

Contents lists available at [SciVerse ScienceDirect](http://SciVerse.Sciencedirect.com)

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

On differential operators of numerical semigroup rings

Valentina Barucci^a, Ralf Fröberg^{b,*}^a University of Roma 1, Italy^b University of Stockholm, Sweden

ARTICLE INFO

Article history:

Received 24 December 2011

Received in revised form 21 May 2012

Available online 20 June 2012

Communicated by A.V. Geramita

MSC:

Primary: 13N10; 16S32

Secondary: 13A30; 13N15

ABSTRACT

If $S = \langle d_1, \dots, d_v \rangle$ is a numerical semigroup, we call the ring $\mathbb{C}[S] = \mathbb{C}[t^{d_1}, \dots, t^{d_v}]$ the semigroup ring of S . We study the ring of differential operators on $\mathbb{C}[S]$, and its associated graded in the filtration induced by the order of the differential operators. We find that these are easy to describe if S is a so-called Arf semigroup. If I is an ideal in $\mathbb{C}[S]$ that is generated by monomials, we also give some results on $\text{Der}(I, I)$ (the set of derivations which map I into I).

© 2012 Elsevier B.V. All rights reserved.

1. Introduction

It is well known that the ring $D(\mathbb{C}[S])$ of differential operators of a semigroup ring $\mathbb{C}[S] = \mathbb{C}[t^{d_1}, \dots, t^{d_v}]$, where \mathbb{C} is the complex field, is a subring of $D(\mathbb{C}[t, t^{-1}])$, the ring of differential operators of the Laurent polynomials $\mathbb{C}[t, t^{-1}]$. $D(\mathbb{C}[t, t^{-1}])$ is a non-commutative $\mathbb{C}[t, t^{-1}]$ -algebra generated by ∂ , the usual derivation d/dt . It is also known that $D(\mathbb{C}[S])$ is a \mathbb{C} -algebra finitely generated and a set of generators was found independently in [4,5]. The ring $D(\mathbb{C}[S])$ inherits a grading from the ring of differential operators of $\mathbb{C}[t, t^{-1}]$, where $\deg(t^s) = s$ and $\deg(\partial) = -1$. Its associated graded is a commutative Noetherian subring of the ring of polynomials in two indeterminates $\mathbb{C}[t, y]$ and it is a semigroup ring $\mathbb{C}[\Sigma]$, where $\Sigma \subseteq \mathbb{N}^2$ is a semigroup, with $|\mathbb{N}^2 \setminus \Sigma|$ finite. It follows that $D(\mathbb{C}[S])$ is right and left Noetherian.

In this paper we study that commutative associated ring $\mathbb{C}[\Sigma]$, in terms of the starting numerical semigroup S . Many properties of the semigroup Σ , including the minimal set of generators, can be predicted looking at S . If S is of maximal embedding dimension, then Σ behaves well with respect to the blowup of the maximal ideal. If, moreover, S is an Arf semigroup, then we show how Σ , and the ring $\mathbb{C}[\Sigma]$ as well, is completely determined by S . In Section 4 we characterize the irreducible ideals of Σ , i.e. the irreducible monomial ideals of $\mathbb{C}[\Sigma]$ and determine the number of components for a principal monomial ideal as irredundant intersection of irreducible ideals. Finally in Section 5 we observe that, if I is a monomial ideal of $\mathbb{C}[S]$, then $\text{Der}(I, I)$, the $\mathbb{C}[S]$ -module of derivations which map I into I is isomorphic to the overring $I : I$ of $\mathbb{C}[S]$. Thus we study the overrings of this form in relation with the monomial ideals I which realize them.

2. Numerical semigroups

We fix for the whole paper the following notation. S is a numerical semigroup, i.e. a subsemigroup of \mathbb{N} , with zero and with finite complement $H(S) = \mathbb{N} \setminus S$ in \mathbb{N} . The numerical semigroup generated by $d_1, \dots, d_v \in \mathbb{N}$ is $S = \langle d_1, \dots, d_v \rangle = \{\sum_{i=1}^v n_i d_i; n_i \in \mathbb{N}\}$. $M = S \setminus \{0\}$ is the maximal ideal of S , $e = e(S)$ is the multiplicity of S , that is the smallest positive integer of S , $g = g(S)$ is the Frobenius number of S , that is the greatest integer which does not belong to S , $n = n(S)$ is the number of elements of S smaller of g . Thus we have $S = \{0 = s_0 < s_1 = e < s_2 \cdots < s_{n-1} < s_n = g + 1, g + 2, \dots\}$.

* Corresponding author.

E-mail addresses: [barucci@mat.uniroma1](mailto:barucci@mat.uniroma1.it) (V. Barucci), ralf@math.su.se (R. Fröberg).

A relative ideal of S is a nonempty subset I of \mathbb{Z} (which is the quotient group of S) such that $I + S \subseteq I$ and $I + s \subseteq S$, for some $s \in S$. A relative ideal which is contained in S is an integral ideal of S .

If I, J are relative ideals of S , then the following is a relative ideal too: $I - J = \{z \in \mathbb{Z} \mid z + J \subseteq I\}$.

If I is a relative ideal of S , then $I - I$ is the biggest semigroup T such that I is an integral ideal of T . There is a chain of semigroups $S \subseteq (I - I) \subseteq (2I - 2I) \subseteq \dots \subseteq \mathbb{N}$, which stabilizes on a semigroup $\mathcal{B}(I) = (hI - hI)$ for $h \gg 0$, called the blowup of I . Setting $S_0 = S, S_1 = \mathcal{B}(M)$ and $S_{i+1} = \mathcal{B}(M_i)$, where M_i is the maximal ideal of S_i , the multiplicity sequence of S is (e_0, e_1, e_2, \dots) , where $e_i = e(S_i)$. If $z \in \mathbb{Z}$, set $S(z) = \{s \in S; s \geq z\}$, which is an ideal of S . If, with the notation above, $x = s_i$, we denote for simplicity $S(s_i)$ with I_i .

Definitions. For each $z \in \mathbb{Z}$, we define the valency of z with respect to a semigroup S as $\text{val}_S(z) = |\{s \in S; z + s \notin S\}|$ (in [4] this numerical function is called σ). Let $V_i(S) = \{a \in \mathbb{Z}; \text{val}_S(a) \leq i\}$. When there is no ambiguity about the semigroup S , we will write simply $\text{val}(z)$ and V_i respectively.

Lemma 2.1. (a) $S = V_0$.

(b) $S - M = V_1$.

(c) For each $z \in \mathbb{Z}$, $\text{val}(-z) = \text{val}(z) + z$.

(d) $I_i - I_i \subseteq V_i$.

(e) If $a \in \mathbb{N}$, then $\text{val}(a) \leq n$.

(f) V_i is a relative ideal of S .

Proof. (a) and (b) follow immediately from the definitions.

(c) is given in [4, Proposition 2], [7, Lemma 3.3] and [5].

(d): Let $a \in I_i - I_i$. Then $a + s_j \in I_i \subseteq S$ if $j \geq i$, so at most $\{a + s_0, a + s_1, \dots, a + s_{i-1}\}$ are not in S .

(e): By (d), $I_n - I_n = \mathbb{N} \subseteq V_n$, so $\text{val}(a) \leq n$ for each $a \in \mathbb{N}$.

(f): Assume $a \in V_i$ and $s \in S$, then $\text{val}(a + s) \leq i$. In fact if $(a + s) + b \notin S$, with $b \in S$, also $a + (s + b) \notin S$, with $s + b \in S$. Moreover by (c) V_i has a minimum $m \leq 0$. Thus $s + V_i \subseteq S$, for some $s \in S$. Take for example $s = g + 1 - m$. \square

We denote by $T(S)$ the set $\{x \in \mathbb{Z}; x \notin S, x + M \subseteq M\}$, called in [8] the set of pseudo-Frobenius numbers. With the notation above, $T(S) = V_1 \setminus V_0$. The cardinality of $T(S)$ is the type t of the semigroup S . It is well known that $t \leq e - 1$ and $t = e - 1$ if and only if the numerical semigroup S is of maximal embedding dimension, i.e. when the number ν of generators equals the multiplicity e , i.e. when $|M \setminus 2M| = e$, cf. e.g. [8, Corollaries 2.23 and 3.2, 3]). It is also well known that S is of maximal embedding dimension if and only if $M - e$ is a semigroup (cf. e.g. [8, Proposition 3.12]). In this case we have $S_1 = M - e$. Setting $\pm H(S) = \{\pm h; h \in \mathbb{N} \setminus S\}$, we have

Lemma 2.2. Let S be a numerical semigroup of maximal embedding dimension. Then $\text{val}_{S_1}(z) = \text{val}_S(z) - 1$, for each $z \in \pm H(S)$.

Proof. Let $z \in \pm H(S)$. Denote by $\Omega_S(z)$ (respectively $\Omega_{S_1}(z)$) the set of pairs $(s, s + z)$, with $s \in S$ and $s + z \notin S$ (resp. the set of pairs $(s_1, s_1 + z)$, with $s_1 \in S_1$ and $s_1 + z \notin S_1$). By definition of valency, $\text{val}_S(z)$ is the cardinality of $\Omega_S(z)$ and $\text{val}_{S_1}(z)$ is the cardinality of $\Omega_{S_1}(z)$. If $(s, s + z) \in \Omega_S(z)$, with $s \neq 0$, then $(s - e, s - e + z) \in \Omega_{S_1}(z)$ and, conversely, if $(s_1, s_1 + z) \in \Omega_{S_1}(z)$, then $(s_1 + e, s_1 + e + z) \in \Omega_S(z)$. Thus there is a one-to-one correspondence between $\Omega_S(z) \setminus \{(0, z)\}$ and $\Omega_{S_1}(z)$ and the conclusion follows. \square

An Arf semigroup is a numerical semigroup S such that $I_i - s_i$ is a semigroup for each $i \geq 0$. Thus, if S is Arf, then $M - e$ is a semigroup and S is of maximal embedding dimension. Given an Arf semigroup $S = \{0 = s_0 < s_1 < s_2 \dots\}$, the multiplicity sequence of S is $\{s_1 - s_0, s_2 - s_1, s_3 - s_2, \dots\}$. It follows that the multiplicity sequence e_0, e_1, \dots of an Arf semigroup is such that for all $i, e_i = \sum_{h=1}^k e_{i+h}$, for some $k \geq 1$. Conversely, any sequence of natural numbers e_0, e_1, \dots such that $e_n = 1$, for $n \gg 0$, and, for all $i, e_i = \sum_{h=1}^k e_{i+h}$, for some $k \geq 1$, is the multiplicity sequence of an Arf semigroup $S = \{0, e_0, e_0 + e_1, e_0 + e_1 + e_2, \dots\}$, [1].

Lemma 2.3. If S is an Arf semigroup, then $I_i - I_i = V_i \cap \mathbb{N}$.

Proof. By Lemma 2.1(d), $I_i - I_i \subseteq V_i \cap \mathbb{N}$. Conversely let $a \in \mathbb{N}$ and observe that, since S is Arf, if $a + s_i \in S$, then also $a + s_j \in S$ for each $j \geq i$. In fact $a + s_i \in S$ if and only if $a + s_i \in I_i$ if and only if $a \in I_i - s_i$. If $j \geq i$, since $a, s_j - s_i \in I_i - s_i$, which is a semigroup, then $a + s_j - s_i \in I_i - s_i$, so that $a + s_j \in S$. Thus $\text{val}(a) = i$ if and only if $a + s_j \in S$ for $j \geq i$. \square

3. Differential operators on numerical semigroup rings

Let R be a commutative \mathbb{C} -algebra. The ring of differential operators $D(R)$ of R is inductively defined in the following way. Setting

$$D^0(R) = \{\theta_a; a \in R\}$$

$$D^i(R) = \{\theta \in \text{Hom}_{\mathbb{C}}(R, R); [\theta, D^0(R)] \subseteq D^{i-1}(R)\}$$

where $\theta_a: R \rightarrow R$ is the multiplication map $r \mapsto ar$, and $[\theta, \phi] = \theta\phi - \phi\theta$ is the commutator, the ring of differential operators is

$$D(R) = \bigcup_{i \geq 0} D^i(R).$$

This is a filtered ring, $D^i(R)D^j(R) \subseteq D^{i+j}(R)$ for all i and j , and $D^i(R) \subseteq D^{i+1}(R)$, and its associated graded is $\text{gr}(D(R)) = \bigoplus_{i \geq 0} D^i(R)/D^{i-1}(R)$, where $D^{-1}(R) = 0$.

The module of derivations $\text{Der}(R)$ is $\{\vartheta \in \text{Hom}_k(R, R); \vartheta(ab) = \vartheta(a)b + a\vartheta(b), a, b \in R\}$ and it is well known that $\text{Der}(R)$ is $\{\vartheta \in D^1(R); \vartheta(1) = 0\}$.

The rings of differential operators of semigroup rings have been studied by e.g. Perkins [7], Eriksen [4] and Eriksson [5]. It is shown that the ring of differential operators of a semigroup ring $\mathbb{C}[S]$ is a subring of $D(\mathbb{C}[t, t^{-1}]) = \mathbb{C}[t, t^{-1}](\partial)$. The ring of differential operators of a semigroup ring inherits a grading from $D(\mathbb{C}[t, t^{-1}])$, where $\text{deg}(t^s) = s$ and $\text{deg}(\partial) = -1$. Its associated graded is a commutative Noetherian subring of the ring of polynomials $\mathbb{C}[t, y]$ and it is a semigroup ring $\mathbb{C}[\Sigma]$, where $\Sigma \subseteq \mathbb{N}^2$ is a semigroup, with $|\mathbb{N}^2 \setminus \Sigma|$ finite. More precisely, by [4, Lemma 5] or by [5] and Lemma 2.1(a), (c), it follows that:

Theorem 3.1 ([4,5]). *If $S = \langle d_1, \dots, d_v \rangle$ is a numerical semigroup and $\mathbb{C}[S]$ its semigroup ring, then $\text{gr}(D(\mathbb{C}[S]))$ is a \mathbb{C} -subalgebra of $\mathbb{C}[t, y]$ minimally generated by the monomials*

$$\{t^{d_1}, \dots, t^{d_v}, y^{d_1}, \dots, y^{d_v}\} \cup \{ty\} \cup \{t^{\text{val}(-h)}y^{\text{val}(h)}; h \in \pm H(S)\}.$$

Thus $\text{gr}(D(\mathbb{C}[S]))$ is a semigroup ring $\mathbb{C}[\Sigma]$, where Σ is a subsemigroup of \mathbb{N}^2 minimally generated by

$$(d_1, 0), \dots, (d_v, 0), (0, d_1), \dots, (0, d_v), (1, 1), \{(\text{val}(-h), \text{val}(h)); h \in \pm H(S)\}.$$

We will study this semigroup Σ . For all this section, $S = \langle d_1, \dots, d_v \rangle$ is a numerical semigroup and $\text{gr}(D(\mathbb{C}[S])) = \mathbb{C}[\Sigma]$.

If $z \in \mathbb{Z}$, denote by Δ_z the diagonal $\{(a, b) \in \mathbb{N}^2; a - b = z\}$. Observing the minimal set of generators of Σ , we can immediately say that all the diagonals Δ_s , with $s \in \pm S$ are contained in Σ (here, as above, we set $\pm S = \{\pm s; s \in S\}$). On the other hand the diagonals Δ_h , with $h \in \pm H(S)$ are not contained in Σ in general. In the example after Proposition 3.5 below, for example the elements $(1, 0)$, $(2, 1)$, $(3, 2)$ of Δ_1 are not in Σ .

Since $H(S)$ is finite, Σ is finitely generated and, setting $|H(S)| = \delta$, the number of minimal generators of Σ is $2v + 1 + 2\delta$. Thus Σ is an affine monoid of rank 2, and $\text{gp}(\Sigma)$, the group generated by Σ is \mathbb{Z}^2 . The normalization of Σ , $\bar{\Sigma} = \{\alpha \in \text{gp}(\Sigma); m\alpha \in \Sigma \text{ for some } m \in \mathbb{N}, m \geq 1\}$ is \mathbb{N}^2 . We denote by Σ_+ the maximal ideal $\Sigma \setminus \{(0, 0)\}$, and we set $T(\Sigma) = \{\tau \in \text{gp}(\Sigma); \tau \notin \Sigma, \tau + \Sigma_+ \subseteq \Sigma_+\}$.

Observing that, if $z \in \mathbb{Z}$ and $(\text{val}(-z), \text{val}(z)) = (a, b)$, then $(\text{val}(z), \text{val}(-z)) = (b, a)$ (cf. Lemma 2.1 (c)), we get for Σ the following symmetric property:

Corollary 3.2 ([4,5]). *If $a, b \in \mathbb{N}$, then $(a, b) \in \Sigma$ if and only if $(b, a) \in \Sigma$.*

Lemma 3.3. *Let S be a numerical semigroup. Let $a, b \in \mathbb{N}$. Then $(a, b) \in \Sigma$ if and only if $a - b \in V_b$.*

Proof. Note first that:

- (i) If $(a, b) \in \Sigma$, then $(a, b) \in \Delta_z$, with either $z \in \pm S$ or $z \in \pm H(S)$.
- (ii) If $(a, b) \in \Delta_h$, with $h \in H(S)$, then $(a, b) = (\text{val}(-h), \text{val}(h)) + c(1, 1)$, with $c = b - \text{val}(h)$.

We already observed that Σ contains all the diagonals Δ_s , for $s \in \pm S$. If $s \in S$, then for each element $(a, b) \in \Delta_s$ (respectively $(a, b) \in \Delta_{-s}$), we get $\text{val}(a - b) = \text{val}(s) = 0$ and $0 \leq b$ (respectively $\text{val}(a - b) = \text{val}(-s) = \text{val}(s) + s = 0 + s = s$ and $s \leq b$). Thus in both cases $\text{val}(a - b) \leq b$, i.e. $a - b \in V_b$.

Further, if $h \in \pm H(S)$, then $(\text{val}(-h), \text{val}(h)) = (\text{val}(h) + h, \text{val}(h)) \in \Delta_h$ is a minimal generator of Σ . Since $(1, 1) \in \Sigma$, it follows that an element (a, b) of Δ_h is in Σ if and only if $b - \text{val}(h) \geq 0$, i.e. $b \geq \text{val}(h) = \text{val}(a - b)$, i.e. $a - b \in V_b$. \square

Definition Let Γ be a subsemigroup of \mathbb{N}^2 , and let $\gamma \in \Gamma$. The Apéry set of Γ with respect to γ is $\text{Ap}_\gamma(\Gamma) = \{\alpha \in \Gamma; \alpha - \gamma \notin \Gamma\}$.

Lemma 3.4. $\text{Ap}_{(1,1)}(\Sigma) = \{(0, s); s \in S\} \cup \{(s, 0); s \in S\} \cup \{(\text{val}(-h), \text{val}(h)); h \in \pm H(S)\}$.

Proof. Let $(a, b) \in \text{Ap}_{(1,1)}(\Sigma)$. We can suppose that $a \geq b$ due to the symmetry of Σ . It is clear that $(s, 0) \in \text{Ap}_{(1,1)}(\Sigma)$ if and only if $s \in S$. Suppose $h \in H(S)$. We know that $\alpha = (h + \text{val}(h), \text{val}(h)) \in \Delta_h$ is a minimal generator of Σ . Thus $\alpha \in \text{Ap}_{(1,1)}(\Sigma)$ and no other element σ of $\Delta_h \cap \Sigma$ is in $\text{Ap}_{(1,1)}(\Sigma)$, because $\sigma - (1, 1) \in \Sigma$. \square

Proposition 3.5. *Let $\tau \in \mathbb{Z}^2$. Then the following conditions are equivalent:*

- (1) $\tau \in T(\Sigma)$.
- (2) $\tau + (1, 1)$ is a minimal generator of Σ of the form $(\text{val}(-h), \text{val}(h))$ with $h \in \pm H(S)$.

Proof. (2) \Rightarrow (1). It is clear that $\tau \notin \Sigma$ since $\tau + (1, 1)$ is a minimal generator. It is also clear that if $\tau = (\tau_1, \tau_2)$, then $\tau_i \geq 0, i = 1, 2$. We can suppose that $\tau_1 \geq \tau_2$ due to the symmetry of Σ . Thus $\tau = (h + i, i)$ for some $h \in H(S), i \geq 0$. We have $\sigma = (h + \text{val}(h), \text{val}(h)) \in \text{Ap}_{(1,1)}(\Sigma)$ according to Lemma 3.4. Let σ' be a minimal generator. Then $\sigma + \sigma'$ is not a minimal generator, so $\sigma + \sigma' \notin \text{Ap}_{(1,1)}(\Sigma)$, since certainly $\sigma + \sigma' \notin \{(0, s); s \in S\} \cup \{(s, 0); s \in S\}$. Thus $\sigma + \sigma' - (1, 1) \in \Sigma$, so $\tau = \sigma - (1, 1) \in T(\Sigma)$.

(1) \Rightarrow (2). If $\tau = (\tau_1, \tau_2) \in T(\Sigma)$, then $\tau_i \geq 0, i = 1, 2, \tau \notin \Sigma$, and $\tau + (1, 1) \in \Sigma$. Thus $\tau + (1, 1) \in \text{Ap}_{(1,1)}(\Sigma)$, so $\tau + (1, 1) = (\text{val}(-h), \text{val}(h))$ for some $h \in \pm H(S)$ according to Lemma 3.4. \square

Example. Let $S = \langle 3, 5 \rangle$. Then $H(S) = \{7, 4, 2, 1\}$ and $\text{val}(7) = 1, \text{val}(4) = 2, \text{val}(2) = 2, \text{val}(1) = 3$. Thus, if $\text{gr}(D(\mathbb{C}[S])) = \mathbb{C}[\Sigma]$, then Σ is minimally generated by $(3, 0), (5, 0), (1, 1), (0, 3), (0, 5), (8, 1), (6, 2), (4, 2), (4, 3), (1, 8), (2, 6), (2, 4), (3, 4)$. Thus $T(\Sigma) = \{(7, 0), (5, 1), (3, 1), (3, 2), (0, 7), (1, 5), (1, 3), (2, 3)\}$.

Proposition 3.5 gives a one-to-one correspondence between $T(\Sigma)$ and the minimal generators of the form $(\text{val}(-h), \text{val}(h))$ with $h \in \pm H(S)$, so we get:

Corollary 3.6. *The minimal set of generators of Σ has cardinality $2\nu + 1 + |T(\Sigma)|$.*

We know by Lemma 3.3 that $\Sigma = \cup_{b \geq 0} (b + V_b, b)$. In a similar way, we can describe $\Sigma \cup T(\Sigma)$:

Proposition 3.7. $\Sigma \cup T(\Sigma) = \cup_{b \geq 0} (b + V_{b+1}, b)$.

Proof. If $\sigma \in \Sigma$, then, for some $b \geq 0, \sigma \in (b + V_b, b) \subseteq (b + V_{b+1}, b)$. If $\sigma \in T(\Sigma)$, then $\sigma + (1, 1) \in (b + V_b, b)$, for some $b \geq 1$, thus $\sigma \in ((b - 1) + V_b, b - 1)$, for some $b \geq 1$. Finally, if $\sigma \in \mathbb{N}^2 \setminus (\Sigma \cup T(\Sigma))$, then $\sigma + (1, 1) \notin (b + V_b, b)$, for any $b \geq 1$, so $\sigma \notin ((b - 1) + V_b, b - 1)$, for any $b \geq 1$. \square

Proposition 3.8. *Let S be a numerical semigroup of maximal embedding dimension and let $\text{gr}(D(\mathbb{C}[S_1])) = \mathbb{C}[\Sigma_1]$. Then*

- (i) $\Sigma_1 = \Sigma \cup T(\Sigma)$.
- (ii) If $a, b \in \mathbb{N}$, then $(a, b) \in \Sigma_1$ if and only if $(a + 1, b + 1) \in \Sigma$.

Proof. The proof follows combining Lemma 2.2 and Proposition 3.7. \square

Example. Let $S = \langle 4, 6, 9, 11 \rangle$. Then

$$T(\Sigma) = \{(3, 2), (2, 0), (4, 1), (5, 0), (7, 0), (2, 3), (0, 2), (1, 4), (0, 5), (0, 7)\}$$

and $\mathbb{N}^2 \setminus \Sigma = T(\Sigma) \cup \{(1, 0), (3, 0), (2, 1), (0, 1), (0, 3), (1, 2)\}$. The blowup of S is $S_1 = \langle 2, 5 \rangle, \Sigma_1 \setminus \Sigma = T(\Sigma)$, and $T(\Sigma_1) = \{(2, 1), (3, 0), (1, 2), (0, 3)\}$. The blowup of S_1 is $S_2 = \langle 2, 3 \rangle, \Sigma_2 \setminus \Sigma_1 = T(\Sigma_1)$, and $T(\Sigma_2) = \{(1, 0), (0, 1)\}$. The blowup of S_2 is $\mathbb{N}^2, \mathbb{N}^2 \setminus \Sigma_2 = T(\Sigma_2)$.

Now let's consider the case when the starting numerical semigroup S is Arf.

Lemma 3.9. *Let S be an Arf numerical semigroup. Let $a, b \in \mathbb{N}, a \geq b$. Then $(a, b) \in \Sigma$ if and only if $a - b \in I_b - s_b$.*

Proof. By Lemma 3.3 (a, b) , with $a \geq b$ is an element of Σ when $a - b \in V_b$ and $a - b \geq 0$. By Lemma 2.3 this is equivalent to $a - b \in I_b - I_b$. But $I_b - I_b = I_b - s_b$, because S is Arf. \square

Proposition 3.10. *If S is an Arf semigroup, $S \neq \mathbb{N}$, then*

$$T(\Sigma) = \bigcup_{b=0}^{n-1} ((T(S_b) + b) \times \{b\}) \cup \bigcup_{a=0}^{n-1} (\{a\} \times (T(S_a) + a)).$$

Proof. Let $\tau = (t + b, b)$, with $t \in T(S_b)$ and $b \in \mathbb{N}$. Since $\tau \in \mathbb{N}^2$, to show that $\tau \in T(\Sigma)$, by Lemma 3.4 and Proposition 3.5 it is enough to show that $\tau \notin \Sigma$ and $\tau + (1, 1) \in \Sigma$. Since $t \in T(S_b)$, we have $t \in S_{b+1} \setminus S_b$. So $\tau = (t + b, b) \notin (S_b + b) \times \{b\}$ and $(t + b, b) + (1, 1) \in S_{b+1} \times \{b + 1\} \subseteq \Sigma$ (cf. Lemma 3.9). By the symmetry of Σ it follows that each element of the form $(a, t + a)$, with $t \in T(S_a)$ and $a \in \mathbb{N}$ is in $T(\Sigma)$. For the opposite inclusion, note that, by Lemma 3.5, $|T(\Sigma)| = 2|H(S)|$. Counting the elements in $\cup_{b=0}^{n-1} ((T(S_b) + b) \times \{b\}) \cup \cup_{a=0}^{n-1} (\{a\} \times (T(S_a) + a))$, we have $2((e_0 - 1) + (e_1 - 1) + \dots + (e_{n-1} - 1)) = 2|H(S)|$ elements, since for an Arf semigroup S with multiplicity e , $|T(S)| = e - 1$. \square

Proposition 3.11. *If S is an Arf semigroup, $S \neq \mathbb{N}$ with multiplicity sequence $e_0, e_1, \dots, e_{n-1} \neq 1, e_n = 1, e_{n+1} = 1, \dots$, then*

- (a) $|\mathbb{N}^2 \setminus \Sigma| = 2(e_0 + 2e_1 + \dots + ne_{n-1} - \binom{n+1}{2}) = 2(ns_n - (s_1 + s_2 + \dots + s_{n-1}) - \binom{n+1}{2})$.
- (b) Σ is minimally generated by $2e_0 + 1 + 2((e_0 - 1) + (e_1 - 1) + \dots + (e_{n-1} - 1))$ elements.

Proof. (a) By the symmetry of Σ and since $(1, 1) \in \Sigma$, it suffices to consider the set $\{(a, b) \in \mathbb{N}^2; a > b\}$. The number of $(a, 0) \in \mathbb{N}^2 \setminus \Sigma$ is $s_n - n = (e_0 + e_1 + \dots + e_{n-1}) - n$. The number of $(a, 1) \notin \Sigma, a > 1$, is $e_1 + e_2 + \dots + e_{n-1} - (n - 1)$. The number of $(a, 2) \notin \Sigma, a > 2$, is $e_2 + \dots + e_{n-1} - (n - 2)$, and so on. Thus we get the left hand side. Since $e_0 + e_1 + \dots + e_{i-1} = s_i$, we have $ns_n - (s_1 + s_2 + \dots + s_{n-1}) = n(e_0 + e_1 + \dots + e_{n-1}) - (e_0 + (e_0 + e_1) + \dots + (e_0 + e_1 + \dots + e_{n-2})) = e_0 + 2e_1 + \dots + ne_{n-1}$. (b) We have seen that Σ is minimally generated by $2\nu + 1 + 2\delta$ elements. Since S is Arf, we have $\nu = e_0$ and $\delta = (e_0 - 1) + (e_1 - 1) + \dots + (e_{n-1} - 1)$. \square

Proposition 3.12. *If S is any numerical semigroup, $S \neq \mathbb{N}$, the number μ of minimal generators of Σ satisfies $g + 6 \leq \mu \leq 4g + 3$, with equality to the left if and only if S is 2-generated and equality to the right if and only if $S = \langle g + 1, g + 2, \dots, 2g + 1 \rangle$.*

Proof. We know that the number of minimal generators of Σ is $2\nu + 1 + 2\delta$, where ν is the number of generators for S , and δ is the number of gaps. The number of gaps is at least $(g + 1)/2$, and the number of generators is at least 2. The number of gaps is at most g , and the number of generators at most $g + 1$. If $\nu = 2$, then S is symmetric, thus $\delta = (g + 1)/2$ and we have equality to the left. On the other hand, it is $\delta = g$ if and only if $S = \langle g + 1, g + 2, \dots, 2g + 1 \rangle$. This is a semigroup of maximal embedding dimension of multiplicity $e = g + 1$, so $\nu = e = g + 1$ and we have equality to the right. \square

We give two examples of the ring of differential operators and its associated graded ring for Arf semigroup rings.

Example 1. If $S = \langle 2, 5 \rangle$, then Σ is given by all $(a, b) \in \mathbb{N}^2$ except

$$\{(1, 0), (3, 0), (0, 1), (0, 3), (2, 1), (1, 2)\}.$$

A minimal set of generators for $\mathbb{C}[\Sigma]$ is

$$\{t^2, t^5, y^2, y^5, ty, t^4y, t^3y^2, t^2y^3, ty^4\}$$

so $T(\Sigma) = \{(3, 0), (2, 1), (0, 3), (1, 2)\}$. A corresponding set of generators for the ring of differential operators $D(\mathbb{C}[S])$ is

$$\{t^2, t^5, \partial^2 - 4t^{-1}\partial, \partial^5 - 10t^{-1}\partial^4 + 45t^{-2}\partial^3 - 105t^{-3}\partial^2 + 105t^{-4}\partial, t\partial, t^4\partial, t^3\partial^2 - t^2\partial, t^2\partial^3 - 3t\partial^2 + 3\partial, t\partial^4 - 6\partial^3 + 15t^{-1}\partial^2 - 15t^{-2}\partial\}.$$

This is not a minimal generating set, e.g. $[t^4\partial, t^2] = 2t^5$. The blowup of S is $S_1 = \langle 2, 3 \rangle$. All monomials except t and y belong to $\mathbb{C}[\Sigma_1] = \text{gr}(D(\mathbb{C}[S_1]))$. In general, if $S = \langle 2, 2k + 1 \rangle$, $|\mathbb{N}^2 \setminus \Sigma| = k(k + 1)$. All elements (a, b) , $a, b \in \mathbb{N}^2$, except those where $a + b = 2i - 1$, $i = 1, \dots, k$, belong to Σ .

Example 2 (cf. [4, Section 8]). If $S = \langle 3, 4, 5 \rangle$, then Σ is given by all $(a, b) \in \mathbb{N}^2$ except

$$\{(1, 0), (2, 0), (0, 1), (0, 2)\}.$$

A minimal set of generators for $\mathbb{C}[\Sigma]$ is

$$\{t^3, t^4, t^5, y^3, y^4, y^5, ty, t^2y, t^3y, ty^2, ty^3\}$$

so $T(\Sigma) = \{(1, 0), (2, 0), (0, 1), (0, 2)\}$. A corresponding set of generators for $D(\mathbb{C}[S])$ is

$$\{t^3, t^4, t^5, \partial^3 - 6t^{-1}\partial^2 + 12t^{-3}\partial, \partial^4 - 8t^{-1}\partial^3 + 28t^{-2}\partial^2 - 40t^{-3}\partial, \partial^5 - 10t^{-1}\partial^4 + 50t^{-2}\partial^3 - 140t^{-3}\partial^2 + 180t^{-4}\partial, t\partial, t^2\partial, t^3\partial, t\partial^2 - 2\partial, t\partial^3 - 4\partial^2 + 6t^{-1}\partial\}.$$

This is not a minimal generating set, e.g. $[t^2\partial, t^3] = 3t^4$. The blowup of S is \mathbb{N} and $\text{gr}(D(\mathbb{C}[\mathbb{N}])) = \mathbb{C}[t, y]$.

4. Irreducible ideals

It is well known that, for a numerical semigroup S , the cardinality of $T(S)$ is the CM type of $\mathbb{C}[S]$, i.e. $t = |T(S)|$ is the number of components of a decomposition of a principal ideal as irredundant intersection of irreducible ideals. We want to study whether $|T(\Sigma)|$ has a similar meaning in the ring $\mathbb{C}[\Sigma]$.

Let I be a proper ideal of Σ i.e. a proper subset I of Σ such that $I + \Sigma \subseteq I$. I is *irreducible* if it is not the intersection of two (or, equivalently, a finite number of) ideals which properly contain I . I is *completely irreducible* if it is not the intersection of any set of ideals which properly contain I .

Consider the partial order on Σ given by

$$\sigma_1 \leq \sigma_2 \Leftrightarrow \sigma_1 + \sigma_3 = \sigma_2 \quad , \quad \text{for some } \sigma_3 \in \Sigma \tag{*}$$

and for $x \in \Sigma$, set

$$B(x) = \{\sigma \in \Sigma \mid \sigma \leq x\}.$$

Lemma 4.1. *If I is a proper ideal of Σ , then the following conditions are equivalent:*

- (1) I is completely irreducible.
- (2) I is maximal as ideal with respect to the property of not containing an element x , for some $x \in \Sigma$.
- (3) $I = \Sigma \setminus B(x)$, for some $x \in \Sigma$.

Proof. (1) \Rightarrow (2). Let H be the intersection of all the ideals properly containing I . Then there is $x \in H \setminus I$, so I is maximal with respect to the property of not containing x .

(2) \Rightarrow (1). Each ideal J properly containing I contains x , so I is not the intersection of all such ideals J and it is completely irreducible. (2) \Leftrightarrow (3) is trivial. \square

Lemma 4.2. *For each $a, b \in \mathbb{N}$, $a, b > 0$, the following are irreducible, non-completely irreducible ideals of Σ :*

$$N_{(a,0)} := \Sigma \cap \{(x, y) \in \mathbb{N}^2; x \geq a\}$$

$$N_{(0,b)} := \Sigma \cap \{(x, y) \in \mathbb{N}^2; y \geq b\}.$$

Proof. Any ideal J of Σ properly containing $N_{(a,0)}$ contains $(a - 1, s)$, for some $s \in S$, so it contains $(a - 1, s + S)$. It follows that, if J_1, J_2 are ideals properly containing $N_{(a,0)}$, then $J_1 \cap J_2$ contains $(a - 1, \max(s, s') + g + 1 + S)$ (if $(a - 1, s) \in J_1$ and $(a - 1, s') \in J_2$), so $J_1 \cap J_2 \neq N_{(a,0)}$ and $N_{(a,0)}$ is irreducible. On the other hand $N_{(a,0)}$ is the intersection of all the ideals properly containing it, so it is not completely irreducible. \square

In all the results of this section max has to be intended with respect to the partial order (*) on Σ .

Proposition 4.3. *Let I be an ideal of Σ generated by $(a_1, b_1), \dots, (a_h, b_h)$ and let $a = \min\{a_i\}, b = \min\{b_i\}$. Then*

$$I = \bigcap_{x \in \max(\Sigma \setminus I)} (\Sigma \setminus B(x)) \cap N_{(a,0)} \cap N_{(0,b)}$$

is the unique irredundant decomposition of the ideal I as intersection of irreducible ideals.

Proof. \subseteq : let $\alpha \in I$. Then $\alpha \notin B(x)$ for each $x \in \Sigma \setminus I$ (otherwise $\alpha + \beta = x$, for some $\beta \in \Sigma$ and so $x \in I$, a contradiction). Thus $\alpha \in \Sigma \setminus B(x)$, for each $x \in \max(\Sigma \setminus I)$. Moreover $\alpha \in N_{(a,0)} \cap N_{(0,b)}$. \supseteq : observe first that

$$\bigcap_{x \in \max(\Sigma \setminus I)} (\Sigma \setminus B(x)) = \bigcap_{x \in (\Sigma \setminus I)} (\Sigma \setminus B(x))$$

in fact $(\Sigma \setminus B(x_1)) \subseteq (\Sigma \setminus B(x_2))$ if and only if $B(x_1) \supseteq B(x_2)$ if and only if $x_2 \leq x_1$. Suppose that $\alpha \in N_{(a,0)} \cap N_{(0,b)}$, i.e. that $\alpha = (c, d) \in \Sigma$, with $c \geq a$ and $d \geq b$. We have to show that, if $\alpha \in \bigcap_{x \in (\Sigma \setminus I)} (\Sigma \setminus B(x))$, then $\alpha \in I$. In fact, if $\alpha \notin I$, then (since trivially $\alpha \in B(\alpha)$) $\alpha \notin \Sigma \setminus B(x)$, for some $x \in \Sigma \setminus I$ (take $x = \alpha$).

To show that the decomposition is irredundant, it's easy to see that $N_{(a,0)}$ (respectively $N_{(0,b)}$) does not contain the intersection of the other components. Moreover, if $x \in \max(\Sigma \setminus I)$, the only component of the intersection which does not contain x is $\Sigma \setminus B(x)$. Thus this component is not superfluous. \square

The following result agrees with [6, Theorem 11.3]:

Corollary 4.4. *The unique irreducible ideals of Σ are $N_{(a,0)}$, for some $a > 0, N_{(0,b)}$, for some $b > 0$ and those of the form $\Sigma \setminus B(x)$, for some $x \in \Sigma$.*

Corollary 4.5. *If $(0, 0) \neq \sigma = (a, b) \in \Sigma$, then*

$$\sigma + \Sigma = \bigcap_{x \in \max \text{Ap}_\sigma(\Sigma)} (\Sigma \setminus B(x)) \cap N_{(a,0)} \cap N_{(0,b)}$$

is the unique irredundant decomposition of the principal ideal $\sigma + \Sigma$ as intersection of irreducible ideals.

Proof. It follows from the Proposition 4.3, observing that $\text{Ap}_\sigma(\Sigma) = \Sigma \setminus (\sigma + \Sigma)$. \square

What we got for the ideals of the semigroup Σ can be read in terms of monomial ideals of $\mathbb{C}[\Sigma]$. In fact each ideal I of Σ corresponds to the monomial ideal of $\mathbb{C}[\Sigma]$ generated by $\{t^a y^b; (a, b) \in I\}$. Moreover if a monomial ideal of $\mathbb{C}[\Sigma]$ is not the intersection of two strictly larger monomial ideals, then it is not the intersection of two strictly larger ideals, even if non-monomial ideals are allowed [6, Proposition 11, p. 41]. Thus the results above characterize the irreducible monomial ideals of $\mathbb{C}[\Sigma]$ as well.

Corollary 4.6. *Each principal monomial ideal of $\mathbb{C}[\Sigma]$ is an irredundant intersection of $|T(\Sigma)| + 2 = 2\delta + 2$ irreducible ideals (where $\delta = |H(S)|$).*

Proof. It follows from the previous corollary, recalling that, for each $(0, 0) \neq \sigma \in \Sigma, 2\delta = |T(\Sigma)| = |\max \text{Ap}_\sigma(\Sigma)|$, because there is a one to one correspondence between the sets $T(\Sigma)$ and $\max \text{Ap}_\sigma(\Sigma)$, more precisely it is proved in [3, Proposition 4.1] that $\tau \in T(\Sigma)$ if and only if $\tau + \sigma \in \max \text{Ap}_\sigma(\Sigma)$. \square

5. Derivations

Let I be an ideal in $\mathbb{C}[S]$. Then we denote by $\text{Der}(I, I)$ the set of derivations which map I into I .

Lemma 5.1. *If I is generated by monomials in $\mathbb{C}[S]$, then $\text{Der}(I, I) \simeq I : I$ as $\mathbb{C}[S]$ -module. Thus $\text{Der}(I, I)$ is isomorphic to a semigroup ring $\mathbb{C}[T]$, where T is a semigroup, $S \subseteq T \subseteq \mathbb{N}$.*

Proof. If I is generated by monomials also $I : I$, which is a fractional ideal of $\mathbb{C}[S]$, is generated by monomials. Let $\{t^{n_i}\}$ be the generators of $I : I$, then $\text{Der}(I, I)$ is generated by $\{t^{n_i+1}\partial\}$, and $t^k \mapsto t^{k+1}\partial$ induces an isomorphism as $\mathbb{C}[S]$ -modules. Moreover $I : I$ is a semigroup ring $\mathbb{C}[T]$, for some semigroup $T, S \subseteq T \subseteq \mathbb{N}$ and so $\text{Der}(I, I)$ is isomorphic to $\mathbb{C}[T]$ as $\mathbb{C}[S]$ -module. \square

Observe that, if $I : I = \mathbb{C}[T]$, then also $xI : xI = \mathbb{C}[T]$, for each nonzero $x \in \mathbb{C}[S]$. In particular we can say that, for each monomial principal ideal I of $\mathbb{C}[S]$, $\text{Der}(I, I) \simeq \mathbb{C}[S]$ as $\mathbb{C}[S]$ -modules.

If I is not generated by monomials, the statement in the proposition is no longer true. If $I = (t^4+t^5, t^4+t^6)$ in $k[t^4, t^5, t^6]$, then $\text{Der}(I, I)$ is generated by $t^5\partial, t^6\partial, t^7\partial$.

For a monomial ideal I of $\mathbb{C}[S]$, we denote by $\min(I)$ the minimal degree of the monomials in I .

Proposition 5.2. Let T be a semigroup, $S \subseteq T \subseteq \mathbb{N}$ and let C be the conductor ideal $C = \mathbb{C}[S] : \mathbb{C}[T]$. Then:

- (i) If J is a monomial ideal of $\mathbb{C}[S]$ such that $J : J = \mathbb{C}[T]$, then $J \subseteq C$ and $\min(J) \geq \min(C)$.
- (ii) The overring $\mathbb{C}[T]$ of $\mathbb{C}[S]$ is of the form $I : I$, for some monomial ideal I of $\mathbb{C}[S]$.
- (iii) If J is a monomial ideal of $\mathbb{C}[S]$, with $\min(J) = j$, such that $J : J = \mathbb{C}[T]$, then $J \supseteq t^j \mathbb{C}[T]$ and $t^j \in C$.

Proof. (i) If J is a monomial ideal of $\mathbb{C}[S]$ such that $J : J = \mathbb{C}[T]$, then, since J is an ideal of $\mathbb{C}[T]$ too, it is contained in C which is the biggest ideal that $\mathbb{C}[S]$ and $\mathbb{C}[T]$ share. Thus $\min(J) \geq \min(C)$.

(ii) We have that $\mathbb{C}[T] \subseteq \overline{\mathbb{C}[S]} = \mathbb{C}[t] = \mathbb{C}[\mathbb{N}]$, so $\mathbb{C}[T]$ is a fractional ideal of $\mathbb{C}[S]$ and $r\mathbb{C}[T] = I \subseteq \mathbb{C}[S]$, for some nonzero $r \in \mathbb{C}[S]$. Now $I : I = r\mathbb{C}[T] : r\mathbb{C}[T] = \mathbb{C}[T]$ and $\mathbb{C}[T]$ is of the requested form.

(iii) Suppose now that J is a monomial ideal of $\mathbb{C}[S]$, with $\min(J) = j$, such that $J : J = \mathbb{C}[T]$. We claim that $J \supseteq t^j \mathbb{C}[T]$. Indeed the principal ideal $t^j \mathbb{C}[S]$ is contained in J and so $\mathbb{C}[T] = J : J \subseteq J : t^j \mathbb{C}[S] = t^{-j}(J : \mathbb{C}[S]) = t^{-j}J$, hence $t^j \mathbb{C}[T] \subseteq J$. Then also $t^j \mathbb{C}[T] \subseteq \mathbb{C}[S]$ and $t^j \in C$. \square

It is known that the ring $\mathbb{C}[[T]]$ is Gorenstein if and only if the numerical semigroup T is symmetric. The extension of that result to the non-local case is not difficult:

Lemma 5.3. Let T be a numerical semigroup. Then the ring $\mathbb{C}[T]$ is Gorenstein if and only if T is symmetric.

Proof. $\mathbb{C}[T] = \mathbb{C}[t^{n_1}, \dots, t^{n_h}]$ is Gorenstein if and only if each localization at a prime ideal is Gorenstein. For the localization at $P = (t^{n_1}, \dots, t^{n_h})$, we have that $\mathbb{C}[T]_P$ is Gorenstein if and only if T is symmetric, arguing similarly to the local case $\mathbb{C}[[T]]$. For the other nonzero prime ideals Q , we have that Q does not contain the conductor $\mathbb{C}[T] : \mathbb{C}[t]$, so $\mathbb{C}[T]_Q \cong \mathbb{C}[t]_{Q'}$ is a DVR, thus a Gorenstein ring, where Q' is the unique prime ideal of $\mathbb{C}[t]$ lying over Q . \square

Corollary 5.4. Let $\mathbb{C}[T]$ be a Gorenstein overring of $\mathbb{C}[S]$. If J is a monomial ideal of $\mathbb{C}[S]$, then $J : J = \mathbb{C}[T]$ if and only if $J = t^j \mathbb{C}[T]$, where $j = \min(J)$.

Proof. We know by Proposition 5.2(iii) that $J \supseteq t^j \mathbb{C}[T]$. In order to prove the opposite inclusion, we show that, if H is a monomial ideal of $\mathbb{C}[S]$ with $\min(H) = j$ properly larger than $t^j \mathbb{C}[T]$, then $H : H \neq \mathbb{C}[T]$. Let g be the Frobenius number of the semigroup T , which is symmetric because $\mathbb{C}[T]$ is Gorenstein (cf. Lemma 5.3). We can argue equivalently on the fractional ideals of $\mathbb{C}[S]$. So, let H be a fractional monomial ideal of $\mathbb{C}[S]$ with $\min(H) = 0$ and strictly larger than $\mathbb{C}[T]$ and let $t^h \in H \setminus \mathbb{C}[T]$. Then $t^{g-h} \in \mathbb{C}[T]$. Since $t^{g-h} t^h = t^g \notin \mathbb{C}[T]$, we have $t^{g-h} \notin H : H$ and so $H : H \neq \mathbb{C}[T]$. \square

If I, J are ideals of $\mathbb{C}[S]$, we say that I and J are *equivalent* ($I \sim J$) if $xI = yJ$, for some nonzero elements $x, y \in \mathbb{C}[S]$. Recall also that an ideal I is called *stable* if it is principal in the overring $I : I$. In a similar way a semigroup ideal I of S is called *stable* if it is principal in the over-semigroup $I - I = \{z \in Z \mid z + I \subseteq I\}$.

Example. If $S = \langle 3, 4, 5 \rangle$, the only semigroups T for which $S \subseteq T \subseteq \mathbb{N}$ are $S, T_1 = \langle 2, 3 \rangle$, and \mathbb{N} , so that the only proper semigroup overrings of $\mathbb{C}[S]$ are $\mathbb{C}[T_1]$ and $\mathbb{C}[\mathbb{N}] = \mathbb{C}[t]$, which are both Gorenstein rings. The conductor ideals coincide, $\mathbb{C}[S] : \mathbb{C}[T_1] = \mathbb{C}[S] : \mathbb{C}[\mathbb{N}] = t^3 \mathbb{C}[t]$. By Corollary 5.4 and Proposition 5.2(iii), if J is a monomial ideal of $\mathbb{C}[S]$, then $J : J = \mathbb{C}[T_1]$ if and only if $J = t^j \mathbb{C}[T_1]$ with $j \geq 3$, i.e. $J \sim (t^3, t^5)$ and $J : J = \mathbb{C}[t]$ if and only if $J = t^j \mathbb{C}[t]$ with $j \geq 3$, i.e. $J \sim (t^3, t^4, t^5)$. On the other hand, the trivial overring $\mathbb{C}[S]$ is not Gorenstein and $\mathbb{C}[S] : \mathbb{C}[S] = \mathbb{C}[S]$. We have that $J : J = \mathbb{C}[S]$ if and only if J is a principal ideal of $\mathbb{C}[S]$ or $J \sim (t^3, t^4)$, so here two equivalence classes of monomial ideals correspond to the same overring.

Proposition 5.5. The following conditions are equivalent:

- (1) There exists a one-to-one correspondence between the semigroup overrings of $\mathbb{C}[S]$ and the equivalence classes of monomial ideals of $\mathbb{C}[S]$.
- (2) $\mathbb{C}[S] = \mathbb{C}[t^2, t^{2k+1}]$, for some $k \in \mathbb{N}$.
- (3) Each semigroup overring of $\mathbb{C}[S]$ is Gorenstein.

Proof. (1) \Leftrightarrow (2). By Proposition 5.2, (ii) and (iii), there exists a one-to-one correspondence between the semigroup overrings of $\mathbb{C}[S]$ and the classes of stable monomial ideals of $\mathbb{C}[S]$ (cf. also [2, Proposition II.4.3]). So we get the requested one-to-one correspondence if and only if each monomial ideal of $\mathbb{C}[S]$ is stable, i.e. if and only if each semigroup ideal of S is stable. By [2, Theorems I.5.13, (i) \Leftrightarrow (iii) and I.4.2 (i) \Leftrightarrow (v)], this is equivalent to $S = \langle 2, 2k+1 \rangle$, for some $k \in \mathbb{N}$. (2) \Leftrightarrow (3). By Lemma 5.3, condition (3) means that each semigroup $T, S \subseteq T \subseteq \mathbb{N}$ is symmetric and that holds if and only if $S = \langle 2, 2k+1 \rangle$, for some $k \in \mathbb{N}$ (cf. [2, Theorem I.4.2(v) \Leftrightarrow (ix)]). \square

Example. If $S = \langle 2, 5 \rangle$ there are three equivalence classes of ideals generated by monomials with representatives $\mathbb{C}[S]$, (t^2, t^5) and (t^4, t^5) . These correspond to $\mathbb{C}[S]$, $\mathbb{C}[t^2, t^3]$ and $\mathbb{C}[t]$, respectively. More generally, if $S = \langle 2, 2k+1 \rangle$, there are $k+1$ equivalence classes of ideals generated by monomials with representatives $\mathbb{C}[S]$, (t^2, t^{2k+1}) , (t^4, t^{2k+1}) , \dots , (t^{2k}, t^{2k+1}) . These correspond to $\mathbb{C}[S]$, $\mathbb{C}[t^2, t^{2k-1}]$, $\mathbb{C}[t^2, t^{2k-3}]$, \dots , $\mathbb{C}[t^2, t] = \mathbb{C}[t]$, respectively.

Acknowledgment

We want to thank the anonymous referee for a careful reading.

References

- [1] V. Barucci, M. D'Anna, R. Fröberg, ARF characters of an algebroid curve, *J. Algebra Number Theory Appl.* 3 (2) (2003) 219–243.
- [2] V. Barucci, D. Dobbs, M. Fontana, Maximality properties in numerical semigroups and applications to one-dimensional analytically irreducible local domains, *Memoirs of the American Mathematical Society* 125 (598) (1997).
- [3] E. Emtander, On positive affine monoids, preprint.
- [4] E. Eriksen, Differential operators on monomial curves, *J. Algebra* 264 (2003) 186–198.
- [5] A. Eriksson, Differential operators on some classes of rings, Ph.D. thesis, Stockholm university, 2000.
- [6] E. Miller, B. Sturmfels, *Combinatorial Commutative Algebra*, Springer, 2005.
- [7] P.T. Perkins, Commutative subalgebras of the ring of differential operators on a curve, *Pacific J. Math.* 139 (2) (1989) 279–302.
- [8] J.C. Rosales, P.A. García-Sánchez, *Numerical Semigroups*, Springer, 2009.