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Conditionally Poised Birkhoff Interpolation Problems

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1. INTRODUCTION AND BACKGROUND

Much attention has centered recently on Birkhoff interpolation, which can be described as follows. Let $E = (e_{ij})_{i=0}^{k+1}\,{}_{j=0}^{n}$ be a matrix of zeros and ones, with exactly $n + 1$ ones. (For the sake of convenience, our notation will differ slightly from the usual.) Let $x_0 < x_1 < \cdots x_{k+1}$ be interpolation nodes. Without loss of generality, it is possible to specify two of the nodes, and we set $x_0 = -1$, $x_{k+1} = 1$. Denote by P_n the set of algebraic polynomials of degree $\leqslant n$. Let $X = (x_1, x_2, \ldots, x_k)$, and suppose that the system of equations

$$p^{(j)}(x_i) = c_{ij} \qquad \text{for} \quad e_{ij} = 1, \qquad p(x) = \sum_{\nu=0}^{n} b_\nu x^\nu, \tag{1}$$

has a unique solution for all c_{ij}. Then E is said to be *X-poised.* Since we seek a unique solution, it is clear that we need only consider the homogeneous case, $c_{ij} = 0$. Since $p(x) \equiv 0$ is always a solution in this case, E will be X-poised if $p(x) \equiv 0$ is the only polynomial in P_n which annihilates E (i.e., satisfies the zero data). If E is X-poised for all X, then E is said to be *poised*, whereas if for some X, E is X-poised and for others it is not, then E is said to be *conditionally poised.* (In the literature, the term nonpoised has generally been used for matrices which are not poised, without making a distinction between those that are conditionally poised and those that are not X-poised for any X. The latter type, however, is easily distinguished: they fail to satisfy the Pólya conditions (see below), [4, 20]). Examples of poised matrices include Lagrange matrices, where $e_{ij} = 0$ for $j \geqslant 1$, $i = 0, 1, \ldots, k + 1$, and Hermite matrices, where each row begins with a single sequence of ones followed by a single sequence of zeros. In the latter case, each row is said to contain *Hermite data.*

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The intriguing problem of Birkhoff interpolation is to find a characterization of poised matrices. As of now, no such characterization has been found, and the possibility of obtaining one seems remote. As a result, attention has turned to finding classes of poised and nonpoised matrices [1, 2, 6–9, 12, 13, 18, 22]. We mention just two examples of these types of studies, which will be needed in this paper.

Let $m_j = \sum_{i=0}^{k+1} e_{ij}$, $j = 0, 1,..., n$, and $M_r = \sum_{j=0}^{r} m_j$, $r = 0, 1,..., n$. E satisfies the *Polya conditions* if $M_r \geq r + 1$, $r = 0, 1,..., n$. The Polya conditions are known to be a necessary, but not sufficient condition for the poisedness of E [20].

A maximal sequence of ones in row i of E, beginning in column j, is said to be *supported* if there exist $i_1 < i$, $i_2 > i$, $j_1 < j$, $j_2 < j$, such that $e_{i_1 j_1} = e_{i_2 j_2} = 1$. Such a sequence is *odd* (*even*) if if consists of an odd (even) number of ones. The following results illustrate the importance of this concept.

THEOREM A. [1]. *If E satisfies the Polya conditions and has no odd supported sequences, then E is poised.*

THEOREM B [6, 7]. *If $M_r \geq r + 2$, $r = 0, 1,..., n - 1$ and if some row of E has exactly one odd supported sequence, then E is conditionally poised.*

Many other examples of nonpoised matrices are given in [2, 8, 9, 13, 18, 22]. Although new classes continue to be found, the basic problem remains unsolved. It may be useful, therefore, to investigate other aspects of the subject. For example, very little has been done on conditionally poised matrices. There is some work on lacunary interpolation (known as 0–2, 0–1–3, etc.) [19, 25] and on symmetric interpolation [24, 28]. In these cases, however, only certain configurations of nodes and derivatives are considered. The purpose of this paper is to study conditionally poised matrices in greater generality.

2. FURTHER RESULTS AND EXAMPLES

One important result on conditionally poised matrices is due to D. Ferguson. In this setting, complex interpolation nodes are allowed.

THEOREM C [4]. *If E is conditionally poised, then the set of vectors, X, for which E is not X-poised is a closed, nowhere dense set in complex k-space.*

Because of its generality, one shouldn't expect this theorem to yield information about the set of vectors for which a *particular* matrix is not X-poised. But it does give a clue to what one may expect; that is, it would not be surprising to find what one may call "intervals of poisedness." We illustrate with some simple examples.

EXAMPLE 1. Let

$$E_1 = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{Vmatrix}$$

It is easily seen that E_1 is X-poised if and only if $x_1 \neq 0$, i.e., if and only if $x_1 \in (-1, 0) \cup (0, 1)$.

EXAMPLE 2. Let

$$E_2 = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{Vmatrix}$$

A calculation of the determinant of the linear system shows that E_2 is X-poised unless $x_1 x_2 = -1/3$. Hence, if $x_1, x_2 \in (-1/\sqrt{3}, 1/\sqrt{3})$ or $x_1, x_2 \notin [-1/\sqrt{3}, 1/\sqrt{3}]$, then E is X-poised. Moreover, if $x_2 < 1/3$ or $x_1 > -1/3$, then again E_2 is X-poised. Our results in Section 3 will help us understand these examples.

3. THE MAIN RESULT

We consider the following class of matrices. Let rows 0 and $k + 1$ be arbitrary, except for a one in column 0. Let row i, $i = 1, 2,..., k$, begin with a zero, followed by a sequence of ones, and then by a sequence of zeros. A typical matrix would thus be

$$\begin{Vmatrix} 1 & 0 & 0 & 1 & 0 & \\ 0 & 1 & 1 & 0 & 0 & \\ 0 & 1 & 1 & 1 & 0 & \mathbf{0} \\ 0 & 1 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 1 & \end{Vmatrix}.$$

Suppose also that E satisfies the Polya conditions, and let l_i be the number of ones in row i, $i = 0, 1,..., k + 1$. It follows from Theorems A and B and [1, p. 231] that E is poised if l_i is even, $i = 1, 2,..., k$, and E is conditionally poised if some l_i, $1 \leqslant i \leqslant k$, is odd. Suppose the latter is the case and that X is a fixed set of nodes. Let $p \in P_n$ annihilate E and let $q = p'$. Then $q \in P_{n-1}$ and $\int_{-1}^{1} q = p(1) - p(-1) = 0$.

Now let

$$\int_{-1}^{1} f \approx \sum_{j=1}^{N} a_j f(y_j) \tag{2}$$

be any quadrature formula, exact for P_{n-1}, with $a_j > 0, j = 1, 2,..., N$. Let $Y = \{y_1, y_2,..., y_N\}$, $I_j = (y_j, y_{j+1}), j = 1, 2,..., N-1$, and let $\nu_j = \sum l_i$, $j = 1, 2,..., N-1$, where the sum extends over those indices i for which $x_i \in I_j$.

DEFINITION. (E, X) is *evenly distributed* with respect to Y if

$$\nu_j \text{ is even}, j = 1, 2,..., N-1, \text{ and} \tag{3}$$

$$\text{if } x_r \in X \cap Y, \text{ then } l_r \text{ is even.} \tag{4}$$

THEOREM 1. *Suppose there exists a quadrature formula* (2) *exact for* P_{n-1}, *with* $a_j > 0, j = 1, 2,..., N$, *such that* (E, X) *is evenly distributed with respect to* Y. *Then* E *is* X*-poised.*

Proof. We show first that if $q = p'$ is not identically 0, then q can't have any zeros in $(-1, 1)$, other than the ones specified by E. This is clear if rows 0 and $k + 1$ consist of Hermite data only, since then q will have $n - 1$ zeros specified by E (counting multiplicities). In the more general case, we invoke the Budan–Fourier Theorem [14], used in the following form: If q is a polynomial of exact degree $n - 1$, then

$$Z[q; (-1, 1)] + S^{+}((-1)^i q^{(i)}(-1))_{i=0}^{n-1} + S^{+}(q^{(i)}(1))_{i=0}^{n-1} \leqslant n - 1. \tag{5}$$

Here $Z[q; (-1, 1)]$ is the number of zeros of q in $(-1, 1)$, counting multiplicities, and $S^{+}(b_i)_{i=0}^{n-1}$ is the maximal number of sign changes in the sequence $b_0, b_1,..., b_{n-1}$ which can be obtained by replacing zero terms by terms of arbitrary sign. Each one in row 0 and $k + 1$ corresponds to a zero of some derivative of q. It may easily be seen that every zero term in the sequence $b_0, b_1,..., b_{n-1}$ contributes *at least* one variation in sign to $S^{+}(b_i)_{i=0}^{n-1}$. Now q annihilates E', which is the matrix E with the first column deleted. Since there are a total of $n - 1$ conditions specified by E', each one contributing at least one to the left-hand side of (5), q can have no other zeros in $(-1, 1)$. In particular, q can change sign only at the points of X.

We next show that $Y - X \neq \varnothing$. Note first that we must have $2N - 1 \geqslant n - 1$, since $2N - 1$ is the maximal degree of precision possible in a quadrature formula based on N nodes [23, p. 136]. Now suppose $Y \subset X$. Then, since (E, X) is evenly distributed with respect to Y, q must have a zero of even degree at y_j, $j = 1, 2,..., N$. But this yields a total of at least $2N \geqslant n$ zeros for q, which implies that $q \equiv 0$, so that E is X-poised.

Hence, we can suppose that there exists $y_i \notin X$, so that $q(y_i) \neq 0$. Assume, without loss of generality, that $q(y_i) > 0$. Since (E, X) is evenly distributed with respect to Y, q changes sign an even number of times in I_j, $j = 1, 2,...,$ $N - 1$, while if $y_j \in X \cap Y$, then q does not change sign at y_j. Thus $q(y_s) \geqslant 0$ for all s, and $q(y_i) > 0$. We obtain $0 = \int_{-1}^{1} q = \sum_{j=1}^{N} a_j\, q(y_j) > a_i\, q(y_i) > 0$. It follows that $q \equiv 0$, so that E is X-poised. ∎

COROLLARY 1. *If all x_i's lie outside $[y_1, y_N]$, then E is X-poised.*

We can also obtain a special case of Theorem A.

COROLLARY 2. *If l_i is even, $i = 1, 2,..., k$, (so that there are no odd sequences), then E is poised.*

To illustrate our results, we return to our examples of Section 2.

EXAMPLE 1. For E_1, we have $q \in P_1$. We use Gaussian quadrature [5, p. 390], $\int_{-1}^{1} q = 2q(0)$, exact for P_1. Clearly, if $x_1 \neq 0$, then E_1 is X-poised.

EXAMPLE 2. For E_2, we have $q \in P_2$. Using Gaussian quadrature, we obtain $\int_{-1}^{1} q = q(-1/\sqrt{3}) + q(1/\sqrt{3})$, exact for P_3. Here $Y = \{-1/\sqrt{3}, 1/\sqrt{3}\}$, so that if $x_1, x_2 \in (-1/\sqrt{3}, 1/\sqrt{3})$ or $x_1, x_2 \notin [-1/\sqrt{3}, 1/\sqrt{3}]$, then (E_2, X) is evenly distributed with respect to Y. Hence, E_2 is X-poised. We can, however, go a step further. The strength of Theorem 1 is that it can be used with *any* quadrature formula with positive coefficients. Thus, if we apply Radau quadrature [5, p. 406], $\int_{-1}^{1} q = 1/2q(-1) + 3/2q(1/3)$, exact for P_2, we obtain $Y = \{-1, 1/3\}$, so that $x_2 < 1/3$ implies E_2 is X-poised. Similarly, since $\int_{-1}^{1} q = 3/2q(-1/3) + 1/2q(1)$, we see that E_2 is X-poised if $x_1 > -1/3$. These results are in agreement with our earlier observations.

4. REMARKS AND EXTENTIONS

1. A class of quadrature formulae with positive coefficients, based on N nodes and exact for P_{2N-3}, has been obtained in [17]. Many other such formulae are known [3], such as Newton–Coates for $N < 8$ [23, p. 113] and Lobatto quadrature [5, p. 409]. Other results connecting Birkhoff interpolation and quadrature formulae (in different contexts) include [11, 16, 21].

2. It is tempting to think that the converse of Theorem 1 is true; i.e., if E is X-poised then there exists *some* quadrature formula with positive coefficients, $\sum_{j=1}^{N} a_j f(y_j)$, exact for P_{n-1}, such that (E, X) is evenly distributed with

respect to Y. This can be demonstrated for the matrix E_2 by a calculation, and the conjecture seems plausible in general.

3. The ideas of Theorem 1 can also be used to show that certain matrices are not X-poised, when the interpolation takes place *at* the nodes of an appropriate quadrature formula. Specifically, we have the following:

THEOREM 2. *Let $Y = \{y_1, y_2, ..., y_N\}$ be the nodes of a quadrature formula (not necessarily with positive coefficients) which is exact for P_{n-1}. Let $X = \{-1, 1\} \cup Y$ and suppose that E is an $(N + 2) \times (n + 1)$ matrix in the class of Section 3 which has Hermite data in rows 0 and $N + 1$. Then E is not X-poised.*

Proof. As in the proof of Theorem 1, we have $n \leqslant 2N$, so that at least one interior row of E has just one 1. By Theorem B, E is conditionally poised. Now let $q(x) = (x + 1)^{l_0-1}(x - 1)^{l_{N+1}-1} \prod_{i=1}^{N} (x - y_i)^{l_i}$, and let $p(x) = \int_{-1}^{x} q(t)\, dt$. Then $p \in P_n$, $p(-1) = 0$, and $\int_{-1}^{1} q = \sum_{j=1}^{N} a_j\, q(y_j) = 0$, so that $p(1) = 0$. Thus, p is a non-trivial polynomial in P_n which annihilates E, so that E is not X-poised. ∎

As a special case of Theorem 2, let Y be the set of zeros of the Legendre polynomial of degree N and let $n = 2N$. Let $X = \{-1, 1\} \cup Y$ and let E satisfy the conditions of Theorem 2. Then E is not X-poised. For example,

$$E = \begin{Vmatrix} 1 & 1 & 0 & \\ 0 & 1 & 0 & \\ 0 & 1 & 1 & \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & \\ 1 & 0 & 0 & \end{Vmatrix}$$

is not X-poised for $X = \{-1, 1\} \cup \{y_1, y_2, y_3, y_4\}$, where the y's are the zeros of the Legendre polynomial of degree 4.

4. We noted that E_2 is X-poised unless $x_1 x_2 = -1/3$. In general, to obtain the algebraic equation for which a matrix is nonpoised requires evaluating a determinant. If, however, rows 0 and $k + 1$ contain Hermite data, then this equation may be obtained simply, since

$$q(x) = (x + 1)^{l_0-1} (x - 1)^{l_{k+1}-1} \prod_{j=1}^{k} (x - x_j)^{l_j} = \sum_{i=0}^{n-1} (-1)^i\, \sigma_i x^{n-1-i},$$

where $\{\sigma_i\}$ are the elementary symmetric functions based on $-1, x_1, x_2, ..., x_k, 1$, counting multiplicities. Since E is not X-poised if and only if $\int_{-1}^{1} q = 0$, we obtain the following result.

THEOREM 3. *Suppose E contains Hermite data in rows 0 and $k+1$. Then E is not X-poised if and only if*

$$\frac{\sigma_1}{n-1} + \frac{\sigma_3}{n-3} + \cdots + \sigma_{n-1} = 0, \qquad n \text{ even},$$

$$\frac{\sigma_2}{n-2} + \frac{\sigma_4}{n-4} + \cdots + \sigma_{n-1} = -\frac{1}{n}, \qquad n \text{ odd}.$$

5. We saw in the case of E_2 that additional information can be obtained about the poisedness of a matrix by considering various quadrature formulae. Similarly, quadrature formulae with multiple nodes can also be useful, although the arguments become more delicate and a general theory remains to be developed. We illustrate with an example. Let

$$E = \left\| \begin{matrix} 1 & 0 & 0 & 0 & \\ 0 & 1 & 1 & 1 & \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & \end{matrix} \right\|$$

Here $q \in P_6$, so that we can use Gaussian quadrature based on the four nodes $-.86, -.34, .34, .86$ (rounded to 2 places). In particular, if $x_1, x_2 \notin [-.87, .87]$, then E is X-poised. We now consider the quadrature formula

$$\int_{-1}^{1} q = A_0 q(y_1) + A_1 q'(y_1) + A_2 q''(y_1) + B_0 q(y_2) + B_1 q'(y_2) + B_2 q''(y_2), \tag{6}$$

which is exact for P_7 [24, 26, 27]. The nodes and coefficients have been calculated [24] and been determined to be $y_1 = -.63$, $y_2 = .63$, $B_0 = A_0$, $B_1 = -A_1$, $B_2 = A_2$, where $A_i > 0$, $i = 0, 1, 2$. Now suppose $x_1 < y_1$, $x_2 > y_2$, and let $q(x) = c(x - x_1)^3(x - x_2)^3$, where $c < 0$. It is easily seen that $q(y_1) > 0$, $q(y_2) > 0$, $q'(y_1) > 0$, $q'(y_2) < 0$. Moreover, a calculation shows that $q''(y_1) > 0$ and $q''(y_2) > 0$. Hence, each term in the quadrature formula is positive, so that $\int_{-1}^{1} q > 0$, and E is X-poised. A similar, but more subtle calculation yields the same result if $x_1 < y_1$, $x_2 < y_1$, or $x_1 > y_2$, $x_2 > y_2$ (here the *magnitudes* of the coefficients become important, not just the signs). We thus see that E is X-poised if $x_1, x_2 \notin [-.63, .63]$, which is a substantial improvement over the result obtained using simple Gaussian quadrature.

A similar calculation yields analogous results for the matrix

$$\left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 1 & 1 & 1 & 1 & \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \end{matrix} \right\|.$$

It thus seems that quadrature formulae with multiple nodes can be of use in determining X-poisedness. In order to apply this method to more general cases, however, a good deal of analysis of such formulae is still needed. In particular, knowledge of the signs and magnitudes of the coefficients will likely be important factors. In connection with this, see [15, p. 429].

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