, then*

$$\operatorname{sr}(K_n,k)\geqslant\binom{m}{t}.$$

Proof. Consider the complete graph $K(m,t)$ whose vertices are all the t-element subsets of $\{1,2,\dots,m\}$, in which the edge SS' has colour $S\cup S'$. Then this colouring of $K(m,t)$ is k-bounded, since $k\geqslant U(t)$. At the same time, there is no polychromatic K_n, since $n>F(m,t)$. □

Corollary 4. *If $k\geqslant 15$, then (for a positive constant c)*

$$\operatorname{sr}(K_n,k)\geqslant cn^{3/2}$$

and, if $3\leqslant k<15$, then

$$\operatorname{sr}(K_n,k)\geqslant cn^{4/3}.$$

Proof. If $k\geqslant 15=U(3)$, then we can choose $t=3$ in Theorem 3. For $t=3$, any choice of m with $n>c_t m^{\lceil\frac{4t}{3}\rceil/2}=c_3m^2$ assures that we have $n>F(m,3)$. We may choose $m=\lceil(c_3^{-1/2}-\varepsilon)n^{1/2}\rceil$, for a small positive ε. Such a value of m yields $\binom{m}{3}\geqslant cn^{3/2}$, thus $\operatorname{sr}(K_n,k)\geqslant cn^{3/2}$ (for a positive constant c). For $k\geqslant 3=U(2)$, we proceed analogously, taking $t=2$ and m satisfying $n>c_2m^{3/2}$, yielding $\operatorname{sr}(K_n,k)\geqslant\binom{m}{3}\geqslant cn^{4/3}$. □

We remark that using [6], in which $F(m,3)$ is evaluated exactly as $F(m,3)=\lfloor m(m-1)/6\rfloor$, we find, for $k\geqslant 15$, that

$$\operatorname{sr}(K_n,k)\geqslant\left(\sqrt{6}-\varepsilon\right)n^{3/2}.$$

Addendum

We have just learned that similar results were obtained in [13]. Our upper bounds are slightly better than the corresponding bounds of [13]; this improvement is particularly pronounced for small values of k. (We note that our upper bounds correspond to the lower bounds in [13], and vice versa, due to a dual formulation of the problem.) On the other hand, our lower bounds are not as good as those of [13], where random graphs are used to obtain a lower bound of the order n^2 divided by a polylogarithmic factor. However, our lower bounds are constructive, and also apply to small values of n.

References

[1] M. Albert, A.M. Frieze, B. Reed, Multicoloured Hamilton cycles, Electron. J. Combin. 2 (1995) R10.
[2] B. Alspach, M. Gerson, G. Hahn, P. Hell, On sub-Ramsey numbers, Ars Combin. 22 (1986) 199–206.
[3] P. Dirac, Extension of Turan's Theorem on Graphs, Acta Math. Sci. Hungar. 14 (1963) 417–422.
[4] P. Erdős, J. Nešetřil, V. Rödl, On some problems related to partitions of edges of a graph, in: M. Fiedler (Ed.), Graphs and Other Combinatorial Topics, Teubner, Leipzig, 1983, pp. 54–63.
[5] P. Fraisse, G. Hahn, D. Sotteau, Star sub-Ramsey numbers, Ann. Discrete Math. 34 (1987) 153–163.
[6] P. Frankl, Z. Füredi, A new extremal property of Steiner triple-systems, Discrete Math. 48 (1984) 205–212.
[7] P. Frankl, Z. Füredi, Union-free families of sets and equations over fields, J. Number Theory 23 (1986) 210–218.
[8] A. Frieze, B. Reed, Polychromatic Hamilton cycles, Discrete Math. 118 (1993) 69–74.
[9] F. Galvin, Advanced Problem number 6034, Amer. Math. Monthly 82 (1975) 529.
[10] G. Hahn, Some star anti-Ramsey numbers, Congr. Numer. 19 (1977) 303–310.
[11] G. Hahn, More star sub-Ramsey numbers, Discrete Math. 43 (1981) 131–139.
[12] G. Hahn, C. Thomassen, Path and cycle sub-Ramsey numbers and an edge-colouring conjecture, Discrete Math. 62 (1986) 29–33.
[13] H. Lefmann, V. Rödl, B. Wysocka, Multicolored subsets in colored hypergraphs, J. Combin. Theory Ser. A 74 (1996) 209–248.
[14] M. Maamoun, H. Meyniel, On a problem of G. Hahn about coloured Hamilton paths in K_2n, Discrete Math. 51 (1984) 213–214.