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Successive Approximation Method for Quasilinear Impulsive Differential Equations with Control

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Abstract—We introduce a technique to define successive approximations to solutions of the control problem with implulse actions on surfaces

$$\begin{aligned} \frac{dx}{dt} &= A(t)x(t) + C(t)u + f(t) + \mu g(t,x,u,\mu), && t \neq \zeta_i, \\ \Delta x(\zeta_i) &= B_i x + D_i v_i + J_i + \mu W_i(x, v_i, \mu), && i = 1,2,\dots,p, \\ & x(\alpha) = a, \qquad x(\beta) = b, \end{aligned}$$

where μ is a small positive parameter, $\zeta_i = \theta_i + \mu\tau_i(x(\zeta_i),\mu)$, $x \in R^n$ and $\Delta x(\theta) := x(\theta+) - x(\theta)$. A sequence of piecewise continuous functions with discontinuities of the first kind that converges to a solution of the above problem is constructed.

Keywords—Impulse, Control, Quasilinear system, Successive approximation.

1. INTRODUCTION

Let α and β be fixed real numbers such that $\alpha < \beta$, and r and p be fixed positive integers. Denote by $L_2^r[\alpha,\beta]$ the set of all square integrable and bounded functions $\eta : [\alpha,\beta] \to R^r$ and by $D^r[1,p]$ the set of all finite sequences $\{\xi_i\}$, $\xi_i \in R^r$, $i = 1,2,\dots,p$. We define a space $\Pi^r = L_2^r \times D^r$ and denote its elements by $\{\eta,\xi\}$.

We consider the controllability problem of solutions of differential equations with impulse actions on surfaces of the form

$$\begin{aligned} \frac{dx}{dt} &= A(t)x(t) + C(t)u + f(t) + \mu g(t,x,u,\mu), && t \neq \zeta_i, \\ \Delta x(\zeta_i) &= B_i x + D_i v_i + J_i + \mu W_i(x, v_i, \mu), && i = 1,2,\dots p, \end{aligned} \tag{1}$$

subject to

$$x(\alpha) = a, \qquad x(\beta) = b, \tag{2}$$

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where μ is a small positive parameter, $\zeta_i = \theta_i + \mu\tau_i(x(\zeta_i), \mu)$, $x \in R^n$ and $\Delta x(\theta) := x(\theta+) - x(\theta)$.

With regard to (1),(2), we assume without further mention that the following conditions are satisfied:

(C1) A and C are matrix functions of the sizes $(n \times n)$ and $(n \times m)$, respectively, the elements of which belong to $L_2^1[\alpha, \beta]$,

(C2) $\{\theta_i\}$, $i = 1, 2, \ldots, p$, is a strictly increasing sequence of real numbers in (α, β),

(C3) B_i and D_i are, respectively, $(n \times n)$ and $(n \times m)$ constant matrices with $\det(I + B_i) \neq 0$, $i = 1, 2, \ldots, p$,

(C4) $\{f, J\} \in \Pi^n[\alpha, \beta]$,

(C5) g, W_i, and τ_i are continuous and continuously differentiable functions in x, u, and v.

DEFINITION 1. *The problem (1),(2), which we shall denote by Σ_μ, is said to be solvable if given any bounded set $G \subset R^n$ there exists a positive $\mu_0 \in R$, $\mu_0 = \mu_0(G)$, such that for all arbitrary $a, b \in G$ and $\mu < \mu_0$ there is a control $\{u, v\} \in \Pi^m$ for which system (1) admits a solution $x(t)$ satisfying (2).*

It should be noted that the system under investigation involves impulse effects at nonfixed points, and due to this fact it is possible for the integral curve of our system to meet more than one time, and even infinitely many times, one and the same surface of discontinuity. This phenomenon is called beating [1]. Clearly, investigation of systems with impulse actions on surfaces needs conditions for the absence of beating. In [2], we have shown that if μ is sufficiently small then beating is not possible. Therefore, we may assume in this paper that beating is absent.

2. NOTATIONS AND FORMULATION OF THE PROBLEM

Let $X(t), X(\alpha) = I$, be a fundamental matrix of

$$\frac{dx}{dt} = A(t)x, \qquad t \neq \theta_i,$$
$$\Delta x|_{t=\theta_i} = B_i x,$$

and define

$$\Psi(t) = \int_\alpha^t Q(t)Q^\top(t)\,dt + \sum_{\alpha<\theta_i<t} P_i P_i^\top,$$

where $Q(t) = X^{-1}(t)C(t)$ and $P_i = X^{-1}(\theta_i+)D_i$.

Let s be a positive real number and $|\cdot|$ the euclidean norm in R^n. We denote by Π_s the subspace of elements (x, u, v) satisfying the inequality $|x| + |u| + |v| \leq s$ and let

$$G_s = \{(x, u, v, t, i, \mu) : (x, u, v) \in \Pi_s,\ \alpha \leq t \leq \beta,\ i = 1, 2, \ldots, p,\ \mu \leq \mu_1\},$$

where μ_1 is a fixed positive real number.

We fix a positive real number H, and let

$$m_1 = \max\left\{\sup_t |A(t)|, \sup_t |C(t)|, \max_i |B_i|\right\},$$

$$m_2 = \max\left\{\sup_t |f(t)|, \max_i |J_i|\right\},$$

$$m_3 = \max\left\{\max_{G_H} |g|, \max_{G_H} |W|, \max_{G_H} |\tau|\right\},$$

$$m_4 = \max\left\{\max_{t,s} \left|Q(t)^\top \Psi^{-1}(\beta) X^{-1}(s)\right|, \max_{i,s} \left|P_i^\top \Psi^{-1}(\beta) X^{-1}(s)\right|,\right.$$
$$\left.\max_{t,s} \left|X(t)X^{-1}(s)\right|, \max_{t,s} \left|X(t)\Psi(t)\Psi^{-1}(\beta)X^{-1}(s)\right|\right\}.$$

In view of (C5), there exists an $L > 0$ such that

$$\left|g\left(t, x_1, u_1, v^1, \mu\right) - g\left(t, x_2, u_2, v^2, \mu\right)\right| \le L\left\{|x_1 - x_2| + |u_1 - u_2| + \left|v^1 - v^2\right|\right\},$$
$$\left|W_i\left(x_1, v^1, \mu\right) - W_i\left(x_2, v^2, \mu\right)\right| \le L\left\{|x_1 - x_2| + \left|v^1 - v^2\right|\right\},$$
$$|\tau_i(x_1, \mu) - \tau_i(x_2, \mu)| \le L\,|x_1 - x_2|,$$

uniformly for $t, \mu \in G_H$.

We also let

$$\begin{aligned}
h_1(H,\mu) &= 2m_1H + m_2 + \mu m_3,\\
h_2(H,\mu) &= L(1 - \mu L h_1(H,\mu))^{-1},\\
\delta_1(\mu) &= m_3(m_1 + \mu L)(2 + m_1) + L,\\
\delta_2(\mu) &= (m_1 + \mu L)(m_3(1 + m_1) + h_1(H,\mu)h_2(H,\mu)) + 2(m_1H + \mu m_3)h_2(H,\mu) + L,\\
\delta_3(\mu) &= 4m_4(L(\beta - \alpha) + p\delta_1(\mu)),\\
\delta_4(\mu) &= 4m_4 p\delta_2(\mu),\\
\delta_5(\mu) &= 1 + \mu m_3(1 + \mu L),\\
\delta_6(\mu) &= m_3(m_1 + \mu L) + h_1(H,\mu)h_2(H,\mu).
\end{aligned}$$

To investigate system (1), we have used a comparison with the following system:

$$\begin{aligned}
\frac{dx}{dt} &= A(t)x(t) + C(t)u + f(t) + \mu g(t, x, u, \mu), && t \ne \theta_i,\\
\Delta x|_{t=\theta_i} &= B_i x + D_i v_i + J_i + S_i(x, u, v, \mu), && i = 1, 2, \ldots p,
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
S_i(x, u, v, \mu) &= (I + B_i)\int_{\theta_i}^{\zeta_i} \left[A(t)x_0(t) + C(t)u(t) + f(t) + \mu g(t, x_0(t), u(t), \mu)\right] dt\\
&+ \mu W_i(x_0(\zeta_i), v_i, \mu) + \int_{\zeta_i}^{\theta_i} \left[A(t)x_1(t) + C(t)u(t) + f(t) + \mu g(t, x_1(t), u(t), \mu)\right] dt.
\end{aligned} \tag{4}$$

In (4), $x_0(t)$ is a solution of system

$$\frac{dx}{dt} = A(t)x(t) + C(t)u + f(t) + \mu g(t, x, u, \mu) \tag{5}$$

satisfying $x_0(\theta_i) = x$, $t = \zeta_i$ is the instant of meeting of solution $x_0(t)$ with the surface $t = \theta_i + \mu\tau_i(x, \mu)$, and $x_1(t)$ is a solution of (5) such that $x_1(\zeta_i) = (I + B_i)x_0(\zeta_i) + D_i v_i + J_i + \mu W_i(x_0(\zeta_i), v_i, \mu)$. We denote the control problem (3),(2) by γ_μ.

As in [1], one can easily show that the systems (1) and (3) have a property Ω in G_H in the sense described below. Without any loss of generality, we let $\zeta_i \ge \theta_i$ for $i = 1, 2, \ldots, p$.

DEFINITION 2. *The systems (1) and (3) are said to enjoy a property Ω in G_H if for a fixed positive real number $h < H$ and a sufficiently small μ, it is true that: given any solution $x(t)$ of (1), $|x(t)| < h$, $t \in [\alpha, \beta]$, there is a solution $y(t)$ of (3), $|y(t)| < H$, such that $x(t) = y(t)$ for all $t \in [\alpha, \beta]$ except possibly at points $t \in [\theta_i, \zeta_i]$, $i = 1, 2, \ldots, p$, and conversely, given any solution $y(t)$ of (3), $|y(t)| < h$, $t \in [\alpha, \beta]$, there is a solution $x(t)$ of (1), $|x(t)| < H$, such that $x(t) = y(t)$ for all $t \in [\alpha, \beta]$ except possibly at points $t \in [\theta_i, \zeta_i]$, $i = 1, 2, \ldots, p$.*

By the help of problem γ_μ and Definition 2, the following result was obtained in [2].

THEOREM 1. *If $\Psi(\beta)$ is nonsingular, then Σ_μ is solvable and the solution is the limit of a uniformly convergent sequence obtained by the method of successive approximations.*

This theorem provides conditions for the existence of a solution of problem Σ_μ. However, since no method was described in [2] for obtaining the successive approximations, it cannot be used effectively. Therefore, our objective in this paper is to construct a method for obtaining these successive approximations and thereby complement the results obtained in [2].

3. THE METHOD OF SUCCESSIVE APPROXIMATIONS

We define

$$u_0(t) = Q^\top(t)\Psi(\beta)^{-1}K + \hat{u}(t), \qquad v_i^0 = P_i^\top \Psi(\beta)^{-1}K + \hat{v}_i,$$

$$x_0(t) = X(t)\left\{a + \int_\alpha^t \left[Q(s)u_0(s) + X^{-1}(s)f(s)\right] ds + \sum_{\alpha<\theta_i<t} P_i v_i^0 + X^{-1}(\theta_i)J_i\right\},$$

$$\phi_0(t) = \left(x_0(t), u_0(t), v_i^0\right), \qquad \phi(t) = (x(t), u(t), v_i),$$

$$\kappa(t, \phi, \mu) = \int_\alpha^t X^{-1}(s)g(s, x(s), u(s), \mu)\, ds,$$

$$\psi(t, \phi, \mu) = \frac{1}{\mu}\sum_{\alpha<\theta_i<t} X^{-1}(\theta_i)S_i(x(\theta_i), v_i, \mu),$$

where $\{\hat{u}, \hat{v}\} \in \Pi^m$ is orthogonal to all columns of $[Q^\top, P_i^\top]$ and

$$K = X^{-1}(\beta)b - X^{-1}(\alpha)a - \int_\alpha^\beta X^{-1}(t)f(t)\, dt - \sum_{i=1}^p X^{-1}(\theta_i+)J_i.$$

It follows that solution ϕ of γ_μ satisfies

$$\phi = \phi_0 + \mu\mathcal{P}(\phi, \mu), \tag{6}$$

where $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}^i)$,

$$\mathcal{P}_1(t, \phi) = X(t)\left[\kappa(t, \phi, \mu) + \psi(t, \phi, \mu) - \Psi(t)\Psi^{-1}(\beta)(\kappa(\beta, \phi, \mu) + \psi(\beta, \phi, \mu))\right],$$

$$\mathcal{P}_2(t, \phi) = Q(t)^\top \Psi^{-1}(\beta)\left[\kappa(\beta, \phi, \mu) + \psi(\beta, \phi, \mu)\right],$$

$$\mathcal{P}^i(\phi) = P_i^\top \Psi^{-1}(\beta)\left[\kappa(\beta, \phi, \mu) + \psi(\beta, \phi, \mu)\right].$$

We refer the reader to Theorem 3 in [2] for further details.

We shall first consider the construction of successive approximations $\{\varphi_n(t)\}$ for problem γ_μ.

We introduce the norm $\|\cdot\|$ defined by

$$\|\phi\| = \max_t |x(t)| + \max_t |u(t)| + \max_i \left|v^i\right|,$$

in the space of all functions $\phi(t)$ of the form $\phi(t) = (x(t), u(t), v_i)$.

Without any loss of generality, we may assume that $\tau_i \geq 0$ for $i = 1, 2, \dots, p$.

We take $\varphi_0(t) = \phi_0(t)$ as the first approximation and suppose that $\|\varphi_0\| \leq h$, where $h < H$ is fixed. Let $x_0^i(t) = x(t, \theta_i, x_0(\theta_i))$ be a solution of equation

$$\frac{dx}{dt} = A(t)x(t) + C(t)u_0(t) + f(t) + \mu g(t, x, u_0, \mu) \tag{7}$$

defined for $t \geq \theta_i$, and let θ_i^0 be the moment of meeting of $x_0^i(t)$ with the surface $t = \theta_i + \mu\tau_i(x, \mu)$. We claim that $|x_0^i(t)| < H$ for all $t \in [\theta_i, \theta_i + \mu m_3]$ if μ is sufficiently small. To see this fact, we first fix a positive real number $\mu_2 \leq \mu_1$ such that

$$\mu_2 m_3 h_1(H, \mu_2) < H - h. \tag{8}$$

Next, let $\mu \leq \mu_2$ and assume on the contrary that there exists a $t^* \in [\theta_i, \theta_i + \mu m_3)$ such that $|x_0^i(t)| < H$ for all $t \in [\theta_i, t^*)$ but $|x_0^i(t^*)| = H$. Since $x_0^i(t)$ is a solution of (7) and (8) is satisfied, we obtain $|x_0^i(t^*)| \leq h + \mu_2 m_3 h_1(H, \mu_2) < H$, a contradiction.

Assume that the approximation $\varphi_n(t) = (x_n(t), u_n(t), v_i^n)$ is available and $\|\varphi_n\| \leq h$. To determine $\varphi_{n+1}(t)$ we proceed as follows. Let $x_n^i(t) = x(t, \theta_i, x_n(\theta_i))$ be a solution of the equation

$$\frac{dx}{dt} = A(t)x_{n-1}^i(t) + C(t)u_n(t) + f(t) + \mu g(t, x_{n-1}^i, u_n(t), \mu) \tag{9}$$

defined for $t \geq \theta_i$, and let θ_i^n be the moment of meeting of solution $x_n^i(t)$ with the surface $t = \theta_i + \mu\tau_i(x, \mu)$. We may assume that $|x_{n-1}^i(t)| < H$ for all $t \in [\theta_i, \theta_i + \mu m_3]$. Then it follows as above that $|x_n^i(t)| < H$ for all $t \in [\theta_i, \theta_i + \mu m_3]$. We set

$$\begin{aligned} S_i^{n+1}(\varphi_n, \mu) &= (I + B_i)\int_{\theta_i}^{\theta_i^n} \left[A(t)x_n^i(t) + C(t)u_n(t) + f(t) + \mu g\left(t, x_n^i(t), u_n(t), \mu\right)\right] dt \\ &\quad + \mu W_i\left(x_n^i(\dot{\theta}_i), v_i^n, \mu\right) + \int_{\theta_i^n}^{\theta_i} [A(t)x_n(t) + C(t)u_n(t) + f(t) \\ &\quad + \mu g(t, x_n(t), u_n(t), \mu)]\, dt. \end{aligned} \tag{10}$$

$$\kappa_{n+1}(t, \varphi_n, \mu) = \int_\alpha^t X^{-1}(s) g(s, x_n(s), u_n(s), \mu)\, ds, \tag{11}$$

$$\psi(t, \varphi_{n+1}, \mu) = \frac{1}{\mu}\sum_{\alpha<\theta_i<t} X^{-1}(\theta_i+) S_i^{n+1}(\varphi_n, \mu), \tag{12}$$

$$\begin{aligned} \mathcal{P}_1^{n+1}(t, \varphi_n) &= X(t)\left[\kappa_{n+1}(t, \varphi_n, \mu) + \psi_{n+1}(t, \varphi_n, \mu)\right. \\ &\quad \left. - \Psi(t)\Psi^{-1}(\beta)(\kappa_{n+1}(\beta, \varphi_n, \mu) + \psi_{n+1}(\beta, \varphi_n, \mu))\right], \end{aligned} \tag{13}$$

$$\mathcal{P}_2^{n+1}(t, \varphi_n) = Q(t)^\top \Psi^{-1}(\beta)\left[\kappa_{n+1}(\beta, \varphi_n, \mu) + \psi_{n+1}(\beta, \varphi_n, \mu)\right], \tag{14}$$

$$\mathcal{P}_{n+1}^i(\varphi_n) = P_i^\top \Psi^{-1}(\beta)\left[\kappa_{n+1}(\beta, \varphi_n, \mu) + \psi(\beta, \phi, \mu)\right], \tag{15}$$

$$\mathcal{P}_{n+1}(t, \varphi_n) = \left(\mathcal{P}_1^{n+1}(t, \varphi_n), \mathcal{P}_2^{n+1}(t, \varphi_n), \mathcal{P}_{n+1}^i(\varphi_n)\right), \tag{16}$$

and define

$$\varphi_{n+1}(t) = \varphi_0(t) + \mu\mathcal{P}_{n+1}(t, \varphi_n). \tag{17}$$

It follows from (10) through (17) that

$$\|\varphi_{n+1}\| \leq \|\varphi_0\| + 4\mu m_3 m_4(\beta - \alpha + 1 + (2 + m_1)h_1(H, \mu_2)). \tag{18}$$

Choosing $\mu_3 \leq \mu_2$ as a positive real number satisfying

$$4\mu_3 m_3 m_4(\beta - \alpha + 1 + (2 + m_1)h_1(H, \mu_2)) \leq h - \|\varphi_0\|,$$

we can easily deduce from (18) that $\|\varphi_{n+1}\| < h$ for all $\mu \leq \mu_3$.

Thus, we have constructed a sequence $\{\varphi_n\}$. We will show below that the sequence $\{\varphi_n\}$ is uniformly convergent.

Assume without any loss of generality that the sequence $\{\theta_n^i\}$ is nondecreasing, and let

$$a_k = \|\varphi_k - \varphi_{k-1}\|, \quad b_k = \max_i \max_{\theta_i \leq t \leq \mu m_3} \left|x_k^i(t) - x_{k-1}^i(t)\right|, \qquad k = 1, 2, \dots.$$

It is not difficult to see from $\theta_i^{k+1} - \theta_i^k = \mu\left[\tau_i(x_{k+1}^i(\theta_i^{k+1}), \mu) - \tau_i(x_k^i(\theta_i^k), \mu)\right]$ that

$$\theta_i^{k+1} - \theta_i^k \leq \mu h_2(H, \mu) b_{k+1}. \tag{19}$$

We shall now estimate a_{k+1} and b_{k+1} in terms of a_k and b_k. We first notice from (10) that

$$\begin{aligned} S_i^{k+1}(\varphi_k,\mu) - S_i^k(\varphi_k,\mu) = (I+B_i)\int_{\theta_i}^{\theta_i^{k-1}} & \left[A(t)\left(x_k^i(t) - x_{k-1}^i(t)\right) + C(t)\left(u_k(t) - u_{k-1}(t)\right)\right. \\ & \left. + \mu\left(g(t,x_k^i(t),u_k(t),\mu) - g\left(t,x_{k-1}^i(t),u_{k-1}(t),\mu\right)\right)\right] dt \\ + (I+B_i)\int_{\theta_i^{k-1}}^{\theta_i^k} & \left[A(t)x_k^i(t) + C(t)u_k(t) + f(t) + \mu g\left(t,x_k^i(t),u_k(t),\mu\right)\right] dt \\ & + \mu\left(W_i\left(x_k^i\left(\theta_i^k\right), v_i^k,\mu\right) - W_i\left(x_{k-1}^i\left(\theta_i^{k-1}\right), v_i^{k-1},\mu\right)\right) \\ + \int_{\theta_i^k}^{\theta_i^{k-1}} & \left[A(t)x_k(t) + C(t)u_k(t) + f(t) + \mu g\left(t,x_k(t),u_k(t),\mu\right)\right] dt \\ + \int_{\theta_i^{k-1}}^{\theta_i} & \left[A(t)\left(x_k(t) - x_{k-1}(t)\right) + C(t)\left(u_k(t) - u_{k-1}(t)\right)\right. \\ & \left. + \mu\left(g\left(t,x_k(t),u_k(t),\mu\right) - g\left(t,x_{k-1}(t),u_{k-1}(t),\mu\right)\right)\right] dt, \end{aligned}$$

and hence, on using the definitions introduced in Section 2 and (19), we have

$$|S_i^{k+1}(\varphi_k,\mu) - S_i^k(\varphi_k,\mu)| \le \mu\delta_1(\mu)a_k + \mu\delta_2(\mu)b_k. \tag{20}$$

By using (11)–(20) one can easily show that

$$a_{k+1} = \mu\|\mathcal{P}_{k+1}(t,\varphi_k) - \mathcal{P}_k(t,\varphi_{k-1})\| \le \mu\delta_3(\mu)a_k + \mu\delta_4(\mu)b_k. \tag{21}$$

It follows also from (9) that $b_{k+1} \le \mu\delta_5(\mu)a_{k+1} + \mu\delta_6(\mu)b_k$, and so by (21), we get

$$b_{k+1} \le \mu\delta_7(\mu)a_k + \mu\delta_8(\mu)b_k, \tag{22}$$

where $\delta_7(\mu) = \delta_5(\mu)\delta_3(\mu)$ and $\delta_8(\mu) = \delta_5(\mu)\delta_4(\mu) + \delta_6(\mu)$.

Letting $\delta(\mu) = \max_{1\le j\le 6}\{\delta_j(\mu)\}$, we obtain from (21) and (22) that

$$a_{k+1} \le (\mu\delta(\mu))^k \max\{a_1,b_1\}$$

and

$$b_{k+1} \le (\mu\delta(\mu))^k \max\{a_1,b_1\}.$$

Now if we fix a positive real number $\mu_4 \le \mu_3$ sufficiently small such that $\mu\delta(\mu) < 1$ for all $\mu \le \mu_4$, then the above estimates lead to

$$\lim_{k\to\infty} a_k = 0 \tag{23}$$

and

$$\lim_{k\to\infty} b_k = 0. \tag{24}$$

Thus, if $\mu < \mu_4$, then (23) implies that the sequence $\{\varphi_n\}$ is uniformly convergent. One can easily verify that the limiting function $\varphi^0(t)$ satisfies the operator equation (6). But this means that $\varphi^0(t)$ is a solution of problem γ_μ.

On the basis of the construction of Ω-equivalent system (3), it follows that the pair $\{u^0, v_0\}$ is a solving control for Σ_μ.

In view of (19) and (24), if $\mu < \mu_4$ then there exists $\theta_i^* \in (\alpha,\beta)$ so that $\lim_{n\to\infty}\theta_i^n = \theta_i^*$ for each $i = 1,2,\dots,p$.

Let

$$y_n(t) = \begin{cases} x_n^i(t), & \text{for } t \in [\theta_i,\theta_i^*], \\ x_n(t), & \text{for } t \in [\alpha,\beta]\setminus[\theta_i,\theta_i^*]. \end{cases}$$

It is clear that the sequence $\{y_n(t)\}$ is piecewise continuous with discontinuities of the first kind at points θ_i^* for $i = 1,2,\dots$, and moreover, in view of (23) and (24), it converges to a solution y^* of problem Σ_μ with the solving control $\{u^0, v_0\}$.

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