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Divisors on rational normal scrolls

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ABSTRACT

Let A be the homogeneous coordinate ring of a rational normal scroll. The ring A is equal to the quotient of a polynomial ring S by the ideal generated by the two by two minors of a scroll matrix ψ with two rows and ℓ catalecticant blocks. The class group of A is cyclic, and is infinite provided ℓ is at least two. One generator of the class group is $[J]$, where J is the ideal of A generated by the entries of the first column of ψ. The positive powers of J are well-understood; if ℓ is at least two, then the nth ordinary power, the nth symmetric power, and the nth symbolic power coincide and therefore all three nth powers are resolved by a generalized Eagon–Northcott complex. The inverse of $[J]$ in the class group of A is $[K]$, where K is the ideal generated by the entries of the first row of ψ. We study the positive powers of $[K]$. We obtain a minimal generating set and a Gröbner basis for the preimage in S of the symbolic power $K^{(n)}$. We describe a filtration of $K^{(n)}$ in which all of the factors are Cohen–Macaulay S-modules resolved by generalized Eagon–Northcott complexes. We use this filtration to describe the modules in a finely graded resolution of $K^{(n)}$ by free S-modules. We calculate the regularity of the graded S-module $K^{(n)}$ and we show that the symbolic Rees ring of K is Noetherian.

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0. Introduction

Fix a field k and positive integers ℓ and $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_\ell \geqslant 1$. The rational normal scroll $\operatorname{Scroll}(\sigma_1, \ldots, \sigma_\ell)$ is the image of the map

$$\boldsymbol{\Sigma} : (\mathbb{A}^2 \setminus \{0\}) \times (\mathbb{A}^\ell \setminus \{0\}) \to \mathbb{P}^N,$$

where $N = \ell - 1 + \sum_{i=1}^{\ell} \sigma_i$ and

$$\boldsymbol{\Sigma}(x, y; t_1, \ldots, t_\ell) = (x^{\sigma_1} t_1, x^{\sigma_1 - 1} y t_1, \ldots, y^{\sigma_1} t_1, x^{\sigma_2} t_2, x^{\sigma_2 - 1} y t_2, \ldots, y^{\sigma_\ell} t_\ell).$$

From this one sees that the homogeneous coordinate ring of $\operatorname{Scroll}(\sigma_1, \ldots, \sigma_\ell) \subseteq \mathbb{P}^N$ is the subalgebra

$$A = k[x^{\sigma_1} t_1, x^{\sigma_1 - 1} y t_1, \ldots, y^{\sigma_1} t_1, x^{\sigma_2} t_2, x^{\sigma_2 - 1} y t_2, \ldots, y^{\sigma_\ell} t_\ell]$$

of the polynomial ring $k[x, y, t_1, \ldots, t_\ell]$. This algebra has a presentation $A = S/I_2(\psi)$, where S is the polynomial ring

$$S = k[\{T_{i,j} \mid 1 \leqslant i \leqslant \ell \text{ and } 1 \leqslant j \leqslant \sigma_i + 1\}],$$

ψ is the matrix $\psi = [\psi_1 \mid \cdots \mid \psi_\ell]$, and for each u, ψ_u is the generic catalecticant matrix

$$\psi_u = \begin{bmatrix} T_{u,1} & T_{u,2} & \ldots & T_{u,\sigma_u - 1} & T_{u,\sigma_u} \\ T_{u,2} & T_{u,3} & \ldots & T_{u,\sigma_u} & T_{u,\sigma_u + 1} \end{bmatrix}.$$

Further information about rational normal scrolls, with alternative descriptions and many applications, may be found in [7,9,15,6].

The class group of the normal domain A is cyclic, and is infinite provided $\ell \geqslant 2$. One generator of $\mathcal{C}\ell(A)$ is $[J]$, where J is the ideal of A generated by the entries of the first column of ψ. The positive powers of J are well-understood, in the sense that the nth ordinary power J^n and the nth symmetric power $\operatorname{Sym}_n(J)$ are equal, and they coincide with the nth symbolic power $J^{(n)}$ in case $\ell \geqslant 2$. Therefore all three nth powers are resolved by a generalized Eagon–Northcott complex, at least when $\ell \geqslant 2$. The inverse of $[J]$ in the class group of A is $[K]$, where K is the ideal generated by the entries of the first row of ψ. The positive powers of $[K]$ are less well understood. The purpose of the present paper is to rectify this. In Section 1 we obtain a minimal generating set for $K^{(n)}$; the graded components of this ideal can also be read from [18, 1.3]. In Section 2 we exhibit a Gröbner basis for the preimage of $K^{(n)}$ in S. The Gröbner basis is obtained from a minimal generating set of $I_2(\psi)$ in S and a monomial minimal generating set of $K^{(n)}$ in A. In Section 3, we describe a filtration of $K^{(n)}$ in which all of the factors are Cohen–Macaulay S-modules resolved by generalized Eagon–Northcott complexes. We use this filtration in Section 4 to describe the modules in a finely graded resolution of $K^{(n)}$ by free S-modules. More generally, though less explicitly, resolutions of homogeneous coordinate rings of subvarieties of rational normal scrolls have been approached in [18, 3.2 and 3.5] in terms of resolutions by locally free sheaves having a filtration by generalized "Eagon–Northcott sheaves". We calculate the regularity of the graded S-module $K^{(n)}$ in Section 5. The interest in this topic is reflected by the existence of papers like [14] and Hoa's conjecture [13]. As it turns out, the regularity of $K^{(n)}$ is $\lceil \frac{n-1}{\sigma_\ell} \rceil + 1$ and, in particular, only depends on the size of the smallest block in the matrix of the scroll. In Section 6 we show that the symbolic Rees ring of K is Noetherian. (Of course, some symbolic Rees rings are not Noetherian [16,17,8] and positive results have been obtained in [10,4,11] for instance, but in general the question of when a symbolic Rees ring is Noetherian remains wide open.)

Our computation of the regularity in Section 5 uses a second filtration of $K^{(n)}$ that is coarser than the filtration of Section 3. The factors of the coarser filtration are still Cohen–Macaulay modules resolved by generalized Eagon–Northcott complexes. These resolutions give rise to a resolution of $K^{(n)}$

that is sufficiently close to a minimal resolution to allow for a computation of the regularity. On the other hand, if minimality of resolutions is not an issue, like in the calculation of Hilbert series (see, for example [12]), then it is advantageous to use the finer filtration of Section 3 as it is easier to describe.

Let I be a homogeneous ideal of height two in $k[x,y]$. Suppose that the presenting matrix of I is almost linear in the sense that the entries of one column have degree n and all of the other entries are linear. In [12] we prove that the Rees ring and the special fiber ring of I both have the form $A/\mathcal{A}$, where A is the coordinate ring of a rational normal scroll and the ideals $\mathcal{A}$ and $K^{(n)}$ of A are isomorphic. We use the results of the present paper to identify explicit generators for $\mathcal{A}$, to resolve the powers I^s of I, to compute the regularity of I^s, and to calculate the reduction number of I.

1. The generators of $K^{(n)}$

Data 1.1. We are given integers $\sigma_1 \geqslant \cdots \geqslant \sigma_\ell \geqslant 1$ and an integer $n \geqslant 2$. Let S be the polynomial ring

$$S = k\big[\{T_{i,j} \mid 1 \leqslant i \leqslant \ell \text{ and } 1 \leqslant j \leqslant \sigma_i + 1\}\big].$$

For each u, with $1 \leqslant u \leqslant \ell$, let ψ_u be the generic catalecticant matrix

$$\psi_u = \begin{bmatrix} T_{u,1} & T_{u,2} & \dots & T_{u,\sigma_u-1} & T_{u,\sigma_u} \\ T_{u,2} & T_{u,3} & \dots & T_{u,\sigma_u} & T_{u,\sigma_u+1} \end{bmatrix}. \tag{1.2}$$

Define ψ to be the matrix

$$\psi = [\, \psi_1 \quad | \quad \dots \quad | \quad \psi_\ell \,]. \tag{1.3}$$

Let H be the ideal $I_2(\psi)$ of S and A the ring S/H. We will write $T_{i,j}$ for a variable in S and also for its image in A – the meaning will be clear from context. Recall that A is a Cohen–Macaulay ring of Krull dimension $\ell+1$ with isolated singularity. In particular, it is a normal domain. Let K be the ideal in A generated by the entries of the top row of ψ. Notice that K is a height one prime ideal of A.

In Theorem 1.5 we identify a generating set for $K^{(n)}$ and in Proposition 1.20 we identify a minimal generating set for $K^{(n)}$.

Ultimately, we will put three gradings on the rings S and A. The first grading on S is defined by setting

$$\mathrm{Deg}(T_{i,j}) = \sigma_i + 1 - j. \tag{1.4}$$

Notice that H is a homogeneous ideal with respect to this grading and thus Deg induces a grading on A, which we also denote by Deg. Let $A_{\geqslant n}$ be the ideal of A generated by all monomials M with $\mathrm{Deg}(M) \geqslant n$.

Theorem 1.5. *The nth symbolic power, $K^{(n)}$, of K is equal to $A_{\geqslant n}$.*

Proof. Calculate in A. First observe that

$$T_{i,\sigma_i+1}^{\sigma_i - j} T_{i,j} = T_{i,\sigma_i}^{\sigma_i+1-j} \in K^{\sigma_i+1-j}, \tag{1.6}$$

for all i, j with $1 \leqslant i \leqslant \ell$ and $1 \leqslant j \leqslant \sigma_i$. Indeed, the statement is obvious when $j = \sigma_i$. If $j = \sigma_i - 1$, then the assertion holds because

$$0 = \det \begin{bmatrix} T_{i,\sigma_i-1} & T_{i,\sigma_i} \\ T_{i,\sigma_i} & T_{i,\sigma_i+1} \end{bmatrix}.$$

The proof of (1.6) is completed by decreasing induction on j. Since T_{i,σ_i+1} is not in the prime ideal K, from (1.6) we obtain $T_{i,j} \in K^{(\sigma_i+1-j)} = K^{(\operatorname{Deg} T_{i,j})}$. Thus,

$$K^n \subseteq A_{\geqslant n} \subseteq K^{(n)}.$$

Observe that $\operatorname{Deg}(T_{i,\sigma_i+1}) = 0$ for $1 \leqslant i \leqslant \ell$; hence, T_{i,σ_i+1} is a non-zerodivisor on $A/A_{\geqslant n}$, because A is a domain and T_{i,σ_i+1} is not zero in A. On the other hand, the localization $A[T_{i,\sigma_i+1}^{-1}]$ is a regular ring; and hence, in this ring, $K^{(n)}$ coincides with K^n, thus with $A_{\geqslant n}$. Since, T_{i,σ_i+1} is regular modulo $A_{\geqslant n}$, we conclude that $A_{\geqslant n}$ is equal to $K^{(n)}$. $\square$

Observation 1.7. *Let $R = k[x, y]$ be a polynomial ring with homogeneous maximal ideal $\mathfrak{m}$ and write $B = k[x, y, t_1, \ldots, t_\ell]$. Define the homomorphism of k-algebras $\pi : S \to B$ with*

$$\pi(T_{i,j}) = x^{\sigma_i-j+1} y^{j-1} t_i.$$

(a) *The image of π is the k-subalgebra $k[R_{\sigma_1}t_1, \ldots, R_{\sigma_\ell}t_\ell]$ of B. In particular, the image of π is the special fiber ring of the R-module $\mathfrak{m}^{\sigma_1} \oplus \cdots \oplus \mathfrak{m}^{\sigma_\ell}$.*
(b) *The homomorphism $\pi : S \to B$ induces an isomorphism $A \cong \pi(S)$, which we use to identify A with the subring $\pi(S)$ of B.*
(c) *We have $K \subseteq Bx \cap A = A_{\geqslant 1}$.*
(d) *A monomial $x^\alpha y^\beta \prod_{u=1}^{\ell} t_u^{c_u}$ of B belongs to A if and only if*

$$\alpha + \beta = \sum_{u=1}^{\ell} c_u \sigma_u. \tag{1.8}$$

(e) *The ring A is a direct summand of B as an A-module.*

Proof. The first assertion in part (a) is obvious; the statement about the special fiber ring can be shown by considering the Rees ring $R[\mathfrak{m}^{\sigma_1}t_1, \ldots, \mathfrak{m}^{\sigma_\ell}t_\ell]$ and giving the variables t_i degree $-\sigma_i$. Since the quotient field of $\pi(S)$ is

$$k\left(\frac{y}{x}, x^{\sigma_1}t_1, \ldots, x^{\sigma_\ell}t_\ell\right),$$

adjoining x one sees that this field has transcendence degree $\ell + 1$ over k. Thus, the Krull dimension of $\pi(S)$ is $\ell + 1$, which is also the dimension of A. Hence, the prime ideals $H \subseteq \ker \pi$ have the same height and $\pi(S) \cong A$. This is (b). On B, we define a grading by giving x degree 1 and the other variables degree 0. The map π is homogeneous with respect to this grading on B and the grading Deg on S. Hence, the grading on B induces a grading on the subalgebra $\pi(S) = A$ that coincides with Deg as defined in (1.4). Notice that $K \subseteq Bx \cap A = A_{\geqslant 1}$, which is (c). (This provides an alternative proof that $K^{(n)} \subseteq Bx^n \cap A = A_{\geqslant n}$.) Assertion (d) is obvious and (e) follows because a complementary summand is the A-module generated by all monomials of B that do not satisfy (1.8). $\square$

Now we move in the direction of identifying a minimal generating set for $K^{(n)}$. In this discussion, we also use the standard grading, where each variable has degree one, we will refer to it as the total degree.

Observation 1.9. *If f is a monomial of S with $\operatorname{Deg}(f) > 0$, then there exists a monomial of the form*

$$M = T_{1,1}^{a_1} \cdots T_{k,1}^{a_k} T_{k,v} T_{k,\sigma_k+1}^{b_k} \cdots T_{\ell,\sigma_\ell+1}^{b_\ell} \tag{1.10}$$

in S with $1 \leqslant v \leqslant \sigma_k$, $\operatorname{Deg} f = \operatorname{Deg} M$, and $f - M \in H$.

Proof. We will use this calculation later in the context of Gröbner bases; so, we make our argument very precise. Order the variables of S with

$$T_{1,1} > T_{1,2} > \cdots > T_{1,\sigma_1+1} > T_{2,1} > \cdots > T_{2,\sigma_2+1} > T_{3,1} > \cdots > T_{\ell,\sigma_\ell+1}. \tag{1.11}$$

Observe that there exists $\alpha \leqslant \beta$ such that $f = f_1 f' f_2$ where $f_1 = T_{1,1}^{a_1} \cdots T_{\alpha,1}^{a_\alpha}$, $f_2 = T_{\beta,\sigma_\beta+1}^{b_\beta} \cdots T_{\ell,\sigma_\ell+1}^{b_\ell}$ and

$$T_{i,j} \mid f' \quad \Longrightarrow \quad T_{\alpha,1} > T_{i,j} > T_{\beta,\sigma_\beta+1}.$$

Let $T_{i,j}$ be the largest variable which divides f' and $T_{u,v}$ the smallest variable which divides f'. We may shrink f', if necessary, and insist that $1 < j$ and $v < \sigma_u + 1$. If f' has total degree at most one, then one easily may write f in the form of M. We assume that f' has total degree at least two. Take f'' with $f' = T_{i,j} f'' T_{u,v}$. Notice that

$$h = -\det\begin{bmatrix} T_{i,j-1} & T_{u,v} \\ T_{i,j} & T_{u,v+1} \end{bmatrix} = T_{i,j}T_{u,v} - T_{i,j-1}T_{u,v+1} \tag{1.12}$$

is in H and

$$f - f_1 h f'' f_2 = f_1 T_{i,j-1} f'' T_{u,v+1} f_2 \tag{1.13}$$

is more like the desired M than f is. Replacing f by the element of (1.13) does not change Deg because the element h of (1.12) is homogeneous with respect to this grading. Proceed in this manner until M is obtained. □

We use the notion of eligible tuples when we identify a minimal generating set for $K^{(n)}$ in Proposition 1.20. We also use this notion in Section 3 when we describe a filtration of $K^{(n)}$ whose factors are Cohen–Macaulay modules.

Definition 1.14.

(a) We say that $\boldsymbol{a}$ is an *eligible* k-tuple if $\boldsymbol{a}$ is a k-tuple, $(a_1, \ldots, a_k)$, of non-negative integers with $0 \leqslant k \leqslant \ell - 1$ and $\sum_{u=1}^{k} a_u\sigma_u < n$.

(2) Let $\boldsymbol{a}$ be an eligible k-tuple. The non-negative integer $f(\boldsymbol{a})$ is defined by

$$\sum_{u=1}^{k} a_u\sigma_u + f(\boldsymbol{a})\sigma_{k+1} < n \leqslant \sum_{u=1}^{k} a_u\sigma_u + \big(f(\boldsymbol{a}) + 1\big)\sigma_{k+1};$$

and the positive integer $r(\boldsymbol{a})$ is defined to be

$$r(\boldsymbol{a}) = \sum_{u=1}^{k} a_u\sigma_u + \big(f(\boldsymbol{a}) + 1\big)\sigma_{k+1} - n + 1.$$

Be sure to notice that

$$1 \leqslant r(\boldsymbol{a}) \leqslant \sigma_{k+1}. \tag{1.15}$$

(3) We write $T^{\boldsymbol{a}}$ to mean $\prod_{u=1}^{k} T_{u,1}^{a_u}$ for each eligible k-tuple $\boldsymbol{a} = (a_1, \ldots, a_k)$.

Remark 1.16. The empty tuple, $\emptyset$, is always eligible, and we have

$$f(\emptyset)=\left\lceil \frac{n}{\sigma_1}\right\rceil-1, \qquad r(\emptyset)=\sigma_1\left\lceil \frac{n}{\sigma_1}\right\rceil-n+1, \quad \text{and} \quad T^{\emptyset}=1.$$

Notation. If θ is a real number, then $\lceil\theta\rceil$ and $\lfloor\theta\rfloor$ are the "round up" and "round down" of θ, respectively; that is, $\lceil\theta\rceil$ and $\lfloor\theta\rfloor$ are the integers with

$$\lceil\theta\rceil-1<\theta\leqslant\lceil\theta\rceil \quad \text{and} \quad \lfloor\theta\rfloor\leqslant\theta<\lfloor\theta\rfloor+1.$$

Definition 1.17. Let $\mathcal{L}$ be the following list of elements of S,

$$\mathcal{L}=\bigcup_{k=0}^{\ell-1}\bigl\{T^{\boldsymbol{a}}T_{k+1,1}^{f(\boldsymbol{a})}T_{k+1,u}\bigm| \boldsymbol{a}\text{ is an eligible }k\text{-tuple and }1\leqslant u\leqslant r(\boldsymbol{a})\bigr\}.$$

Observation 1.18. *Let M be the monomial $T_{1,1}^{a_1}\cdots T_{k,1}^{a_k}T_{k,v}T_{k,\sigma_k+1}^{b_k}\cdots T_{\ell,\sigma_\ell+1}^{b_\ell}$ of S. If $\operatorname{Deg}M\geqslant n$, then M is divisible by an element of $\mathcal{L}$.*

Proof. We have

$$n\leqslant\operatorname{Deg}(M)=\sum_{u=1}^{k}a_u\sigma_u+\sigma_k+1-v.$$

If $\sum_{u=1}^{k}a_u\sigma_u<n$, then let $\boldsymbol{a}$ be the eligible $(k-1)$-tuple $(a_1,\dots,a_{k-1})$. In this case, $f(\boldsymbol{a})=a_k$, $1\leqslant v\leqslant r(\boldsymbol{a})$, and M is divisible by $T^{\boldsymbol{a}}T_{k,1}^{f(\boldsymbol{a})}T_{k,v}\in\mathcal{L}$. If $n\leqslant\sum_{u=1}^{k}a_u\sigma_u$, then identify the least index j with $n\leqslant\sum_{u=1}^{j}a_u\sigma_u$ and let $\boldsymbol{a}$ be the eligible $(j-1)$-tuple $(a_1,\dots,a_{j-1})$. In this case, $f(\boldsymbol{a})<a_j$ and M is divisible by $T^{\boldsymbol{a}}T_{j,1}^{f(\boldsymbol{a})}T_{j,1}\in\mathcal{L}$. $\square$

Observation 1.19. *The ideals $K^{(n)}$ and $\mathcal{L}A$ are equal.*

Proof. Recall that $K^{(n)}=A_{\geqslant n}$ according to Theorem 1.5. The elements of $\mathcal{L}$ have $\operatorname{Deg}\geqslant n$, which gives $\mathcal{L}A\subseteq A_{\geqslant n}=K^{(n)}$. To prove the other inclusion, let f be a monomial in S with $\operatorname{Deg}(f)\geqslant n$. By Observation 1.9 there exists a monomial M with $f-M\in H$ and $\operatorname{Deg}M=\operatorname{Deg}f\geqslant n$. Now Observation 1.18 shows that M is divisible by an element of $\mathcal{L}$. $\square$

Proposition 1.20. *The elements of $\mathcal{L}$ form a minimal generating set for the ideal $K^{(n)}$.*

Proof. From Observation 1.19 we know that $\mathcal{L}$ is a generating set for $K^{(n)}$. To show it is a minimal generating set, we use the map $\pi:S\to B$ of Observation 1.7 that identifies A with the monomial subring $k[\{x^{\sigma_i-j+1}y^{j-1}t_i\}]$ of $B=k[x,y,t_1,\dots,t_\ell]$. The elements of $\pi(\mathcal{L})$ are monomials in the polynomial ring B, and it suffices to show that if $h\in\pi(\mathcal{L})$ divides $g\in\pi(\mathcal{L})$ in B, then $h=g$ in B.

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be eligible k and j tuples, respectively, and let

$$g=\pi\bigl(T^{\boldsymbol{a}}T_{k+1,1}^{f(\boldsymbol{a})}T_{k+1,v}\bigr)=x^Gy^{v-1}t_1^{a_1}\cdots t_k^{a_k}t_{k+1}^{f(\boldsymbol{a})+1} \quad \text{and}$$
$$h=\pi\bigl(T^{\boldsymbol{b}}T_{j+1,1}^{f(\boldsymbol{b})}T_{j+1,w}\bigr)=x^Hy^{w-1}t_1^{b_1}\cdots t_j^{b_j}t_{j+1}^{f(\boldsymbol{b})+1},$$

for some v and w with $1\leqslant v\leqslant r(\boldsymbol{a})$ and $1\leqslant w\leqslant r(\boldsymbol{b})$, where

$$G = \sum_{u=1}^{k} a_u\sigma_u + \big(f(\boldsymbol{a})+1\big)\sigma_{k+1} - v + 1 \quad \text{and}$$

$$H = \sum_{u=1}^{j} b_u\sigma_u + \big(f(\boldsymbol{b})+1\big)\sigma_{j+1} - w + 1.$$

The hypothesis that h divides g ensures that $j \leqslant k$ and $b_u \leqslant a_u$ for $1 \leqslant u \leqslant j$. If $j < k$, then $f(\boldsymbol{b})+1 \leqslant a_{j+1}$ and

$$n \leqslant \sum_{u=1}^{j} b_u\sigma_u + \big(f(\boldsymbol{b})+1\big)\sigma_{j+1} \leqslant \sum_{u=1}^{j+1} a_u\sigma_u \leqslant \sum_{u=1}^{k} a_u\sigma_u < n.$$

This contradiction guarantees that $j = k$. Again, the hypothesis ensures that $f(\boldsymbol{b}) \leqslant f(\boldsymbol{a})$, and $b_i \leqslant a_i$, for all i. If $b_i < a_i$, for some i, then

$$n \leqslant \sum_{u=1}^{k} b_u\sigma_u + \big(f(\boldsymbol{b})+1\big)\sigma_{k+1} \leqslant \sum_{u=1}^{k} a_u\sigma_u + f(\boldsymbol{b})\sigma_{k+1} \leqslant \sum_{u=1}^{k} a_u\sigma_u + f(\boldsymbol{a})\sigma_{k+1} < n,$$

since

$$\sigma_{k+1} \leqslant \sigma_i. \tag{1.21}$$

This contradiction guarantees that $\boldsymbol{b} = \boldsymbol{a}$. Again, since h divides g, we also have $w \leqslant v$ and $H \leqslant G$. As $\boldsymbol{b} = \boldsymbol{a}$, the definition of H and G forces $w = v$. Thus, indeed, $h = g$. □

Remark 1.22. The proof of Proposition 1.20 uses the hypothesis

$$\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_\ell \tag{1.23}$$

in an essential way at (1.21). Furthermore, Proposition 1.20 is false if one removes hypothesis (1.23). Indeed, the recipe of Definition 1.17 would produce a set of three elements if it were applied to $\sigma_1 = 1$, $\sigma_2 = n = 2$; however, the minimal number of generators of $K^{(n)}$ in this case is two.

Inspired by Observation 1.7 and the proof of Proposition 1.20, we introduce the "fine grading" on S. Let

$$\varepsilon_u = (0, \dots, 0, 1, 0, \dots, 0) \tag{1.24}$$

be the ℓ-tuple with 1 in position u and 0 in all other positions. The variable $T_{i,j}$ has "fine degree" given by

$$\operatorname{fdeg}(T_{i,j}) = (\sigma_i - j + 1, j - 1; \varepsilon_i). \tag{1.25}$$

The variables of S have distinct fine degrees. Notice that H is homogeneous with respect to fine degree and therefore fdeg induces a grading on A. Observe that the grading fdeg on A is simply the grading induced on A by the embedding $A \hookrightarrow B = k[x, y, t_1, \dots, t_\ell]$ of Observation 1.7, where the polynomial ring B is given the usual multigrading.

The two previous gradings that we have considered (Deg and total degree) can be read from fdeg. Let $\boldsymbol{\sigma}$ represent the ℓ-tuple $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_\ell)$. If M is the monomial

$$M = \prod_{i=1}^{\ell} \prod_{j=1}^{\sigma_i+1} T_{i,j}^{a_{i,j}}$$

of S, then

$$\operatorname{fdeg} M = (\operatorname{Deg} M, \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} - \operatorname{Deg} M; \boldsymbol{\varepsilon}),$$

where $\boldsymbol{\varepsilon}$ is the ℓ-tuple $\boldsymbol{\varepsilon} = (e_1, \dots, e_\ell)$, with $e_i = \sum_{j=1}^{\sigma_i+1} a_{i,j}$, and $\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}$ is the dot product. The total degree of M is $e_1 + \ldots + e_\ell = \mathbf{1} \cdot \boldsymbol{\varepsilon}$, where $\mathbf{1} = (1, \dots, 1)$ is an ℓ-tuple of ones. We return to the notion of fine degree in (3.3).

2. Gröbner basis

In Theorem 2.4, we identify a Gröbner basis for the preimage of $K^{(n)}$ in S; and as an application, in Corollary 2.6, we show that $\operatorname{depth} A/K^{(n)} = 1$. Sometimes it is convenient to label the variables using a single subscript. That is, we write T_j for $T_{1,j}$; T_{σ_1+1+j} for $T_{2,j}$; $T_{\sigma_1+\sigma_2+2+j}$ for $T_{3,j}$, etc. In this notation, the matrix ψ of (1.3) is

$$\psi = \begin{bmatrix} T_1 & \cdots & T_{\sigma_1} & T_{\sigma_1+2} & \cdots & T_{\sigma_1+\sigma_2+1} & T_{\sigma_1+\sigma_2+3} & \cdots \\ T_2 & \cdots & T_{\sigma_1+1} & T_{\sigma_1+3} & \cdots & T_{\sigma_1+\sigma_2+2} & T_{\sigma_1+\sigma_2+4} & \cdots \end{bmatrix}. \tag{2.1}$$

Order the variables of S with $T_1 > T_2 > \cdots$, as was done in (1.11). Impose the reverse lexicographic order on the monomials of S. In other words, for two monomials

$$M_1 = T_1^{\alpha_1} \cdots T_N^{\alpha_N} \quad \text{and} \quad M_2 = T_1^{\beta_1} \cdots T_N^{\beta_N}$$

one has $M_1 > M_2$ if and only if either $\sum \alpha_i > \sum \beta_i$, or else $\sum \alpha_i = \sum \beta_i$ and the right most non-zero entry of $(\alpha_1 - \beta_1, \dots, \alpha_N - \beta_N)$ is negative. When we study a homogeneous polynomial from S we underline its leading term. The next result is well-known, see [2, Theorem 4.11]. We give a proof for the sake of completeness. This proof provides good practice in using the Buchberger criterion for determining when a generating set G of an ideal is a Gröbner basis for the ideal. It entails showing that the S-polynomial of any two elements of G reduces to zero modulo G; see, for example, [3, Section 2.9, Theorem 3].

Lemma 2.2. *The set G of 2×2 minors of ψ forms a Gröbner basis for $I_2(\psi)$.*

Proof. Select four columns from ψ,

$$\psi' = \begin{bmatrix} T_a & T_b & T_c & T_d \\ T_{a+1} & T_{b+1} & T_{c+1} & T_{d+1} \end{bmatrix},$$

with $a \leqslant b \leqslant c \leqslant d$. For $i < j$, let

$$\Delta_{i,j} = -\det \begin{bmatrix} T_i & T_j \\ T_{i+1} & T_{j+1} \end{bmatrix} = \underline{T_{i+1} T_j} - T_i T_{j+1}.$$

We first assume that $a < b < c < d$. For most partitions of $\{a, b, c, d\}$ into $p < q$ and $r < s$, the leading terms of $\Delta_{p,q}$ and $\Delta_{r,s}$ are relatively prime; and therefore, the S-polynomial $S(\Delta_{p,q}, \Delta_{r,s})$

reduces to zero modulo G (see, for example, [3, Section 2.9, Proposition 4]). The only interesting S-polynomial is $S(\Delta_{a,c}, \Delta_{b,d})$ when $c = b + 1$. In this case, the greatest common divisor of the leading terms of

$$\Delta_{a,c} = \underline{T_{a+1}T_c} - T_aT_{c+1} \quad \text{and} \quad \Delta_{b,d} = \underline{T_{b+1}T_d} - T_bT_{d+1}$$

is $T_c = T_{b+1}$; thus

$$S(\Delta_{a,c}, \Delta_{b,d}) = T_d\Delta_{a,c} - T_{a+1}\Delta_{b,d} = \underline{-T_aT_{c+1}T_d} + T_{a+1}T_bT_{d+1}.$$

We know the generalized Eagon–Northcott complex associated to ψ'; and therefore, we know that the product

$$P = \psi' \begin{bmatrix} 0 & -\Delta_{c,d} & \Delta_{b,d} & -\Delta_{b,c} \\ \Delta_{c,d} & 0 & -\Delta_{a,d} & \Delta_{a,c} \\ -\Delta_{b,d} & \Delta_{a,d} & 0 & -\Delta_{a,b} \\ \Delta_{b,c} & -\Delta_{a,c} & \Delta_{a,b} & 0 \end{bmatrix}$$

is identically zero. It follows that

$$0 = -P_{1,2} - P_{2,3} = \begin{cases} T_a\Delta_{c,d} - T_c\Delta_{a,d} + T_d\Delta_{a,c}, \\ -T_{a+1}\Delta_{b,d} + T_{b+1}\Delta_{a,d} - T_{d+1}\Delta_{a,b}, \end{cases}$$

and

$$-T_a\Delta_{c,d} + T_{d+1}\Delta_{a,b} = T_d\Delta_{a,c} - T_{a+1}\Delta_{b,d} = S(\Delta_{a,c}, \Delta_{b,d}). \tag{2.3}$$

The leading term of each summand of the left-hand side of (2.3) is at most the leading term of the right-hand side; hence, the S-polynomial $S(\Delta_{a,c}, \Delta_{b,d})$ reduces to zero modulo G.

There are no complicated calculations to make if some of the indices a, b, c, d are equal. Indeed, it suffices to consider these cases:

$$a = b < c < d \implies S(\Delta_{a,c}, \Delta_{a,d}) = T_d\Delta_{a,c} - T_c\Delta_{a,d} = -T_a\Delta_{c,d},$$

$$a < b = c < d \implies \text{the leading terms of } \Delta_{a,b} \text{ and } \Delta_{b,d} \text{ are relatively prime},$$

$$a < b < c = d \implies S(\Delta_{a,c}, \Delta_{b,c}) = T_{b+1}\Delta_{a,c} - T_{a+1}\Delta_{b,c} = T_{c+1}\Delta_{a,b}.$$

In each case, the relevant S-polynomial reduces to zero modulo G. □

Retain the notation of Data 1.1. Recall the set of binomials G from Lemma 2.2 and the set of monomials $\mathcal{L}$ from Definition 1.17.

Theorem 2.4. *The set of polynomials $G \cup \mathcal{L}$ in S is a Gröbner basis for the preimage of $K^{(n)}$ in S.*

Proof. Observation 1.19 shows that $G \cup \mathcal{L}$ is a generating set of the preimage of $K^{(n)}$ in S. To prove it is a Gröbner basis we again apply the Buchberger criterion. We saw in Lemma 2.2 that every S-polynomial $S(h_1, h_2)$, with $h_1, h_2 \in G$, reduces to zero modulo $G \cup \mathcal{L}$. If M_1, M_2 are in $\mathcal{L}$, then the S-polynomial $S(M_1, M_2)$ is equal to zero. Finally, we study the S-polynomial $f = S(M_1, h_1)$, where M_1 is an element of $\mathcal{L}$ and h_1 is in G. The only interesting case is when M_1 and the leading term of h_1 have a factor in common. Henceforth, we make this assumption. It is clear that f is monomial.

We claim that $\operatorname{Deg}(f) \geqslant n$. Once the claim is established, then Observation 2.5 shows that f reduces to zero modulo $G \cup \mathcal{L}$. We prove the claim. Write

$$h_1 = -\det \begin{bmatrix} T_{i,j-1} & T_{u,v} \\ T_{i,j} & T_{u,v+1} \end{bmatrix} = \underline{T_{i,j} T_{u,v}} - T_{i,j-1} T_{u,v+1}$$

for variables $T_{i,j-1} > T_{i,j} \geqslant T_{u,v} > T_{u,v+1}$ from S. There are three possibilities for the greatest common divisor of M_1 and $T_{i,j} T_{u,v}$,

$$T_{i,j} \text{ or } T_{u,v} \text{ or } T_{i,j} T_{u,v}.$$

In the first case, $f = \frac{M_1}{T_{i,j}} T_{i,j-1} T_{u,v+1}$ and

$$\operatorname{Deg}(f) - \operatorname{Deg}(M_1) = 1 + \operatorname{Deg}(T_{u,v+1}) \geqslant 1.$$

In the second case, $f = \frac{M_1}{T_{u,v}} T_{i,j-1} T_{u,v+1}$ and

$$\operatorname{Deg}(f) - \operatorname{Deg}(M_1) = -1 + \operatorname{Deg}(T_{i,j-1}) \geqslant 0.$$

In the third case, $f = \frac{M_1}{T_{i,j} T_{u,v}} T_{i,j-1} T_{u,v+1}$ and $\operatorname{Deg}(f) = \operatorname{Deg}(M_1)$. In each case, $\operatorname{Deg}(f) \geqslant \operatorname{Deg}(M_1) \geqslant n$. Thus the claim is established and the proof is complete. □

Observation 2.5. *If f is a monomial of S with* $\operatorname{Deg}(f) \geqslant n$*, then f reduces to zero modulo $G \cup \mathcal{L}$.*

Proof. The proof of Observation 1.9 shows that the remainder of f on division by G has the form of M from (1.10) with $\operatorname{Deg}(M) = \operatorname{Deg}(f) \geqslant n$. (The proof of Observation 1.9 does not mention division by G; however, the binomial h of (1.12) is in G and the leading term of h is $T_{i,j} T_{u,v}$. This leading term divides the only term of f with quotient

$$\frac{f}{T_{i,j} T_{u,v}} = f_1 f'' f_2.$$

We calculate the S-polynomial $S(f,h) = f - h f_1 f'' f_2$ in (1.13). Proceed in this manner until M is obtained.) Furthermore, Observation 1.18 shows that M is divisible by an element of $\mathcal{L}$. □

Corollary 2.6. *Adopt the notation of Data* 1.1 *with $n \geqslant 2$, then* $\operatorname{depth} A/K^{(n)} = 1$.

Proof. The variable $T_{\ell,\sigma_\ell+1}$ is regular on $A/K^{(n)}$, because its image in A is not contained in K. It remains to prove that the maximal ideal $\mathfrak{m} = (\{T_{ij}\})A$ of A is an associated prime of $C = A/(K^{(n)}, T_{\ell,\sigma_\ell+1}A) = A/(A_{\geqslant n}, T_{\ell,\sigma_\ell+1}A)$, where the last equality holds by Theorem 1.5. Indeed, from (1.4) and the determinantal relations in A one sees that $\mathfrak{m}^{n-1} T_{\ell,\sigma_\ell} \subseteq (A_{\geqslant n}, T_{\ell,\sigma_\ell+1}A)$, which gives $\mathfrak{m}^{n-1} T_{\ell,\sigma_\ell} C = 0$. On the other hand, $T_{\ell,\sigma_\ell} C \neq 0$. For if $T_{\ell,\sigma_\ell} A \subseteq (A_{\geqslant n}, T_{\ell,\sigma_\ell+1}A)$, then $T_{\ell,\sigma_\ell} A \subseteq T_{\ell,\sigma_\ell+1} A$, because $\operatorname{Deg} T_{\ell,\sigma_\ell} = 1$ and $A_{\geqslant n}$ is generated by homogeneous elements with $\operatorname{Deg} \geqslant n > 1$. But $T_{\ell,\sigma_\ell} A \subset T_{\ell,\sigma_\ell+1} A$ is impossible since $A = S/H$ with H generated by forms of (total) degree 2. □

3. Filtration

In Theorem 3.17, we describe a filtration of the nth symbolic power, $K^{(n)}$, of K. The factors in this filtration are Cohen–Macaulay S-modules. We use this filtration to describe the modules in a fdeg-graded resolution of $K^{(n)}$ by free S-modules, see Theorem 4.5 and (1.25). We calculate the regularity of the graded S-module $K^{(n)}$ in Theorem 5.5.

Definition 3.1. Recall the notation of Definition 1.14.

(1) We put a total order on the set of eligible tuples. If $\boldsymbol{b} = (b_1, \dots, b_j)$ and $\boldsymbol{a} = (a_1, \dots, a_k)$ are eligible tuples, then we say that $\boldsymbol{b} > \boldsymbol{a}$ if either

$$\begin{aligned} &\text{(a) } j < k \quad \text{and} \quad b_i = a_i \quad \text{for } 1 \leqslant i \leqslant j, \text{ or} \\ &\text{(b) } \exists i \leqslant \min\{j,k\} \quad \text{with } b_i > a_i \quad \text{and} \quad b_s = a_s \quad \text{for } 1 \leqslant s \leqslant i-1. \end{aligned} \tag{3.2}$$

If one pretends that $\boldsymbol{b}$ and $\boldsymbol{a}$ have the same length, filled out as necessary on the right by the symbol ∞, $(b_1, \dots, b_j, \infty, \dots, \infty)$ and $(a_1, \dots, a_k, \infty, \dots, \infty)$, then the total order $>$ of (3.2) is simply the lexicographic order, which means it can be tested using only rule (b). Recall from Remark 1.16 that the empty tuple $\emptyset$ is always an eligible tuple. Notice that $\emptyset$ is the largest eligible tuple.

(2) For an eligible tuple $\boldsymbol{a}$ we define the A-ideals

$$\mathcal{D}_{\boldsymbol{a}} = \sum_{\boldsymbol{b} > \boldsymbol{a}} T^{\boldsymbol{b}} T_{j+1,1}^{f(\boldsymbol{b})} (T_{j+1,1}, \dots, T_{j+1,r(\boldsymbol{b})}) \quad \text{and}$$

$$\mathcal{E}_{\boldsymbol{a}} = \sum_{\boldsymbol{b} \geqslant \boldsymbol{a}} T^{\boldsymbol{b}} T_{j+1,1}^{f(\boldsymbol{b})} (T_{j+1,1}, \dots, T_{j+1,r(\boldsymbol{b})}),$$

where $\boldsymbol{b} = (b_1, \dots, b_j)$ is eligible and j is arbitrary. Notice that $\mathcal{D}_{\emptyset} = 0$, and if the tuple $\boldsymbol{a}$ is not empty, then

$$\mathcal{D}_{\boldsymbol{a}} = \sum_{\boldsymbol{b} > \boldsymbol{a}} \mathcal{E}_{\boldsymbol{b}},$$

where the sum is taken over all eligible tuples $\boldsymbol{b}$ with $\boldsymbol{b} > \boldsymbol{a}$. Notice also that if $\boldsymbol{a}$ is an eligible k-tuple, then

$$\mathcal{E}_{\boldsymbol{a}} = \mathcal{D}_{\boldsymbol{a}} + T^{\boldsymbol{a}} T_{k+1,1}^{f(\boldsymbol{a})} (T_{k+1,1}, \dots, T_{k+1,r(\boldsymbol{a})}).$$

This gives a finite filtration

$$(0) \subsetneq \mathcal{E}_{\emptyset} \subsetneq \cdots \subsetneq \mathcal{E}_{0^{\ell-1}} = K^{(n)},$$

of $K^{(n)}$, where 0^s is the s-tuple $(0, \dots, 0)$. We define two parallel collections of ideals $\{\mathcal{E}_{\boldsymbol{a}}\}$ and $\{\mathcal{D}_{\boldsymbol{a}}\}$ simultaneously because there is no convenient way to denote the eligible tuple which is immediately larger than a particular eligible tuple $\boldsymbol{a}$. Notice that the modules $\mathcal{E}_{\boldsymbol{a}}/\mathcal{D}_{\boldsymbol{a}}$ are exactly the factors of the filtration $\{\mathcal{E}_{\boldsymbol{a}}\}$.

Recall the fine grading (1.25) on S and A. Observe that the ideals $\mathcal{D}_{\boldsymbol{a}}$ and $\mathcal{E}_{\boldsymbol{a}}$ are homogeneous in this grading. Define fdeg-graded free S-modules

$$E = \left\{ \begin{array}{c} S(1,-1;0) \\ \oplus \\ S(0,0;0) \end{array} \right. \quad \text{and} \quad F_u = \left\{ \begin{array}{c} S(-\sigma_u + 1, -1; -\varepsilon_u) \\ \oplus \\ S(-\sigma_u + 2, -2; -\varepsilon_u) \\ \oplus \\ \vdots \\ \oplus \\ S(0, -\sigma_u; -\varepsilon_u), \end{array} \right. \tag{3.3}$$

for $1 \leqslant u \leqslant \ell$ and ε_u as defined in (1.24). Notice that each $\psi_u : F_u \to E$ is a homogeneous map, with respect to fdeg; and therefore, for each k, the cokernel of

$$\psi_{>k} = [\ \psi_{k+1} \quad | \quad \dots \quad | \quad \psi_\ell\ ] : \bigoplus_{u=k+1}^{\ell} F_u \to E \tag{3.4}$$

is a graded S-module, with respect to the fdeg-grading. Let $\boldsymbol{a}$ be an eligible k-tuple. In this section we prove that $\mathcal{E}_{\boldsymbol{a}}/\mathcal{D}_{\boldsymbol{a}}$ is a well-known Cohen–Macaulay module. Let P_k be the ideal

$$P_k = \big(\{T_{i,j} \mid 1 \leqslant i \leqslant k \text{ and } 1 \leqslant j \leqslant \sigma_i + 1\}\big)$$

of S, and $\varepsilon_{\boldsymbol{a}}$ the multi-shift

$$\varepsilon_{\boldsymbol{a}} = \sum_{u=1}^{k} a_u \varepsilon_u. \tag{3.5}$$

In Theorem 3.17 we prove that the fdeg-graded S-modules $\mathcal{E}_{\boldsymbol{a}}/\mathcal{D}_{\boldsymbol{a}}$ and

$$\operatorname{Sym}_{r(\boldsymbol{a})-1}^{S/I_2(\psi_{>k})}\big(\operatorname{cok}(\psi_{>k})\big) \otimes_S \frac{S}{P_k}(-\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}, 0; -\boldsymbol{\varepsilon}) \tag{3.6}$$

are isomorphic, for $\boldsymbol{\varepsilon} = \varepsilon_{\boldsymbol{a}} + (f(\boldsymbol{a}) + 1)\varepsilon_{k+1}$. The module of (3.6) might look more familiar if we observe that

$$\operatorname{Sym}_{r(\boldsymbol{a})-1}^{S/I_2(\psi_{>k})}\big(\operatorname{cok}(\psi_{>k})\big) \otimes_S \frac{S}{P_k} \cong (T_{k+1,1}, T_{k+1,2})^{r(\boldsymbol{a})-1} \frac{A}{P_k A}\big((r(\boldsymbol{a}) - 1)(\sigma_{k+1}, 0; \varepsilon_{k+1})\big);$$

see the proof of Lemma 3.14. We have written $\operatorname{Sym}_{r(\boldsymbol{a})-1}^{S/I_2(\psi_{>k})}$ rather than Sym or Sym^S in order to emphasize that when $r(\boldsymbol{a}) - 1 = 0$, then the module of (3.6) is a shift of

$$S/I_2(\psi_{>k}) \otimes_S S/P_k = A/P_k A.$$

Recall, from (1.15), that $r(\boldsymbol{a}) - 1$ is non-negative and is less than the number of columns of $\psi_{>k}$. The ideal $I_2(\psi_{>k})$ has generic height (equal to the number of columns of $\psi_{>k}$ minus 1) and the symmetric power $r(\boldsymbol{a}) - 1$ is small enough that $\operatorname{Sym}_{r(\boldsymbol{a})-1}(\operatorname{cok}(\psi_{>k}))$ is a perfect S-module and is resolved by a generalized Eagon–Northcott complex. (See, for example, the family of complexes studied in and near Theorem 2.16 in [1] or Theorem A2.10 in [5]. Recall that a finitely generated S-module M is perfect if the grade of the annihilator of M in S is equal to the projective dimension of M.)

The module of (3.6) is annihilated by P_k. The first step in the proof of Theorem 3.17 is to show that $\mathcal{E}_{\boldsymbol{a}}/\mathcal{D}_{\boldsymbol{a}}$ is also annihilated by P_k.

Lemma 3.7. *If $\boldsymbol{a}$ is an eligible k-tuple and (κ, r) is a pair of integers with*

$$k + 1 \leqslant \kappa \leqslant \ell \quad \textit{and} \quad 1 \leqslant r \leqslant \sum_{u=1}^{k} a_u \sigma_u + f(\boldsymbol{a})\sigma_{k+1} + \sigma_\kappa - n + 1, \tag{3.8}$$

then $P_k T^{\boldsymbol{a}} T_{k+1,1}^{f(\boldsymbol{a})} T_{\kappa,r} \subseteq \mathcal{D}_{\boldsymbol{a}}$. In particular,

(a) *$P_k \mathcal{E}_{\boldsymbol{a}} \subseteq \mathcal{D}_{\boldsymbol{a}}$, and*
(b) *if $r(\boldsymbol{a}) = \sigma_{k+1}$, then $P_k T^{\boldsymbol{a}} T_{k+1,1}^{f(\boldsymbol{a})} T_{\kappa,r} \subseteq \mathcal{D}_{\boldsymbol{a}}$, for all (κ, r) with $k + 1 \leqslant \kappa \leqslant \ell$ and $1 \leqslant r \leqslant \sigma_\kappa$.*

Proof. We notice that (a) and (b) are applications of the first assertion. Indeed, if $\kappa = k+1$, then the upper bound on r in (3.8) is equal to $r(\boldsymbol{a})$; furthermore,

$$\mathcal{E}_{\boldsymbol{a}} = T^{\boldsymbol{a}} T_{k+1,1}^{f(\boldsymbol{a})}\big(\{T_{k+1,r} \mid 1 \leqslant r \leqslant r(\boldsymbol{a})\}\big) + \mathcal{D}_{\boldsymbol{a}}.$$

In (b), the hypothesis $r(\boldsymbol{a}) = \sigma_{k+1}$ forces $\sum_{u=1}^{k} a_u\sigma_u + f(\boldsymbol{a})\sigma_{k+1} = n-1$, and in this case the bound on r in (3.8) becomes $1 \leqslant r \leqslant \sigma_\kappa$.

We prove the first assertion. Fix i and s with $1 \leqslant i \leqslant k$ and $1 \leqslant s \leqslant \sigma_i + 1$. Let

$$X = T_{i,s} T^{\boldsymbol{a}} T_{k+1,1}^{f(\boldsymbol{a})} T_{\kappa,r}.$$

We will prove that $X \in \mathcal{D}_{\boldsymbol{a}}$.

Define $a_{k+1} = f(\boldsymbol{a})$ and

$$b_u = \begin{cases} a_u & \text{if } 1 \leqslant u \leqslant k+1 \text{ and } u \neq i,\\ a_i + 1 & \text{if } u = i.\end{cases}$$

Notice that for each u, with $1 \leqslant u \leqslant k$, we have

$$(b_1, \ldots, b_u) > \boldsymbol{a},$$

where we define order as in Definition 3.1.1. We know

$$\sum_{u=1}^{k} a_u\sigma_u + f(\boldsymbol{a})\sigma_{k+1} < n \leqslant \sum_{u=1}^{k} a_u\sigma_u + f(\boldsymbol{a})\sigma_{k+1} + \sigma_{k+1} \leqslant \sum_{u=1}^{k+1} b_u\sigma_u,$$

where the last inequality holds because $\sigma_{k+1} \leqslant \sigma_i$. Select the least integer j with

$$n \leqslant \sum_{u=1}^{j} b_u\sigma_u.$$

Notice that $i \leqslant j \leqslant k+1$. Select the largest value b'_j with

$$\sum_{u=1}^{j-1} b_u\sigma_u + b'_j\sigma_j < n.$$

Observe that

$$0 \leqslant b'_j < b_j.$$

Let $\boldsymbol{b} = (b_1, \ldots, b_{j-1})$. We see that $\boldsymbol{b}$ is an eligible $(j-1)$-tuple and $\boldsymbol{b} > \boldsymbol{a}$. We have chosen b'_j so that $b'_j = f(\boldsymbol{b})$. It follows that

$$T^{\boldsymbol{b}} T_{j,1}^{b'_j}(T_{j,1}, \ldots, T_{j,r(\boldsymbol{b})}) \subseteq \mathcal{E}_{\boldsymbol{b}} \subseteq \mathcal{D}_{\boldsymbol{a}}.$$

Write $\rho = \min\{r(\boldsymbol{b}), s+r-1\}$. Since $1 \leqslant \rho \leqslant r(\boldsymbol{b})$, it suffices to prove that

$$X \in T^{\boldsymbol{b}} T_{j,1}^{b'_j} T_{j,\rho} A. \tag{3.9}$$

Notice that

$$\text{if } i<j, \quad \text{then } b'_j<b_j=a_j, \quad \text{and} \tag{3.10}$$

$$\text{if } i=j, \quad \text{then } b'_j=a_j. \tag{3.11}$$

We prove (3.11). The definition of b'_j says that b'_j is the largest integer with

$$\sum_{u=1}^{j-1} b_u\sigma_u+b'_j\sigma_i<n.$$

In other words, b'_j is the largest integer with $\sum_{u=1}^{j-1} a_u\sigma_u+b'_j\sigma_j<n$. On the other hand, we know

$$\sum_{u=1}^{j-1} a_u\sigma_u+a_j\sigma_j=\sum_{u=1}^{j} a_u\sigma_u<n\leqslant\sum_{u=1}^{j} b_u\sigma_u=\sum_{u=1}^{j-1} a_u\sigma_u+(a_j+1)\sigma_j.$$

The first equality uses the fact that $j=i\leqslant k$. The last equality holds because $b_j=b_i=a_i+1=a_j+1$. Assertion (3.11) is established.

To prove (3.9) we use the embedding $A\hookrightarrow B=k[x,y,t_1,\dots,t_\ell]$ induced by the map π of Observation 1.7. Thus (3.9) is equivalent to showing that

$$x^\gamma y^{s+r-2}t_it_\kappa\prod_{u=1}^{k+1} t_u^{a_u}=Fx^\delta y^{\rho-1}t_j^{b'_j+1}\prod_{u=1}^{j-1} t_u^{b_u},$$

for some $F\in A$, with

$$\gamma=\sum_{u=1}^{k+1} a_u\sigma_u+\sigma_i+\sigma_\kappa-s-r+2 \quad \text{and} \quad \delta=\sum_{u=1}^{j-1} b_u\sigma_u+(b'_j+1)\sigma_j-\rho+1.$$

Clearly such F exists in the quotient field of A. According to (3.10) and (3.11) one has

$$F=\begin{cases} x^\alpha y^\beta t_\kappa\prod_{u=j+1}^{k+1} t_u^{a_u} & \text{if } i=j,\\ x^\alpha y^\beta t_\kappa t_j^{a_j-b'_j-1}\prod_{u=j+1}^{k+1} t_u^{a_u} & \text{if } i<j,\end{cases}$$

and F is an element of $k(x,y)[t_1,\dots,t_\ell]$. Notice that

$$\beta=s+r-\rho-1 \quad \text{and} \quad \alpha=\begin{cases}\nu & \text{if } i=j,\\ \nu+(a_j-b'_j-1)\sigma_j & \text{if } i<j,\end{cases}$$

for

$$\nu=\sum_{u=j+1}^{k+1} a_u\sigma_u+\sigma_\kappa-s-r+\rho+1.$$

Recall that F is in the quotient field of A and that A is a direct summand of B according to Observation 1.7(e). Thus, to prove that $F \in A$, it suffices to show that $F \in B$ or, equivalently,

$$\alpha \geqslant 0 \quad \text{and} \quad \beta \geqslant 0. \tag{3.12}$$

Clearly, $\beta \geqslant 0$ by the definition of ρ. Likewise, if $\rho = s + r - 1$, then $\alpha \geqslant 0$ according to (3.10). Thus we may assume that $\rho = r(\boldsymbol{b})$. Use the definition of $r(\boldsymbol{b})$,

$$r(\boldsymbol{b}) = \sum_{u=1}^{j-1} b_u \sigma_u + \big(b'_j + 1\big)\sigma_j - n + 1.$$

Treat the cases $i = j$ and $i < j$ separately. Two straightforward calculations yield

$$\alpha = \left(\sum_{u=1}^{k+1} a_u \sigma_u + \sigma_\kappa - n + 1 - r \right) + (\sigma_i + 1 - s) \geqslant 0,$$

where the first summand is non-negative by assumption (3.8) and the second summand is non-negative because of the choice of s. This completes the proof of (3.12). $\square$

We have established half of Theorem 3.17. The next two lemmas are used in the other half of the proof.

Lemma 3.13. *If $\boldsymbol{a}$ is an eligible k-tuple, then $\mathcal{D}_{\boldsymbol{a}} \subseteq P_k A$ and $T^{\boldsymbol{a}}$ is not zero in $(A/\mathcal{D}_{\boldsymbol{a}})_{P_k A}$.*

Proof. Let $\mathfrak{S}$ be the multiplicative subset of $A \setminus P_k A$ which consists of the non-zero elements of the image in A of the polynomial ring $k[T_{k+1,*}, \ldots, T_{\ell,*}]$. Let Q be the quotient field of this image. We notice that

$$\mathfrak{S}^{-1} A = \frac{Q[T_{1,*}, \ldots, T_{k,*}]}{HQ[T_{1,*}, \ldots, T_{k,*}]}.$$

Furthermore, since $k \leqslant \ell - 1$, $HQ[T_{1,*}, \ldots, T_{k,*}]$ is generated by linear forms, and $T_{i,j}$ is an associate of $T_{i,1}$ in $\mathfrak{S}^{-1}A$, for all i, j with $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant \sigma_i + 1$. Indeed, $\mathfrak{S}^{-1}A$ is naturally isomorphic to the polynomial ring $Q[T_{1,1}, \ldots, T_{k,1}]$ in k variables over the field Q. Observe that the ring $A_{P_k A}$ is equal to the further localization $Q[T_{1,1}, \ldots, T_{k,1}]_{(T_{1,1}, \ldots, T_{k,1})}$ of $\mathfrak{S}^{-1}A$.

We first show that $T^{\boldsymbol{a}}$ is not zero in $\mathfrak{S}^{-1}(A/\mathcal{D}_{\boldsymbol{a}})$. We have seen that $\mathfrak{S}^{-1}A$ is the polynomial ring $Q[T_{1,1}, \ldots, T_{k,1}]$. We now observe that the ideal $\mathcal{D}_{\boldsymbol{a}}$ of $\mathfrak{S}^{-1}A$ is generated by the following set of monomials,

$$\begin{aligned} &\big\{T^{\boldsymbol{b}} T_{j+1,1}^{f(\boldsymbol{b})+1} \bigm| \boldsymbol{b} = (b_1, \ldots, b_j) \text{ is eligible, } j < k, \text{ and (3.2.a) or (3.2.b) is in effect}\big\} \\ &\quad \cup \big\{T^{\boldsymbol{b}} \bigm| \boldsymbol{b} = (b_1, \ldots, b_k) \text{ is eligible and (3.2.b) is in effect}\big\}. \end{aligned}$$

It is obvious that in the polynomial ring $Q[T_{1,1}, \ldots, T_{k,1}]$, none of the monomials in the second set can divide $T^{\boldsymbol{a}}$. If some monomial from the first set divides $T^{\boldsymbol{a}}$, then the definition of $f(\boldsymbol{b})$, together with the fact that $\boldsymbol{a}$ is eligible, yields

$$n \leqslant \sum_{u=1}^{j} b_u \sigma_u + \big(f(\boldsymbol{b}) + 1\big)\sigma_{j+1} \leqslant \sum_{u=1}^{k} a_u \sigma_u < n,$$

and of course, this is impossible. Since $T^{\boldsymbol{a}}$ is a monomial, we deduce that $T^{\boldsymbol{a}}$ is not in the ideal $\mathcal{D}_{\boldsymbol{a}}$ of $Q[T_{1,1},\dots,T_{k,1}]$. Thus, $T^{\boldsymbol{a}}$ is not zero in $\mathfrak{S}^{-1}(A/\mathcal{D}_{\boldsymbol{a}})$, which is a standard graded Q-algebra. We localize at the maximal homogeneous ideal to see that $T^{\boldsymbol{a}}$ is also not zero in $(A/\mathcal{D}_{\boldsymbol{a}})_{P_kA}$. In particular, this is not the zero ring, showing that $\mathcal{D}_{\boldsymbol{a}} \subseteq P_kA$. $\square$

Lemma 3.14. *Let $\boldsymbol{a}$ be an eligible k-tuple, B the ring $A/\mathcal{D}_{\boldsymbol{a}}$, and J the ideal $T_{k+1,1}^{f(\boldsymbol{a})}(T_{k+1,1},\dots,T_{k+1,r(\boldsymbol{a})})$ of A. Then*

(a) *the* fdeg*-graded A-module of* (3.6) *is isomorphic to*

$$J(A/P_kA)(-\boldsymbol{\sigma}\cdot\varepsilon_{\boldsymbol{a}},0;-\varepsilon_{\boldsymbol{a}}),$$

with $\varepsilon_{\boldsymbol{a}}$ as defined in (3.5), *and*

(b) $\mathcal{E}_{\boldsymbol{a}}/\mathcal{D}_{\boldsymbol{a}} = T^{\boldsymbol{a}}JB$.

Proof. Assertion (b) is clear. We prove (a) by establishing the following sequence of homogeneous isomorphisms,

$$\begin{aligned}
&\operatorname{Sym}_{r(\boldsymbol{a})-1}^{S/I_2(\psi_{>k})}\big(\operatorname{cok}(\psi_{>k})\big)\otimes_S \frac{S}{P_k}\\
&\xrightarrow{\alpha_1} (T_{k+1,1},T_{k+1,2})^{r(\boldsymbol{a})-1}\frac{A}{P_kA}\big((r(\boldsymbol{a})-1)(\sigma_{k+1},0;\varepsilon_{k+1})\big)\\
&\xrightarrow{\alpha_2} (T_{k+1,1},\dots,T_{k+1,r(\boldsymbol{a})})\frac{A}{P_kA}(\sigma_{k+1},0;\varepsilon_{k+1})\\
&\xrightarrow{\alpha_3} J\frac{A}{P_kA}\big(\big(f(\boldsymbol{a})+1\big)(\sigma_{k+1},0;\varepsilon_{k+1})\big).
\end{aligned}$$

The ideal $(T_{k+1,1},T_{k+1,2})$ of the domain A/P_kA is generated by the entries of the first column of $\psi_{>k}$. The map

$$E \xrightarrow{[\,T_{k+1,2}\quad -T_{k+1,1}\,]} A(\sigma_{k+1},0;\varepsilon_{k+1}),$$

for E in (3.3), induces a natural surjection

$$\operatorname{cok}(\psi_{>k})\otimes_S\frac{S}{P_k} \twoheadrightarrow (T_{k+1,1},T_{k+1,2})\frac{A}{P_kA}(\sigma_{k+1},0;\varepsilon_{k+1}). \tag{3.15}$$

The map α_1 is the surjection induced by (3.15). Recall that the source of α_1 is Cohen–Macaulay and that it has rank one as a module over the domain A/P_kA. Furthermore, the target of α_1 is, up to a shift, a non-zero ideal in this domain. It follows that α_1 is an isomorphism. The ideals

$$(T_{k+1,1},T_{k+1,2})^{r(\boldsymbol{a})-1}\quad\text{and}\quad T_{k+1,1}^{r(\boldsymbol{a})-2}(T_{k+1,1},\dots,T_{k+1,r(\boldsymbol{a})})$$

of the domain A/P_kA are equal, as can be seen from Observation 1.7(b), for instance. Therefore, the isomorphism α_2 is given by multiplication by the unit $1/T_{k+1,1}^{r(\boldsymbol{a})-2}$ in the quotient field of A/P_kA. Multiplication by the non-zero element $T_{k+1,1}^{f(\boldsymbol{a})}$ of the domain A/P_kA gives the A-module isomorphism α_3. $\square$

The next lemma is the final step in our proof of Theorem 3.17. We will also use the same lemma in the proof of Proposition 5.3.

Lemma 3.16. *Let $\boldsymbol{a}$ be an eligible k-tuple, B the ring $A/\mathcal{D}_{\boldsymbol{a}}$, and $\mathcal{J}$ an ideal of A. Assume that*

(1) *$\mathcal{J}$ is* fdeg*-homogeneous and the elements of some generating set of $\mathcal{J}$ involve only the variables $\{T_{i,j}\}$ with $i \geqslant k+1$, and*
(2) *P_k annihilates $T^{\boldsymbol{a}}\mathcal{J}B$.*

Then the fdeg*-graded A-modules $T^{\boldsymbol{a}}\mathcal{J}B$ and $\mathcal{J}(A/P_kA)(-\boldsymbol{\sigma}\cdot\varepsilon_{\boldsymbol{a}}, 0; -\varepsilon_{\boldsymbol{a}})$ are isomorphic, where $\varepsilon_{\boldsymbol{a}}$ is as defined in* (3.5).

Proof. We exhibit homomorphisms of fdeg-graded A-modules

$$\alpha : T^{\boldsymbol{a}}\mathcal{J}B(\boldsymbol{\sigma}\cdot\varepsilon_{\boldsymbol{a}}, 0; \varepsilon_{\boldsymbol{a}}) \longrightarrow \mathcal{J}\left(\frac{A}{P_kA}\right) \quad \text{and} \quad \beta : \mathcal{J}\left(\frac{A}{P_kA}\right) \longrightarrow T^{\boldsymbol{a}}\mathcal{J}B(\boldsymbol{\sigma}\cdot\varepsilon_{\boldsymbol{a}}, 0; \varepsilon_{\boldsymbol{a}}),$$

which are inverses of one another.

We first show that $\beta : \mathcal{J}(\frac{A}{P_kA}) \to T^{\boldsymbol{a}}\mathcal{J}B(\boldsymbol{\sigma}\cdot\varepsilon_{\boldsymbol{a}}, 0; \varepsilon_{\boldsymbol{a}})$, given by $\beta(X) = T^{\boldsymbol{a}}X$, for all X in $\mathcal{J}$, is a well-defined A-module homomorphism. Consider the composition

$$A \longrightarrow B \longrightarrow T^{\boldsymbol{a}}B(\boldsymbol{\sigma}\cdot\varepsilon_{\boldsymbol{a}}, 0; \varepsilon_{\boldsymbol{a}}),$$

where the first map is the natural quotient map and the second map is multiplication by $T^{\boldsymbol{a}}$. This composition restricts to give $\beta' : \mathcal{J}A \to T^{\boldsymbol{a}}\mathcal{J}B(\boldsymbol{\sigma}\cdot\varepsilon_{\boldsymbol{a}}, 0; \varepsilon_{\boldsymbol{a}})$. The first hypothesis ensures that $\mathcal{J}A \cap P_kA = \mathcal{J}P_kA$ and the second hypothesis ensures that $\mathcal{J}P_kA \subseteq \ker\beta'$. So, β' induces

$$\beta : \mathcal{J}\left(\frac{A}{P_kA}\right) = \frac{\mathcal{J}A}{\mathcal{J}A \cap P_kA} \longrightarrow T^{\boldsymbol{a}}\mathcal{J}B(\boldsymbol{\sigma}\cdot\varepsilon_{\boldsymbol{a}}, 0; \varepsilon_{\boldsymbol{a}}),$$

as described above.

Now we show that $\alpha : T^{\boldsymbol{a}}\mathcal{J}B(\boldsymbol{\sigma}\cdot\varepsilon_{\boldsymbol{a}}, 0; \varepsilon_{\boldsymbol{a}}) \to \mathcal{J}(\frac{A}{P_kA})$, given by $\alpha(T^{\boldsymbol{a}}X) = X$, for all X in $\mathcal{J}$, is a well-defined A-module homomorphism. Let

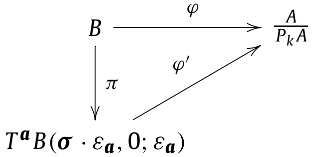
$$\varphi : B = \frac{A}{\mathcal{D}_{\boldsymbol{a}}} \longrightarrow \frac{A}{P_kA}$$

be the natural quotient map which is induced by the inclusion $\mathcal{D}_{\boldsymbol{a}} \subseteq P_kA$ of Lemma 3.13 and let $\pi : B \to T^{\boldsymbol{a}}B(\boldsymbol{\sigma}\cdot\varepsilon_{\boldsymbol{a}}, 0; \varepsilon_{\boldsymbol{a}})$ be multiplication by $T^{\boldsymbol{a}}$.

The kernel of π is the annihilator of $T^{\boldsymbol{a}}$ in B, and the kernel of φ is P_kB. We saw in Lemma 3.13 that $T^{\boldsymbol{a}} \neq 0$ in B_{P_kB}. It follows that

$$\ker\pi \subseteq \ker\varphi.$$

Thus, there exists a unique A-module homomorphism $\varphi' : T^{\boldsymbol{a}}B(\boldsymbol{\sigma}\cdot\varepsilon_{\boldsymbol{a}}, 0; \varepsilon_{\boldsymbol{a}}) \to \frac{A}{P_kA}$ for which the diagram



commutes. The restriction of φ' to $T^{\boldsymbol{a}}\mathcal{J}B(\boldsymbol{\sigma}\cdot\varepsilon_{\boldsymbol{a}}, 0; \varepsilon_{\boldsymbol{a}})$ is the homomorphism α which is described above. □

The next result follows from Lemma 3.14 and Lemma 3.16, applied to the ideal $T_{k+1,1}^{f(\boldsymbol{a})}(T_{k+1,1},\dots,T_{k+1,r(\boldsymbol{a})})$ of A; notice that assumption (2) of Lemma 3.16 is satisfied according to Lemma 3.7(a).

Theorem 3.17. *Adopt the hypotheses of Data* 1.1. *Let* $\{\mathcal{E}_{\boldsymbol{a}}\}$*, as* $\boldsymbol{a}$ *varies over all eligible tuples, be the filtration of* $K^{(n)}$ *from Definition* 3.1. *Then, for each eligible k-tuple* $\boldsymbol{a}$*, the* fdeg*-graded A-modules* $\mathcal{E}_{\boldsymbol{a}}/\mathcal{D}_{\boldsymbol{a}}$ *and* (3.6) *are isomorphic.*

4. Resolution

We first record the minimal homogeneous resolution of the module $\mathcal{E}_{\boldsymbol{a}}/\mathcal{D}_{\boldsymbol{a}}$ by free fdeg-graded S-modules. Recall the free fdeg-graded S-modules E and F_u of (3.3). These modules have rank 2 and σ_u, respectively. Let $F = F_1 \oplus \cdots \oplus F_\ell$. The matrices ψ and ψ_u of (1.3) and (1.2) describe fdeg-homogeneous S-module homomorphisms $\psi : F \to E$ and $\psi_u : F_u \to E$. Let G_u be the free fdeg-graded S-module

$$G_u = \begin{cases} S(-\sigma_u, 0; -\varepsilon_u) \\ \oplus \\ S(-\sigma_u + 1, -1; -\varepsilon_u) \\ \oplus \\ \vdots \\ \oplus \\ S(0, -\sigma_u; -\varepsilon_u) \end{cases}$$

of rank $\sigma_u + 1$, and let $\rho_u : G_u \to S$ be the fdeg-homogeneous S-module homomorphism given by

$$\rho_u = [\, T_{u,1} \quad T_{u,2} \quad \dots \quad T_{u,\sigma_u+1} \,].$$

For any k with $0 \leqslant k \leqslant \ell - 1$, let $F_{>k}$ and $G_{\leqslant k}$ be the free fdeg-graded S-modules

$$F_{>k} = \bigoplus_{u=k+1}^{\ell} F_u \quad \text{and} \quad G_{\leqslant k} = \bigoplus_{u=1}^{k} G_u,$$

and let $\psi_{>k} : F_{>k} \to E$ and $\rho_{\leqslant k} : G_{\leqslant k} \to S$ be the fdeg-homogeneous S-module homomorphisms

$$\psi_{>k} = [\, \psi_{k+1} \quad | \quad \dots \quad | \quad \psi_\ell \,] \quad \text{and} \quad \rho_{\leqslant k} = [\, \rho_1 \quad | \quad \dots \quad | \quad \rho_k \,].$$

The Koszul complex

$$\mathbb{G}_{k,\bullet} = \bigwedge\nolimits^{\bullet} G_{\leqslant k},$$

associated to $\rho_{\leqslant k} : G_{\leqslant k} \to S$, is a minimal homogeneous fdeg-graded resolution of S/P_k by free S-modules. We see that

$$\mathbb{G}_{k,q} = \sum_{i_1+\ldots+i_k=q} \bigwedge\nolimits^{i_1} G_1 \otimes \cdots \otimes \bigwedge\nolimits^{i_k} G_k \quad \text{for } 0 \leqslant q \leqslant \sum_{i=1}^{k} (\sigma_i + 1).$$

The generalized Eagon–Northcott complex $\mathbb{F}_{\boldsymbol{a},\bullet}$, where

$$\mathbb{F}_{\boldsymbol{a},p} = \begin{cases} \operatorname{Sym}_{r(\boldsymbol{a})-1-p} E \otimes \bigwedge^{p} F_{>k} & \text{if } 0 \leqslant p \leqslant r(\boldsymbol{a}) - 1, \\ D_{p-r(\boldsymbol{a})} E^* \otimes \bigwedge^{p+1} F_{>k} & \text{if } r(\boldsymbol{a}) \leqslant p \leqslant \operatorname{rank} F_{>k} - 1, \end{cases}$$

is a minimal homogeneous fdeg-graded resolution of $\mathrm{Sym}_{r(\boldsymbol{a})-1}^{S/I_2(\psi_{>k})}(\mathrm{cok}(\psi_{>k}))$ by free S-modules. See, for example, [1, Theorem 2.16] or [5, Theorem A2.10]. One other generalized Eagon–Northcott complex is of interest to us. For each integer k, with $0 \leqslant k \leqslant \ell - 1$, the complex $(\mathbb{F}_{k,\bullet}, d_{k,\bullet})$, with

$$\mathbb{F}_{k,p} = D_p E^* \otimes \bigwedge^{p+1} F_{>k},$$

is a minimal homogeneous fdeg-graded resolution of

$$\big(\{T_{u,j} \mid k+1 \leqslant u \leqslant \ell \text{ and } 1 \leqslant j \leqslant \sigma_u\}\big)\frac{S}{I_2(\psi_{>k})}(1,-1;0) \tag{4.1}$$

by free S-modules. The complex $\mathbb{F}_{k,\bullet}$ is called $\mathcal{C}^{-1}$ in [5]. The fdeg-homogeneous augmentation map from the complex $\mathbb{F}_{k,\bullet}$ to the module of (4.1) is induced by the map

$$\mathbb{F}_{k,0} = F_{>k} \xrightarrow{[\,\xi_{k+1}\quad \xi_{k+2}\quad \dots\quad \xi_\ell\,]} \frac{S}{I_2(\psi_{>k})}(1,-1;0),$$

where $\xi_u : F_u \to S(1,-1;0)$ is the fdeg-homogeneous map given by

$$[\, T_{u,1} \quad T_{u,2} \quad \dots \quad T_{u,\sigma_u} \,]$$

and the free fdeg-graded S-module F_u is described in (3.3).

With respect to total degree, the maps in $\mathbb{F}_{\boldsymbol{a},\bullet}$ are linear everywhere, except $\mathbb{F}_{\boldsymbol{a},r(\boldsymbol{a})} \to \mathbb{F}_{\boldsymbol{a},r(\boldsymbol{a})-1}$, which is a quadratic map because it involves 2×2 minors of $\psi_{>k}$. All of the maps in $\mathbb{F}_{k,\bullet}$ are linear. In other words, with respect to total degree,

$$\operatorname{reg} \mathrm{Sym}_{r(\boldsymbol{a})-1}^{S/I_2(\psi_{>k})}\big(\mathrm{cok}(\psi_{>k})\big) = \begin{cases} 0 & \text{if } k = \ell - 1 \text{ and } r(\boldsymbol{a}) = \sigma_\ell, \\ 1 & \text{in all other cases.} \end{cases}$$

(A thorough discussion of regularity may be found in Section 5.) Furthermore,

$$\operatorname{reg}\big(\{T_{u,j} \mid k+1 \leqslant u \leqslant \ell \text{ and } 1 \leqslant j \leqslant \sigma_u\}\big)\frac{S}{I_2(\psi_{>k})} = 1$$

because the generators live in degree one and the resolution is linear.

Observation 4.2. *Let k be an integer with $0 \leqslant k \leqslant \ell - 1$.*

(a) *If $\boldsymbol{a}$ is an eligible k-tuple, then*

$$(\mathbb{L}_{\boldsymbol{a},\bullet}, d_{\boldsymbol{a},\bullet}) = (\mathbb{F}_{\boldsymbol{a},\bullet} \otimes_S \mathbb{G}_{k,\bullet})(-\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}, 0; -\boldsymbol{\varepsilon})$$

is the minimal homogeneous fdeg*-graded resolution of the module $\mathcal{E}_{\boldsymbol{a}}/\mathcal{D}_{\boldsymbol{a}}$ by free S-modules, for $\boldsymbol{\varepsilon} = \varepsilon_{\boldsymbol{a}} + (f(\boldsymbol{a})+1)\varepsilon_{k+1}$ and $\varepsilon_{\boldsymbol{a}}$ as defined in* (3.5)*.*

(b) *The complex $\mathbb{L}_{k,\bullet} = \mathbb{F}_{k,\bullet} \otimes_S \mathbb{G}_{k,\bullet}$ is the minimal homogeneous* fdeg*-graded resolution of the module*

$$\big(\{T_{u,j} \mid k+1 \leqslant u \leqslant \ell \textit{ and } 1 \leqslant j \leqslant \sigma_u\}\big)\frac{A}{P_k A}(1,-1;0) \tag{4.3}$$

by free S-modules.

(c) *The S-module $\mathcal{E}_{\boldsymbol{a}}/\mathcal{D}_{\boldsymbol{a}}$ and the S-module of* (4.3) *are Cohen–Macaulay and perfect of projective dimension $\sum_{u=1}^{\ell} \sigma_u + k - 1$.*

Proof. Recall that

$$\mathcal{E}_{\boldsymbol{a}}/\mathcal{D}_{\boldsymbol{a}} \cong \operatorname{Sym}_{r(\boldsymbol{a})-1}^{S/I_2(\psi_{>k})}\big(\operatorname{cok}(\psi_{>k})\big) \otimes_S S/P_k(-\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}, 0; -\boldsymbol{\varepsilon})$$

by Theorem 3.17. We know that $\mathbb{F}_{\boldsymbol{a},\bullet}$ is a minimal homogeneous fdeg-graded resolution of $\operatorname{Sym}_{r(\boldsymbol{a})-1}^{S/I_2(\psi_{>k})}(\operatorname{cok}(\psi_{>k}))$ and $\mathbb{G}_{k,\bullet}$ is a minimal homogeneous fdeg-graded resolution of S/P_k. Furthermore, the generators of P_k are a regular sequence on the S-module $\operatorname{Sym}_{r(\boldsymbol{a})-1}^{S/I_2(\psi_{>k})}(\operatorname{cok}(\psi_{>k}))$; therefore,

$$\operatorname{Tor}_i^S\big(\operatorname{Sym}_{r(\boldsymbol{a})-1}^{S/I_2(\psi_{>k})}\big(\operatorname{cok}(\psi_{>k})\big), S/P_k\big) = 0 \quad \text{for all } i \geqslant 1,$$

and $\mathbb{F}_{\boldsymbol{a},\bullet} \otimes_S \mathbb{G}_{k,\bullet}$ is a minimal homogeneous fdeg-graded resolution of

$$\operatorname{Sym}_{r(\boldsymbol{a})-1}^{S/I_2(\psi_{>k})}\big(\operatorname{cok}(\psi_{>k})\big) \otimes_S S/P_k.$$

Notice that the length of this resolution is

$$\sum_{u=k+1}^{\ell} \sigma_u - 1 + \sum_{u=1}^{k}(\sigma_u + 1) = \sum_{u=1}^{\ell} \sigma_u + k - 1,$$

which is at most the grade of the annihilator of the module it resolves. Assertion (a) and half of assertion (c) have been established. The rest of the result is proved in the same manner. □

Finally, we resolve $K^{(n)}$. Let

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_s = 0$$

be a filtration of a module M. If one can resolve each sub-quotient M_i/M_{i+1}, then one can resolve M by an iterated application of the Horseshoe Lemma, as explained in Lemma 4.4. We apply the lemma to the filtration $\{\mathcal{E}_{\boldsymbol{a}}\}$ of $K^{(n)}$ in Theorem 4.5. One may also apply the lemma to the filtration $\{\mathcal{E}'_{\boldsymbol{a}}\}$ of Section 5 without any difficulty. Neither resolution is minimal.

Lemma 4.4. *Let M be a finitely generated multi-graded module over a multi-graded Noetherian ring and let*

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_s = 0$$

be a finite filtration by graded submodules. Suppose that for each i, with $0 \leqslant i \leqslant s-1$,

$$\mathbb{F}_{i,\bullet}: \quad \cdots \xrightarrow{d_{i,2}} \mathbb{F}_{i,1} \xrightarrow{d_{i,1}} \mathbb{F}_{i,0}$$

is a homogeneous resolution of M_i/M_{i+1}. Then, for each i, j, k, with $0 \leqslant i \leqslant s-1$, $1 \leqslant k \leqslant s-i-1$, and $1 \leqslant j$, there exists a homogeneous map

$$\alpha_{i,j}^{(k)} : \mathbb{F}_{i,j} \longrightarrow \mathbb{F}_{i+k,j-1}$$

such that

$$(\mathbb{M}, D): \quad \cdots \to \mathbb{M}_2 \xrightarrow{D_2} \mathbb{M}_1 \xrightarrow{D_1} \mathbb{M}_0$$

is a homogeneous resolution of M, where $\mathbb{M}_j = \bigoplus_{i=0}^{s-1} \mathbb{F}_{i,j}$ and $D_j : \mathbb{M}_j \to \mathbb{M}_{j-1}$ is the lower triangular matrix

$$D_j = \begin{bmatrix} d_{0,j} & 0 & 0 & 0 & \dots & 0 & 0 \\ \alpha_{0,j}^{(1)} & d_{1,j} & 0 & 0 & \dots & 0 & 0 \\ \alpha_{0,j}^{(2)} & \alpha_{1,j}^{(1)} & d_{2,j} & 0 & \dots & 0 & 0 \\ \alpha_{0,j}^{(3)} & \alpha_{1,j}^{(2)} & \alpha_{2,j}^{(1)} & d_{3,j} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \alpha_{0,j}^{(s-1)} & \alpha_{1,j}^{(s-2)} & \alpha_{2,j}^{(s-3)} & \alpha_{3,j}^{(s-4)} & \dots & \alpha_{s-2,j}^{(1)} & d_{s-1,j} \end{bmatrix}.$$

Proof. By iteration, it suffices to treat the case $s = 2$. In this case the proof is a graded version of the Horseshoe Lemma. □

Theorem 4.5. *Adopt the hypotheses of Data* 1.1 *and recall the resolution $(\mathbb{L}_{\boldsymbol{a},\bullet}, d_{\boldsymbol{a},\bullet})$ of Observation* 4.2*. For each triple $(\boldsymbol{a}, \boldsymbol{b}, j)$, where j is a positive integer and $\boldsymbol{b} > \boldsymbol{a}$ are eligible tuples, there exists an* fdeg*-homogeneous S-module homomorphism*

$$\alpha_{\boldsymbol{a},\boldsymbol{b},j} : \mathbb{L}_{\boldsymbol{a},j} \longrightarrow \mathbb{L}_{\boldsymbol{b},j-1},$$

such that

$$(\mathfrak{L}, D): \quad 0 \to \mathfrak{L}_s \to \dots \to \mathfrak{L}_2 \xrightarrow{D_2} \mathfrak{L}_1 \xrightarrow{D_1} \mathfrak{L}_0$$

is an fdeg*-homogeneous resolution of $K^{(n)}$, where $s = \sum_{u=1}^{\ell} \sigma_u + \ell - 2$, $\mathfrak{L}_j$ is equal to $\bigoplus_{\boldsymbol{a}} \mathbb{L}_{\boldsymbol{a},j}$, and the component*

$$\mathbb{L}_{\boldsymbol{a},j} \hookrightarrow \mathfrak{L}_j \xrightarrow{D_j} \mathfrak{L}_{j-1} \xrightarrow{\text{proj}} \mathbb{L}_{\boldsymbol{c},j-1}$$

of the map $D_j : \mathfrak{L}_j \to \mathfrak{L}_{j-1}$ is equal to

$$\begin{cases} 0 & \text{if } \boldsymbol{a} > \boldsymbol{c}, \\ d_{\boldsymbol{a},j} & \text{if } \boldsymbol{a} = \boldsymbol{c}, \\ \alpha_{\boldsymbol{a},\boldsymbol{c},j} & \text{if } \boldsymbol{c} > \boldsymbol{a}. \end{cases}$$

Proof. Consider the finite decreasing filtration $\{\mathcal{E}_{\boldsymbol{a}}\}$ of $K^{(n)}$ given in Definition 3.1(2). According to Observation 4.2(a), the successive quotients $\mathcal{E}_{\boldsymbol{a}}/\mathcal{D}_{\boldsymbol{a}}$ have fdeg-homogeneous resolutions $\mathbb{L}_{\boldsymbol{a},\bullet}$. Now apply Lemma 4.4. □

Remark. Although the resolution $\mathfrak{L}$ of $K^{(n)}$ may not be minimal, its length is the same as the projective dimension of $K^{(n)}$ as an S-module, as may be calculated from the Auslander–Buchsbaum formula. Indeed, since $n \geqslant 2$ by the hypotheses of Data 1.1, Corollary 2.6 shows that the depth of $K^{(n)}$, as an S-module, is 2 and it is clear that S has depth equal to $\sum_{u=1}^{\ell} \sigma_u + \ell$.

5. Regularity

We turn our attention to the Castelnuovo–Mumford regularity of $K^{(n)}$. In this discussion all of the variables of the polynomial ring S have degree one. In Section 1, we referred to this situation as the grading on S is given by "total degree". If M is a finitely generated non-zero graded S-module and

$$0 \to F_k \to \dots \to F_0 \to M \to 0,$$

with $F_i = \bigoplus_{j=1}^{t_i} S(-a_{i,j})$, is the minimal homogeneous resolution of M by free S-modules, then the regularity of M is equal to

$$\operatorname{reg} M = \max_{i,j}\{a_{i,j} - i\} = \max\{n \mid \mathrm{H}^i_{\mathfrak{m}}(M)_{n-i} \neq 0 \text{ for some } i \geqslant 0\},$$

where $\mathfrak{m}$ is the maximal homogeneous ideal of S. For $M = 0$ one sets $\operatorname{reg} M = -\infty$.

There are two contributions to the regularity of $K^{(n)}$. The highest generator degrees of $K^{(n)}$ and of $\mathcal{E}_{0^{\ell-1}}/\mathcal{D}_{0^{\ell-1}}$ coincide, where $0^{\ell-1}$ is the $(\ell-1)$-tuple of zeros. Also, most of the generalized Eagon–Northcott complexes $\mathbb{F}_{\boldsymbol{a},\bullet}$ are linear in all positions except one position where the maps are quadratic. The rest of the generalized Eagon–Northcott complexes are linear in all positions. For example, the generators of $\mathcal{E}_{0^{\ell-1}}/\mathcal{D}_{0^{\ell-1}}$ have degree $\lceil \frac{n}{\sigma_\ell} \rceil$ and the complex $\mathbb{L}_{0^{\ell-1},\bullet}$ contains some quadratic maps if and only if $\sigma_\ell \nmid (n-1)$. It follows that

$$\operatorname{reg}(\mathcal{E}_{0^{\ell-1}}/\mathcal{D}_{0^{\ell-1}}) = \left\{ \begin{array}{ll} \lceil \frac{n}{\sigma_\ell} \rceil + 1 & \text{if } \sigma_\ell \nmid n-1, \\ \lceil \frac{n}{\sigma_\ell} \rceil & \text{if } \sigma_\ell \mid n-1 \end{array} \right\} = \left\lceil \frac{n-1}{\sigma_\ell} \right\rceil + 1. \tag{5.1}$$

We prove in Theorem 5.5 that $\operatorname{reg} K^{(n)} = \operatorname{reg}(\mathcal{E}_{0^{\ell-1}}/\mathcal{D}_{0^{\ell-1}})$. The filtration $\{\mathcal{E}_{\boldsymbol{a}}\}$ is too fine to allow us to read the exact value of $\operatorname{reg} K^{(n)}$ directly from the factors of $\{\mathcal{E}_{\boldsymbol{a}}\}$. In order to complete our calculation of $\operatorname{reg} K^{(n)}$, we introduce a second filtration $\{\mathcal{E}'_{\boldsymbol{a}}\}$, with $\{\mathcal{E}_{\boldsymbol{a}}\}$ a refinement of $\{\mathcal{E}'_{\boldsymbol{a}}\}$.

Definition 5.2. The k-tuple $\boldsymbol{a}$ is eligible′ if $\boldsymbol{a}$ is eligible and either $k = \ell - 1$ or $r(\boldsymbol{a}) < \sigma_{k+1}$. If $\boldsymbol{a}$ is an eligible′ k-tuple, then

(1) $\mathcal{E}'_{\boldsymbol{a}} = \mathcal{E}_{\boldsymbol{a}}$, and
(2) $\mathcal{D}'_{\boldsymbol{a}} = \sum \mathcal{E}'_{\boldsymbol{b}}$, where the sum varies over all eligible′ tuples $\boldsymbol{b}$, with $\boldsymbol{b} > \boldsymbol{a}$.

Notice that the modules $\mathcal{E}'_{\boldsymbol{a}}/\mathcal{D}'_{\boldsymbol{a}}$ are exactly the factors of the filtration $\{\mathcal{E}'_{\boldsymbol{a}}\}$. The next result, about the filtration $\{\mathcal{E}'_{\boldsymbol{a}}\}$, is comparable to Theorem 3.17 about the filtration $\{\mathcal{E}_{\boldsymbol{a}}\}$. From the point of view of regularity, Proposition 5.3 says that the factors $\mathcal{E}'_{\boldsymbol{a}}/\mathcal{D}'_{\boldsymbol{a}}$ of the filtration $\{\mathcal{E}'_{\boldsymbol{a}}\}$ are either factors $\mathcal{E}_{\boldsymbol{a}}/\mathcal{D}_{\boldsymbol{a}}$ of the filtration $\{\mathcal{E}_{\boldsymbol{a}}\}$ or else have linear resolution. We delay the proof of Proposition 5.3 until after we have used the result to prove Theorem 5.5.

Proposition 5.3. *Let $\boldsymbol{a}$ be an eligible′ k-tuple. The S-module $\mathcal{E}'_{\boldsymbol{a}}/\mathcal{D}'_{\boldsymbol{a}}$ is Cohen–Macaulay and perfect.*

(a) *If $r(\boldsymbol{a}) < \sigma_{k+1}$, then $\mathcal{E}'_{\boldsymbol{a}}/\mathcal{D}'_{\boldsymbol{a}} = \mathcal{E}_{\boldsymbol{a}}/\mathcal{D}_{\boldsymbol{a}}$ and the assertions of Theorem* 3.17 *apply.*
(b) *If $r(\boldsymbol{a}) = \sigma_{k+1}$, then for some non-negative integer j there is an isomorphism of* fdeg*-graded A-modules*

$$\mathcal{E}'_{\boldsymbol{a}}/\mathcal{D}'_{\boldsymbol{a}} \cong J(A/P_jA)(-\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}, 0; -\boldsymbol{\varepsilon}),$$

where J is the A-ideal generated by the entries in the first row of $\psi_{>j}$ and $\boldsymbol{\varepsilon} = \varepsilon_{\boldsymbol{a}} + f(\boldsymbol{a})\varepsilon_{k+1}$ with $\varepsilon_{\boldsymbol{a}}$ as defined in (3.5)*. Furthermore, the complex $\mathbb{L}_{j,\bullet}(-\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon} - 1, 1; -\boldsymbol{\varepsilon})$ of Observation* 4.2(b) *is a resolution of $\mathcal{E}'_{\boldsymbol{a}}/\mathcal{D}'_{\boldsymbol{a}}$. If all of the variables of S are given degree one, then the minimal homogeneous S-resolution of $\mathcal{E}'_{\boldsymbol{a}}/\mathcal{D}'_{\boldsymbol{a}}$ is linear.*
(c) *The modules $\mathcal{E}'_{0^{\ell-1}}/\mathcal{D}'_{0^{\ell-1}}$ and $\mathcal{E}_{0^{\ell-1}}/\mathcal{D}_{0^{\ell-1}}$ are equal.*

Lemma 5.4. *Let R be a standard graded Noetherian ring over a field, M a non-zero finitely generated graded R-module, and $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_s = 0$ a finite filtration by graded submodules with factors $N_i = M_i/M_{i+1}$. If $\operatorname{reg} N_0 \geqslant \operatorname{reg} N_i$ for every i, then $\operatorname{reg} M = \operatorname{reg} N_0$ and $\operatorname{depth} M \leqslant \dim N_0$.*

Proof. Notice that $\operatorname{reg} M_1 \leqslant \operatorname{reg} N_0$ and $\operatorname{reg} M \leqslant \operatorname{reg} N_0$. Let $\mathfrak{m}$ be the maximal homogeneous ideal of R and let d be such that $[\mathrm{H}^d_{\mathfrak{m}}(N_0)]_{\operatorname{reg} N_0 - d} \neq 0$. Clearly $d \leqslant \dim N_0$. We claim that $[\mathrm{H}^d_{\mathfrak{m}}(M)]_{\operatorname{reg} N_0 - d} \neq 0$, which gives $\operatorname{reg} M \geqslant \operatorname{reg} N_0$ as well as $\operatorname{depth} M \leqslant d \leqslant \dim N_0$.

Suppose $[\mathrm{H}^d_{\mathfrak{m}}(M)]_{\operatorname{reg} N_0 - d} = 0$, then the short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow N_0 \longrightarrow 0$$

induces an embedding

$$0 \neq \big[\mathrm{H}^d_{\mathfrak{m}}(N_0)\big]_{\operatorname{reg} N_0 - d} \hookrightarrow \big[\mathrm{H}^{d+1}_{\mathfrak{m}}(M_1)\big]_{\operatorname{reg} N_0 - d}.$$

Hence $[\mathrm{H}^{d+1}_{\mathfrak{m}}(M_1)]_{\operatorname{reg} N_0 - d} \neq 0$, which gives $\operatorname{reg} M_1 \geqslant \operatorname{reg} N_0 + 1$. □

Theorem 5.5. *Adopt the hypotheses of Data* 1.1. *Then* $\operatorname{reg} K^{(n)} = \lceil \frac{n-1}{\sigma_\ell} \rceil + 1$.

Proof. Consider the finite filtration $\{\mathcal{E}'_{\boldsymbol{a}}\}$ of $K^{(n)}$ as described in Definition 5.2. The factors of this filtration are denoted $\mathcal{E}'_{\boldsymbol{a}}/\mathcal{D}'_{\boldsymbol{a}}$, as $\boldsymbol{a}$ varies over all eligible′ tuples. Notice that $0^{\ell-1}$ is the smallest eligible′-tuple and $K^{(n)}/\mathcal{D}'_{0^{\ell-1}} = \mathcal{E}'_{0^{\ell-1}}/\mathcal{D}'_{0^{\ell-1}}$ has regularity $\lceil \frac{n-1}{\sigma_\ell} \rceil + 1$ by Proposition 5.3(c) and (5.1). Hence by Lemma 5.4 it suffices to show that $\operatorname{reg}(\mathcal{E}'_{\boldsymbol{a}}/\mathcal{D}'_{\boldsymbol{a}}) \leqslant \lceil \frac{n-1}{\sigma_\ell} \rceil + 1$ for every eligible′ k-tuple $\boldsymbol{a}$.

The module $\mathcal{E}'_{\boldsymbol{a}}/\mathcal{D}'_{\boldsymbol{a}}$ is generated in degree $\sum a_i + f(\boldsymbol{a}) + 1$ and hence, according to Proposition 5.3, has regularity at most

$$\begin{cases} \sum a_i + f(\boldsymbol{a}) + 2, & \text{if } r(\boldsymbol{a}) < \sigma_{k+1}, \\ \sum a_i + f(\boldsymbol{a}) + 1, & \text{if } r(\boldsymbol{a}) = \sigma_{k+1}. \end{cases}$$

If $r(\boldsymbol{a}) < \sigma_{k+1}$, then $\sum a_i\sigma_i + f(\boldsymbol{a})\sigma_{k+1} < n - 1$. The hypothesis $\sigma_1 \geqslant \cdots \geqslant \sigma_\ell$ ensures that $\sum a_i + f(\boldsymbol{a}) < \frac{n-1}{\sigma_\ell}$, and hence $\operatorname{reg}(\mathcal{E}'_{\boldsymbol{a}}/\mathcal{D}'_{\boldsymbol{a}}) \leqslant \lceil \frac{n-1}{\sigma_\ell} \rceil + 1$. On the other hand, if $r(\boldsymbol{a}) = \sigma_{k+1}$, then $\sum a_i + f(\boldsymbol{a}) \leqslant \frac{n-1}{\sigma_\ell}$, and we still have $\operatorname{reg}(\mathcal{E}'_{\boldsymbol{a}}/\mathcal{D}'_{\boldsymbol{a}}) \leqslant \lceil \frac{n-1}{\sigma_\ell} \rceil + 1$. □

Remark. Recall that $\mathcal{E}'_{0^{\ell-1}}/\mathcal{D}'_{0^{\ell-1}} = \mathcal{E}_{0^{\ell-1}}/\mathcal{D}_{0^{\ell-1}}$ according to Proposition 5.3(c). Since the latter module has dimension two, Lemma 5.4 and the proof of Theorem 5.5 yield an alternative proof of Corollary 2.6, the fact that $\operatorname{depth} K^{(n)} = 2$ for $n \geqslant 2$.

We begin our proof of Proposition 5.3 by making a more detailed study of the totally ordered set of all eligible tuples. In particular, sometimes it is clear when a pair of eligible tuples is adjacent.

Notation 5.6. If $\boldsymbol{a}$ is an eligible k-tuple with $k < \ell - 1$, then let $N(\boldsymbol{a})$ be the $(k+1)$-tuple $(\boldsymbol{a}, f(\boldsymbol{a}))$. If $2 \leqslant h < \ell - k$, then let $N^h(\boldsymbol{a}) = N(N^{h-1}(\boldsymbol{a}))$. We let N^0 denote the identity function.

Lemma 5.7. *Let* $\boldsymbol{a}$ *be an eligible k-tuple with* $k < \ell - 1$.

(a) *The* $(k+1)$*-tuple* $N(\boldsymbol{a})$ *is eligible and the eligible tuples* $\boldsymbol{a} > N(\boldsymbol{a})$ *are adjacent in the sense that if* $\boldsymbol{b}$ *is an eligible tuple with* $\boldsymbol{a} \geqslant \boldsymbol{b} \geqslant N(\boldsymbol{a})$, *then either* $\boldsymbol{a} = \boldsymbol{b}$ *or* $\boldsymbol{b} = N(\boldsymbol{a})$.
(b) *If* $r(\boldsymbol{a}) = \sigma_{k+1}$, *then* $f(N(\boldsymbol{a})) = 0$ *and* $r(N(\boldsymbol{a})) = \sigma_{k+2}$.

Proof. It is clear that

$$\sum_{u=1}^{k+1} N(\boldsymbol{a})_u \sigma_u = \sum_{u=1}^{k} a_u\sigma_u + f(\boldsymbol{a})\sigma_{k+1} < n.$$

We conclude that $N(\boldsymbol{a})$ is an eligible $(k+1)$-tuple. Suppose that $\boldsymbol{b}$ is an eligible j-tuple with $\boldsymbol{a} \gneqq \boldsymbol{b} \geqslant N(\boldsymbol{a})$. Since $\boldsymbol{a} \geqslant \boldsymbol{b} \geqslant N(\boldsymbol{a})$ we have $j \geqslant k$ and $b_u = a_u$ for $u \leqslant k$. As $\boldsymbol{a} \gneqq \boldsymbol{b}$ we also have $j > k$. Now the inequality $\boldsymbol{b} \geqslant N(\boldsymbol{a})$ implies $b_{k+1} \geqslant f(\boldsymbol{a})$. Finally, the definition of $f(\boldsymbol{a})$ ensures $b_{k+1} \leqslant f(\boldsymbol{a})$. Therefore,

$b_{k+1} = f(\boldsymbol{a})$ and then $j = k+1$, again because $\boldsymbol{b} \geqslant N(\boldsymbol{a})$. It follows that $\boldsymbol{b} = N(\boldsymbol{a})$. Assertion (a) is established.

The hypothesis of (b) yields

$$\sigma_{k+1} = \sum_{u=1}^{k} a_u \sigma_u + \big(f(\boldsymbol{a}) + 1\big)\sigma_{k+1} - n + 1;$$

hence,

$$n - 1 = \sum_{u=1}^{k} a_u \sigma_u + f(\boldsymbol{a})\sigma_{k+1} = \sum_{u=1}^{k+1} N(\boldsymbol{a})_u \sigma_u.$$

We now see that $f(N(\boldsymbol{a})) = 0$ and

$$r\big(N(\boldsymbol{a})\big) = \sum_{u=1}^{k+1} N(\boldsymbol{a})_u \sigma_u + \big(f(N(\boldsymbol{a})) + 1\big)\sigma_{k+2} - n + 1 = \sigma_{k+2}. \qquad \square$$

Proof of Proposition 5.3. Once items (a) and (b) are shown, then Observation 4.2 implies that the modules $\mathcal{E}'_{\boldsymbol{a}}/\mathcal{D}'_{\boldsymbol{a}}$ are Cohen–Macaulay, hence perfect.

(a) It suffices to show that $\mathcal{D}'_{\boldsymbol{a}} = \mathcal{D}_{\boldsymbol{a}}$. Let $\boldsymbol{b} > \boldsymbol{a}$ be the eligible tuple which is adjacent to $\boldsymbol{a}$. It suffices to show that $\boldsymbol{b}$ is eligible′. Suppose that $\boldsymbol{b}$ is a j-tuple. If $\boldsymbol{b}$ is not eligible′, then $j < \ell - 1$ and $r(\boldsymbol{b}) = \sigma_{j+1}$. Now, Lemma 5.7 shows $r(\boldsymbol{a}) = \sigma_{k+1}$ and this contradicts the hypothesis.

(b) Notice that k is necessarily equal to $\ell - 1$. Identify the largest non-negative integer s for which there exists an eligible $(\ell - 1 - s)$-tuple $\boldsymbol{b}$ with $\boldsymbol{a} = N^s(\boldsymbol{b})$ and $r(\boldsymbol{b}) = \sigma_{\ell - s}$. Let $j = \ell - 1 - s$. We know, from Lemma 5.7, that

$$\boldsymbol{b} > N(\boldsymbol{b}) > N^2(\boldsymbol{b}) > \cdots > N^s(\boldsymbol{b}) = \boldsymbol{a}$$

are adjacent eligible tuples and that if $0 \leqslant h \leqslant s-1$, then $N^h(\boldsymbol{b})$ is not eligible′. Furthermore, for each integer h, with $1 \leqslant h \leqslant s$, we have

$$f\big(N^h(\boldsymbol{b})\big) = 0 \quad \text{and} \quad r\big(N^h(\boldsymbol{b})\big) = \sigma_{j+1+h}. \tag{5.8}$$

The module $\mathcal{E}'_{\boldsymbol{a}}/\mathcal{D}_{\boldsymbol{b}}$ is defined to be

$$\sum_{h=0}^{s} T^{N^h(\boldsymbol{b})} T_{j+1+h,1}^{f(N^h(\boldsymbol{b}))} (T_{j+1+h,1}, \ldots, T_{j+1+h, r(N^h(\boldsymbol{b}))})(S/\mathcal{D}_{\boldsymbol{b}}).$$

The calculations of (5.8) show that

$$T^{\boldsymbol{b}} T_{j+1,1}^{f(\boldsymbol{b})} = T^{N^h(\boldsymbol{b})} T_{j+1+h,1}^{f(N^h(\boldsymbol{b}))} = T^{\boldsymbol{a}} T_{k+1,1}^{f(\boldsymbol{a})},$$

for $1 \leqslant h \leqslant s$, and

$$\mathcal{E}'_{\boldsymbol{a}}/\mathcal{D}_{\boldsymbol{b}} = T^{\boldsymbol{b}} T_{j+1,1}^{f(\boldsymbol{b})} J(S/\mathcal{D}_{\boldsymbol{b}}),$$

where J is generated by the entries in the first row of $\psi_{>j}$. We also know that $\boldsymbol{\varepsilon}$, which is defined to be $\varepsilon_{\boldsymbol{a}} + f(\boldsymbol{a})\varepsilon_{k+1}$, is equal to $\varepsilon_{\boldsymbol{b}} + f(\boldsymbol{b})\varepsilon_{j+1}$. Lemma 3.7(b) shows that P_j annihilates $T^{\boldsymbol{b}} T_{j+1,1}^{f(\boldsymbol{b})} J(S/\mathcal{D}_{\boldsymbol{b}})$. Apply Lemma 3.16 to the ideal $\mathcal{J} = T_{j+1,1}^{f(\boldsymbol{b})} J$ to see that

$$\begin{aligned}\mathcal{E}'_{\boldsymbol{a}}/\mathcal{D}_{\boldsymbol{b}} &= T^{\boldsymbol{b}} T_{j+1,1}^{f(\boldsymbol{b})} J(S/\mathcal{D}_{\boldsymbol{b}}) \cong T_{j+1,1}^{f(\boldsymbol{b})} J(A/P_jA)(-\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_{\boldsymbol{b}}, 0; -\boldsymbol{\varepsilon}_{\boldsymbol{b}})\\ &\cong J(A/P_jA)(-\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}, 0; -\boldsymbol{\varepsilon}).\end{aligned}$$

The final isomorphism holds because $T_{j+1,1}$ is a non-zero element in the domain A/P_jA.

Let $\boldsymbol{c}$ be the eligible i-tuple so that $\boldsymbol{c} > \boldsymbol{b}$ are adjacent. If $\boldsymbol{c}$ is not eligible′, then $i < \ell - 1$ and $r(\boldsymbol{c}) = \sigma_{i+1}$. Lemma 5.7(a) then says that $\boldsymbol{b} = N(\boldsymbol{c})$ and this contradicts the choice of s. Thus, $\boldsymbol{c}$ is eligible′, $\mathcal{D}'_{\boldsymbol{a}} = \mathcal{D}_{\boldsymbol{b}}$, and the proof is complete.

(c) Notice that $0^{\ell-1}$ is an eligible′-tuple. Let $\boldsymbol{b}$ be the eligible j-tuple with $\boldsymbol{b} > 0^{\ell-1}$ and $\boldsymbol{b}$ adjacent to $0^{\ell-1}$. It suffices to show that $\boldsymbol{b}$ is eligible′. If $\boldsymbol{b}$ is not eligible′, then $j < \ell - 1$ and $r(\boldsymbol{b}) = \sigma_{j+1}$. Lemma 5.7 then shows that $N(\boldsymbol{b}) = 0^{\ell-1}$; hence,

$$f\bigl(0^{\ell-1}\bigr) = f\bigl(N(\boldsymbol{b})\bigr) = 0 \quad\text{and}\quad r\bigl(0^{\ell-1}\bigr) = r\bigl(N(\boldsymbol{b})\bigr) = \sigma_\ell.$$

The definition of r now gives $\sigma_\ell = r(0^{\ell-1}) = \sigma_\ell - n + 1$; or $n = 1$, which is a violation of the ambient hypotheses of Data 1.1. □

6. Symbolic Rees algebra

Retain the notation of Data 1.1.

Proposition 6.1. *The symbolic Rees algebra*

$$\mathcal{R}_s(K) = \bigoplus_{n\geqslant 0} K^{(n)}$$

is finitely generated as an A-algebra.

Proof. View $\mathcal{R}_s(K)$ as the A-subalgebra of the polynomial ring $A[u]$ which is generated by

$$\bigcup_{n=1}^{\infty}\bigl\{\theta u^n \bigm| \theta \in K^{(n)}\bigr\}.$$

Let $\mathcal{S}$ be the following finite subset of $A[u]$,

$$\mathcal{S} = \bigl\{T_{i,j}u^m \bigm| 1 \leqslant i \leqslant \ell,\ 1 \leqslant j \leqslant \sigma_i,\ \text{and } 1 \leqslant m \leqslant \sigma_i + 1 - j\bigr\}.$$

We prove that $\mathcal{R}_s(K)$ is generated as an A-algebra by $\mathcal{S}$. Clearly $\mathcal{S}$ in contained in $\mathcal{R}_s(K)$, as can be seen from Theorem 1.5. Conversely, suppose that θ belongs to the generating set of $K^{(n)}$ given in Definition 1.17. There is an eligible k-tuple $\boldsymbol{a}$ with $\theta = T^{\boldsymbol{a}} T_{k+1,1}^{f(\boldsymbol{a})} T_{k+1,j}$. We have

$$\sum_{u=1}^{k} a_u\sigma_u + f(\boldsymbol{a})\sigma_{k+1} < n \quad\text{and}\quad 1 \leqslant j \leqslant r(\boldsymbol{a}) \leqslant \sigma_{k+1}.$$

Thus,

$$\theta u^n = \prod_{i=1}^{k}\bigl(T_{i,1}u^{\sigma_i}\bigr)^{a_i}\bigl(T_{k+1,1}u^{\sigma_{k+1}}\bigr)^{f(\boldsymbol{a})} T_{k+1,j}u^{\sigma_{k+1}+1-r(\boldsymbol{a})} \in A[\mathcal{S}]. \qquad \square$$

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