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Lower semi-continuity of the Pareto solution map in quasiconvex semi-infinite vector optimization ☆

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ABSTRACT

This paper is concerned with the lower semi-continuity of the efficient (Pareto) solution map for the perturbed quasiconvex semi-infinite vector optimization problem (QCSVO). We establish sufficient conditions for the lower semi-continuous property of the efficient solution map of (QCSVO) under functional perturbations of both the objective function and the constraints. Examples are designed to analyze the obtained results.

1. Introduction

Given a closed convex pointed cone $K \subset \mathbb{R}^m$ with a nonempty interior, one defines a partial order $\preceq_K$ in $\mathbb{R}^m$ as follows

$$y \preceq_K y' \quad \Leftrightarrow \quad y' - y \in K. \tag{1.1}$$

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a *continuous K-quasiconvex* function, i.e., f is continuous on $\mathbb{R}^n$ and for $y \in \mathbb{R}^m$, $x_1, x_2 \in \mathbb{R}^n$, $\lambda \in [0, 1]$,

$$f(x_1), f(x_2) \in y - K \quad \text{implies} \quad f\big(\lambda x_1 + (1-\lambda)x_2\big) \in y - K.$$

We consider the *perturbed quasiconvex semi-infinite vector* optimization problem in $\mathbb{R}^n$,

$$(\mathrm{QCSVO})_{(f,g)}: \quad \min_K f(x) \quad \text{s.t. } g(x,t) \leqslant 0, \quad \forall t \in T, \tag{1.2}$$

where the "minimization" is understood with respect to the ordering relation $\preceq_K$ defined by (1.1), T is an arbitrary (possibly infinite) index set, and $g(\cdot, t) : \mathbb{R}^n \to \mathbb{R}$ is a continuous quasiconvex function for every $t \in T$.

In this way, the pair $(f, g) := p \in P := QC_K[\mathbb{R}^n, \mathbb{R}^m] \times QC[\mathbb{R}^n \times T, \mathbb{R}]$ is regarded as the *parameter* to be perturbed. Here, $QC_K[\mathbb{R}^n, \mathbb{R}^m]$ stands for the set of all continuous K-quasiconvex functions of the form $f : \mathbb{R}^n \to \mathbb{R}^m$, and $QC[\mathbb{R}^n \times T, \mathbb{R}]$ denotes the set of all functions of the form $g : \mathbb{R}^n \times T \to \mathbb{R}$ such that for each $t \in T$, $g(\cdot, t) : \mathbb{R}^n \to \mathbb{R}$ is continuous and quasiconvex.

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Let $p = (f, g) \in P$. We denote by

$$C(p) = \{x \in \mathbb{R}^n \mid g(x,t) \leqslant 0, \ \forall t \in T\} \tag{1.3}$$

the set of all *feasible points* of (1.2), and write $\bar{x} \in \mathcal{S}(p)$ to indicate that $\bar{x}$ is an *efficient* (or Pareto) *solution* of (1.2) if $\bar{x} \in C(p)$ and there is no $x \in C(p)$ satisfying

$$f(x) \preceq_K f(\bar{x}), \quad f(x) \neq f(\bar{x}).$$

The multifunction $\mathcal{S} : P \rightrightarrows \mathbb{R}^n$ assigns to $p \in P$ the set of all efficient solutions $\mathcal{S}(p)$, is called the *efficient* (or Pareto) *solution map* of (QCSVO).

Stability analysis in semi-infinite optimization problems has been investigated intensively by many researchers; see e.g. [1–10,12,14] and the references therein. One of the main problems here is to find sufficient conditions for the efficient (or Pareto) solution map of a semi-infinite *vector optimization problem* to have a certain stability property, such as the lower (upper) semi-continuity, the continuous property, the calmness, the Aubin property (also known as the pseudo-Lipschitz property), and the Lipschitz property. Observe that most of the publications devoted to stability analysis of semi-infinite vector optimization problems of type (1.2) deal with T is a *compact set* and/or f, g are *linear functions* or *convex functions* with respect to a convex cone. Let us briefly discuss some of them.

Starting with T being a compact Hausdorff space, Todorov [14] studied the upper and lower semi-continuous properties of the efficient and/or weakly efficient solution map in the Berge sense and in the Kuratowski sense for the parametric *linear* semi-infinite vector optimization problem undergone the continuous perturbation of the constraints and the linear perturbation of the objective function. Under functional perturbations of both the objective function and the constraint set, the authors in [4] established necessary and/or sufficient conditions for the lower and upper semi-continuous properties of the generalized parametric semi-infinite vector optimization problem with compact constraints. Some of these results have been extended by Fan, Cheng, and Wang in a very recent paper [9] for the *convex* semi-infinite vector optimization problem without compact constraints. Paper [5] gave a sufficient condition for the pseudo-Lipschitz property (a strict property of the lower semi-continuity!) of the Pareto solution map of the *linear* semi-infinite vector optimization problem under the continuous perturbation of the right-hand side of the constraints and the linear perturbation of the objective function. This result was developed in [6] for the *convex* semi-infinite vector optimization problem.

Unlike with aforementioned frameworks, here we consider a semi-infinite vector optimization problem in a more general case, with T being an arbitrary index set, $g(\cdot,t), t \in T$ being quasiconvex functions and f being a quasiconvex vector function with respect to a convex cone.

The main goal of the present paper is to provide sufficient conditions for the lower semi-continuous property of the efficient solution map of (1.2) under functional perturbations of both the objective function and the constraints. In addition, examples are given to analyze the obtained results.

The rest of the paper is organized as follows. In Section 2, we first recall some basic definitions and preliminaries from set-valued analysis. Then we give some auxiliary results which will be used in this work. The main result of the paper is presented in Section 3. Furthermore, examples are designed to analyze the assumptions of the main theorem.

2. Preliminaries and auxiliary results

In this section we provide further the basic definitions and notation widely used in what follows and also present some auxiliary results that play a significant role in establishing our main results in the next section. Let us first recall some standard notions from set-valued analysis. Let X be a metric space. In particular, when $X = \mathbb{R}^k$ with $k \in \mathbb{N} := \{1, 2, \ldots\}$, the metric on $\mathbb{R}^k$ is generated by the Euclidean norm $\|\cdot\|_k$. Meanwhile, 0_k stands for the zero vector of $\mathbb{R}^k$. Given $\Omega \subset X$, the topological closure and the topological interior of Ω will be denoted by $\operatorname{cl}\Omega$ and $\operatorname{int}\Omega$, respectively. We will use $\mathcal{N}(x)$ to denote the set of all neighborhoods of $x \in X$. The positive polar cone to the cone $K \subset \mathbb{R}^m$ is defined by

$$K^* := \{\hbar \in \mathbb{R}^m \mid \langle \hbar, \ y\rangle \geqslant 0 \text{ for all } y \in K\},$$

where $\langle\cdot,\cdot\rangle$ indicates the scalar product in $\mathbb{R}^m$.

Let $F : X \rightrightarrows Y$ be a multifunction between metric spaces. The effective domain and the graph of F are given, respectively, by

$$\operatorname{dom} F := \{x \in X \mid F(x) \neq \emptyset\}, \qquad \operatorname{gph} F := \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

Definition 2.1. (i) F is said to be a *closed multifunction* iff $\operatorname{gph} F$ is a closed set.

(ii) F is said to be *lower semi-continuous* (lsc for brevity) at $x_0 \in \operatorname{dom} F$ iff for any open set $V \subset Y$ satisfying $V \cap F(x_0) \neq \emptyset$ there exists $U_0 \in \mathcal{N}(x_0)$ such that $V \cap F(x) \neq \emptyset$ for all $x \in U_0$.

For each $r \in \mathbb{N}$ we set $K_r := r\mathbb{B}$, where $\mathbb{B}$ is the closed unit ball in $\mathbb{R}^n$. Then $\{K_r\}_{r=1}^{\infty}$ is the sequence of compact sets in $\mathbb{R}^n$ satisfying $K_r \subset \operatorname{int} K_{r+1}$ and $\mathbb{R}^n = \bigcup_{r=1}^{\infty} K_r$. For any $f_1, f_2 \in QC_K[\mathbb{R}^n, \mathbb{R}^m]$, the distance between f_1 and f_2 is defined by

$$\rho(f_1, f_2) := \sum_{r=1}^{\infty} \frac{1}{2^r} \frac{\rho_r(f_1, f_2)}{1 + \rho_r(f_1, f_2)},$$

where $\rho_r(f_1, f_2) := \max_{x \in K_r} \|f_1(x) - f_2(x)\|_m$ for all $r \in \mathbb{N}$. Then

$$\delta(g_1, g_2) := \sup_{t \in T} \rho\big(g_1(\cdot, t), g_2(\cdot, t)\big)$$

is a metric on $QC[\mathbb{R}^n \times T, \mathbb{R}]$.

The *parameter space* P is endowed with the metric

$$d(p_1, p_2) := \max\big\{\rho(f_1, f_2), \delta(g_1, g_2)\big\},$$

for $p_1 = (f_1, g_1), p_2 = (f_2, g_2) \in P$.

Suppose Ω is a set. One says that a sequence of functions $h_k : \Omega \to \mathbb{R}^m, k \in \mathbb{N}$, converges uniformly to $h : \Omega \to \mathbb{R}^m$ on a set $\Omega_0 \subset \Omega$ if for each $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$\big\|h_k(x) - h(x)\big\|_m < \epsilon \quad \forall x \in \Omega_0,\ \forall k \geqslant k_0.$$

Remark 2.2. Let $p := (f, g) \in P$, and let $p_k := (f_k, g_k) \in P$ for all $k \in \mathbb{N}$. Then $\lim_{k\to\infty} d(p_k, p) = 0$ if and only if for each $r \in \mathbb{N}$ the sequence $\{f_k\}_{k\in\mathbb{N}}$ converges uniformly to f on K_r, and the sequence $\{g_k\}_{k\in\mathbb{N}}$ converges uniformly to g on $K_r \times T$.

We say that $C(p)$ satisfies the *strong Slater condition* if there exist $\hat{x} \in \mathbb{R}^n$ and $\ell > 0$ such that $g(\hat{x}, t) \leqslant -\ell$ for all $t \in T$.

The following lemma gives some properties, and a sufficient condition for the lower semi-continuity of the constraint set mapping $C : P \rightrightarrows \mathbb{R}^n$ defined by (1.3).

Lemma 2.3. *The following statements hold*:

(a) *C is a closed multifunction with convex values.*
(b) *If $C(p)$, $p := (f, g) \in P$, satisfies the strong Slater condition and if the local minima of $G(\cdot) := \sup_{t\in T} g(\cdot, t)$ are global, then C is lsc at p.*

Proof. (a) Consider arbitrarily $p := (f, g) \in P$. Since $g(\cdot, t)$ are quasiconvex for all $t \in T$, $C(p)$ is a convex set. Let $\{p_k := (f_k, g_k)\}_{k=1}^{\infty} \subset P$ and $\{x_k\}_{k=1}^{\infty} \subset \mathbb{R}^n$ be sequences satisfying

$$p_k \to p, \quad x_k \to x, \quad \text{and} \quad x_k \in C(p_k) \quad \forall k \in \mathbb{N}.$$

It is sufficient to show that $x \in C(p)$. As $\{x_k\}_{k=1}^{\infty}$ is a convergent sequence, there is no loss of generality in assuming that $\{x_k\}_{k=1}^{\infty} \subset \operatorname{int} K_r$ for $r \in \mathbb{N}$ large enough. Since $\lim_{k\to\infty} d(p_k, p) = 0$, we get by Remark 2.2 that $\{g_k\}_{k\in\mathbb{N}}$ converges uniformly to g on $K_r \times T$ for such $r \in \mathbb{N}$. In addition, $x_k \to x$ and $g(\cdot, t)$ is continuous for every $t \in T$, one has

$$\lim_{k\to\infty} g_k(x_k, t) = g(x, t) \quad \forall t \in T. \tag{2.1}$$

Besides, it follows by $x_k \in C(p_k)$ for all $k \in \mathbb{N}$ that $g_k(x_k, t) \leqslant 0$ for all $t \in T$. Thus, we receive from (2.1) that $g(x, t) \leqslant 0$ for all $t \in T$. The last assertion shows that $x \in C(p)$, which concludes the proof of (a).

(b) Suppose that $V \subset \mathbb{R}^n$ is an open set such that $V \cap C(p) \neq \emptyset$. Let $\tilde{x} \in V \cap C(p)$ and let $\hat{x} \in \mathbb{R}^n$ and $\ell > 0$ such that $g(\hat{x}, t) \leqslant -\ell$ for all $t \in T$. Then $g(\tilde{x}, t) \leqslant 0$ for all $t \in T$ and so, $G(\tilde{x}) \leqslant 0$.

If $G(\tilde{x}) = 0$, then $\tilde{x}$ is not a global minimizer of $G(\cdot)$ because $G(\hat{x}) \leqslant -\ell < 0$. By our assumption, $\tilde{x}$ is not a local minimizer of $G(\cdot)$. This means that there exists $\bar{x} \in V$ such that $G(\bar{x}) < 0$.

If $G(\tilde{x}) < 0$, then we put $\bar{x} := \tilde{x}$.

Set $\lambda := -G(\bar{x}) > 0$ and fix $r \in \mathbb{N}$ such that $\bar{x} \in K_r$. Now we show that $\bar{x} \in C(p')$ for all $p' := (f', g') \in P$ satisfying $d(p', p) < \varepsilon$, where $\varepsilon \in (0, \frac{\lambda}{2^r(1+\lambda)})$. Indeed, it is easy to verify that $d(p', p) < \varepsilon$ implies $\rho_r(g'(\cdot, t), g(\cdot, t)) < \lambda$ for all $t \in T$. So, it holds that

$$\big|g'(\bar{x}, t) - g(\bar{x}, t)\big| \leqslant \max_{x\in K_r}\big|g'(x, t) - g(x, t)\big| = \rho_r\big(g'(\cdot, t), g(\cdot, t)\big) < \lambda \quad \forall t \in T.$$

This yields

$$g'(\bar{x}, t) < g(\bar{x}, t) + \lambda \leqslant G(\bar{x}) + \lambda = 0 \quad \forall t \in T.$$

The proof is complete. $\square$

Next we recall from [13] the *smallest strictly monotonic function* $h_{e,a} : \mathbb{R}^m \to \mathbb{R}$ defined by

$$h_{e,a}(y) = \min\{t \in \mathbb{R} \mid y \in a + te - K\}, \tag{2.2}$$

where $e \in \operatorname{int} K$ and $a \in \mathbb{R}^m$ are fixed points. Many general properties of this function and its applications in vector optimization can be found in [11,13]. In our setting, $h_{e,a}$ can be exhibited as a simpler form by the following proposition.

Proposition 2.4. *Consider $h_{e,a}$ defined by* (2.2) *and set $\Theta := \{\hbar \in K^* \mid \langle \hbar, e\rangle = 1\}$. We have for $y \in \mathbb{R}^m$,*

$$h_{e,a}(y) = \max\{\langle \hbar, y-a\rangle \mid \hbar \in \operatorname{ext}\Theta\},$$

where $\operatorname{ext}\Theta$ *denotes the set of extreme points of Θ.*

Proof. According to [11, Lemma 2.2.17] Θ is a compact base of K^*. So, for each $y \in \mathbb{R}^m$, it holds that

$$\begin{aligned}
h_{e,a}(y) &= \min\{t \in \mathbb{R} \mid -y + a + te \in K\} \\
&= \min\{t \in \mathbb{R} \mid \langle \hbar, -y + a + te\rangle \geqslant 0,\ \forall \hbar \in K^*\} \\
&= \min\{t \in \mathbb{R} \mid \langle \hbar, -y + a + te\rangle \geqslant 0,\ \forall \hbar \in \Theta\} \\
&= \min\{t \in \mathbb{R} \mid t \geqslant \langle \hbar, y-a\rangle,\ \forall \hbar \in \Theta\} \\
&= \min\{t \in \mathbb{R} \mid t \geqslant \max\{\langle \hbar, y-a\rangle,\ \forall \hbar \in \Theta\}\} \\
&= \max\{\langle \hbar, y-a\rangle \mid \hbar \in \Theta\} \\
&= \max\{\langle \hbar, y-a\rangle \mid \hbar \in \operatorname{ext}\Theta\}.
\end{aligned}$$

The proof is complete. □

Remark 2.5. In a more special case, where $K := \mathbb{R}^m_+$, $e := (1, 1, \ldots, 1) \in \operatorname{int}\mathbb{R}^m_+$, and $a := (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$, we get by Proposition 2.4 a very simple formula as follows

$$h_{e,a}(y) = \max\{y_i - a_i \mid i = 1, 2, \ldots, m\}, \quad y := (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m.$$

Let $p := (f, g) \in P$ and $\bar{x} \in \mathcal{S}(p)$, where $\mathcal{S}(p)$ is the set of all efficient solutions of problem (1.2). If there exists $h_{e,a}$ defined by (2.2) such that

$$\bar{x} \in \operatorname{argmin}\{h_{e,a} \circ f(x) \mid x \in C(p)\},$$

then $\bar{x}$ is called a *scalarized solution* by $h_{e,a}$.

It is worth mentioning here that in our framework each $\bar{x} \in \mathcal{S}(p)$ is a scalarized solution by $h_{e,f(\bar{x})}$.

Lemma 2.6. *Let $p := (f, g) \in P$ and $\bar{x} \in \mathcal{S}(p)$. Then, $\bar{x}$ is a scalarized solution by $h_{e,f(\bar{x})}$.*

Proof. Since any efficient solution is weakly efficient, the proof is derived from [13, Theorem 4.2.15]. □

3. Lower semi-continuity of the Pareto solution map

We are now ready to state and prove the main result of the paper.

Theorem 3.1. *Let $p := (f, g) \in P$. Suppose that the following conditions hold:*

(i) *C is lsc at p.*
(ii) *For each $\bar{x} \in \mathcal{S}(p)$ one has*

$$\operatorname{argmin}\{h_{e,f(\bar{x})} \circ f(x) \mid x \in C(p)\} = \{\bar{x}\}.$$

Then $\mathcal{S}$ is lsc at p.

Proof. Suppose contrary to the conclusion of the theorem that $\mathcal{S}$ is not lsc at p. Then there exist an element $\bar{x} \in \mathcal{S}(p)$, an open set $U \in \mathcal{N}(\bar{x})$, and a sequence $\{p_k := (f_k, g_k)\}_{k=1}^{\infty} \subset P$ such that $\{p_k\}$ converges to $p = (f, g)$ and

$$\mathcal{S}(p_k) \cap U = \emptyset \quad \forall k \in \mathbb{N}. \tag{3.1}$$

Pick an open ball with center $\bar{x}$ and radius $\lambda > 0$, say $B(\bar{x}, \lambda)$, such that $\operatorname{cl} B(\bar{x}, \lambda) \subset U$. Since C is lsc at p, there is a sequence $\{v^k\}$ such that $v^k \in C(p_k)$ for all $k \in \mathbb{N}$ and $v^k \to \bar{x}$ as $k \to \infty$. Without loss of generality one may assume that $v^k \in B(\bar{x}, \lambda)$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, we set

$$W_k(\bar{x}) := \left\{ x \in C(p_k) \cap B(\bar{x}, \lambda) \;\middle|\; \|x - \bar{x}\|_n < \left\| v^k - \bar{x} \right\|_n + \frac{1}{k} \right\}.$$

It is obvious that $W_k(\bar{x}) \neq \emptyset$, due to $v^k \in W_k(\bar{x})$, for all $k \in \mathbb{N}$.

Next, let us justify that there exist $k_0 \in \mathbb{N}$, $x^k \in W_k(\bar{x})$, and $z^k \in C(p_k) \backslash B(\bar{x}, \lambda)$ such that

$$f_k\big(z^k\big) - f_k\big(x^k\big) \in -K \backslash \{0_m\} \quad \forall k \geqslant k_0. \tag{3.2}$$

Indeed, argue by contradiction that the claim in (3.2) is false. Then there exists a subsequence $\{k_l\}$ of $\{k\}$, for convenience we shall denote this subsequence by $\{k\}$ again, such that for each $k \in \mathbb{N}$, it holds that

$$f_k(z) - f_k(x) \notin -K \backslash \{0_m\} \quad \forall x \in W_k(\bar{x}),\ \forall z \in C(p_k) \backslash B(\bar{x}, \lambda). \tag{3.3}$$

Denote by $\mathcal{S}(\Omega, f_k), k \in \mathbb{N}$, the set of Pareto solutions of the problem

$$\min_K \big\{ f_k(x) \mid x \in \Omega \big\},$$

where Ω is some subset of $C(p_k)$. In view of Lemma 2.3(a), we have $C(p_k)$ is closed for every $k \in \mathbb{N}$. Hence $\mathcal{S}(C(p_k) \cap \operatorname{cl} B(\bar{x}, \lambda), f_k) \neq \emptyset$ for every $k \in \mathbb{N}$ by virtue of the compactness of $C(p_k) \cap \operatorname{cl} B(\bar{x}, \lambda)$ and the continuity of f_k. Consider the following two possibilities:

If $\mathcal{S}(C(p_k) \cap \operatorname{cl} B(\bar{x}, \lambda), f_k) \cap W_k(\bar{x}) \neq \emptyset$, then there exists

$$\bar{z} \in \mathcal{S}\big(C(p_k) \cap \operatorname{cl} B(\bar{x}, \lambda), f_k\big) \cap W_k(\bar{x}).$$

We have $\bar{z} \in \mathcal{S}(p_k)$. Indeed, if $\bar{z} \notin \mathcal{S}(p_k)$, then by $\bar{z} \in \mathcal{S}(C(p_k) \cap \operatorname{cl} B(\bar{x}, \lambda), f_k)$, there exists $z \in C(p_k) \backslash B(\bar{x}, \lambda)$ such that

$$f_k(z) - f_k(\bar{z}) \in -K \backslash \{0_m\},$$

contrary to (3.3). So $\bar{z} \in \mathcal{S}(p_k)$ and thus,

$$\bar{z} \in \mathcal{S}(p_k) \cap W_k(\bar{x}) \subset \mathcal{S}(p_k) \cap B(\bar{x}, \lambda) \subset \mathcal{S}(p_k) \cap U,$$

which contradicts (3.1).

If $\mathcal{S}(C(p_k) \cap \operatorname{cl} B(\bar{x}, \lambda), f_k) \cap W_k(\bar{x}) = \emptyset$, then we select an element

$$\bar{y} \in W_k(\bar{x}) \backslash \mathcal{S}\big(C(p_k) \cap \operatorname{cl} B(\bar{x}, \lambda), f_k\big).$$

Then there exists $z_{\bar{y}} \in C(p_k) \cap \operatorname{cl} B(\bar{x}, \lambda)$ such that

$$f_k(z_{\bar{y}}) - f_k(\bar{y}) \in -K \backslash \{0_m\}. \tag{3.4}$$

Put $D := \{x \in C(p_k) \cap \operatorname{cl} B(\bar{x}, \lambda) \mid f_k(x) - f_k(z_{\bar{y}}) \in -K\}$. It is easy to verify that $\mathcal{S}(D, f_k) \neq \emptyset$ and

$$\mathcal{S}(D, f_k) \subset \mathcal{S}\big(C(p_k) \cap \operatorname{cl} B(\bar{x}, \lambda), f_k\big).$$

Taking arbitrarily $\bar{z} \in \mathcal{S}(D, f_k)$, we have $\bar{z} \in \mathcal{S}(p_k)$. Indeed, if $\bar{z} \notin \mathcal{S}(p_k)$, then by $\bar{z} \in \mathcal{S}(C(p_k) \cap \operatorname{cl} B(\bar{x}, \lambda), f_k)$, there exists $y \in C(p_k) \backslash B(\bar{x}, \lambda)$ such that

$$f_k(y) - f_k(\bar{z}) \in -K \backslash \{0_m\}. \tag{3.5}$$

By $\bar{z} \in D$, $f_k(\bar{z}) - f_k(z_{\bar{y}}) \in -K$. This together with (3.4) and (3.5) yields

$$f_k(y) - f_k(\bar{y}) \in -K \backslash \{0_m\},$$

contrary to (3.3). Hence $\bar{z} \in \mathcal{S}(p_k)$. It follows from $\bar{z} \in D$ that

$$\bar{z} \in \mathcal{S}(p_k) \cap \operatorname{cl} B(\bar{x}, \lambda) \subset \mathcal{S}(p_k) \cap U,$$

which contradicts (3.1). Combining now the previous arguments gives the assertion in (3.2).

To proceed, we consider the following two possible cases:

If $\{z^k\}_{k \geqslant k_0}$ is bounded, then by taking a subsequence if necessary, we may assume that $z^k \to z^0 \in \mathbb{R}^n \backslash B(\bar{x}, \lambda)$. According to Lemma 2.3(a), C is a closed multifunction. Hence, $z^0 \in C(p)$. As $\{z^k\}_{k \geqslant k_0}$ is a bounded sequence, there is no loss of generality in assuming that $\{z^k\}_{k \geqslant k_0} \subset \operatorname{int} K_r$ for $r \in \mathbb{N}$ large enough. Since $\lim_{k \to \infty} d(p_k, p) = 0$, we get from Remark 2.2

that the sequence $\{f_k\}_{k\in\mathbb{N}}$ converges uniformly to f on K_r for such $r \in \mathbb{N}$. In addition, since $z^k \to z^0$ and f is continuous, one has

$$\lim_{k\to\infty} f_k\big(z^k\big) = f\big(z^0\big).$$

Similarly, we have $\lim_{k\to\infty} f_k(x^k) = f(\bar{x})$. So, by letting $k \to \infty$ in (3.2), we get

$$f\big(z^0\big) - f(\bar{x}) \in -K. \tag{3.6}$$

Hence, $f(z^0) = f(\bar{x})$ due to $\bar{x} \in \mathcal{S}(p)$. In addition, since $\bar{x} \in \mathcal{S}(p)$, it follows from Lemma 2.6 that $\bar{x}$ is a scalarized solution by $h_{e,f(\bar{x})}$. Therefore,

$$z^0 \in \operatorname{argmin}\big\{h_{e,f(\bar{x})} \circ f(x) \mid x \in C(p)\big\}. \tag{3.7}$$

Combining (3.7) with assumption (ii) of the theorem gives $z^0 = \bar{x}$, which contradicts the fact that $z^0 \in \mathbb{R}^n \backslash B(\bar{x}, \lambda)$.

If $\{z^k\}_{k\geqslant k_0}$ is unbounded, then we may assume that

$$\big\|z^k\big\|_n \to \infty \quad \text{and} \quad \frac{z^k}{\|z^k\|_n} \to \hat{z} \in \mathbb{R}^n \quad \text{with } \|\hat{z}\|_n = 1.$$

Note that $z^k, x^k \in C(p_k)$ for all $k \geqslant k_0$. It follows from Lemma 2.3(a) that

$$\frac{1}{\|z^k\|_n} z^k + \left(1 - \frac{1}{\|z^k\|_n}\right) x^k \in C(p_k) \tag{3.8}$$

for all k large enough. Moreover, since C is a closed multifunction by Lemma 2.3(a), (3.8) entails that $\hat{z} + \bar{x} \in C(p)$. Obviously, $f_k(x^k) \in f_k(x^k) - K$ for all $k \in \mathbb{N}$ and from (3.2) one has $f_k(z^k) \in f_k(x^k) - K$ for all $k \geqslant k_0$. So, using the K-quasiconvex property of f_k for each k, we obtain for k large enough,

$$f_k\left(\frac{1}{\|z^k\|_n} z^k + \left(1 - \frac{1}{\|z^k\|_n}\right) x^k\right) - f_k\big(x^k\big) \in -K. \tag{3.9}$$

By the same argument as above, we can show that

$$\lim_{k\to\infty} f_k\left(\frac{1}{\|z^k\|_n} z^k + \left(1 - \frac{1}{\|z^k\|_n}\right) x^k\right) = f(\hat{z} + \bar{x}) \quad \text{and} \quad \lim_{k\to\infty} f_k\big(x^k\big) = f(\bar{x}).$$

Now, letting $k \to \infty$ in (3.9), one gets $f(\hat{z} + \bar{x}) - f(\bar{x}) \in -K$. Therefore, $f(\hat{z} + \bar{x}) = f(\bar{x})$ because $\bar{x} \in \mathcal{S}(p)$. Following the same manner as above we arrive at $\hat{z} + \bar{x} = \bar{x}$, which is absurd. The proof is complete. □

As a consequence of Theorem 3.1 and Lemma 2.3, we have the following result.

Corollary 3.2. *Let $p = (f, g) \in P$. Suppose that the following conditions hold*:

(i) *$C(p)$ satisfies the strong Slater condition and the local minima of $G(\cdot) := \sup_{t\in T} g(\cdot, t)$ are global.*
(ii) *For each $\bar{x} \in \mathcal{S}(p)$ one has*

$$\operatorname{argmin}\big\{h_{e,f(\bar{x})} \circ f(x) \mid x \in C(p)\big\} = \{\bar{x}\}.$$

Then $\mathcal{S}$ is lsc at p.

Remark 3.3. Assumption (ii) in Theorem 3.1 is fulfilled by [13, Theorem 2.15] when f is *strictly K-quasiconvex*, i.e., for $y \in \mathbb{R}^m$, $x_1, x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$, $\lambda \in (0, 1)$,

$$f(x_1), f(x_2) \in y - K \quad \text{implies} \quad f\big(\lambda x_1 + (1-\lambda)x_2\big) \in y - \operatorname{int} K.$$

We close this section by two examples that show the importance of the hypotheses imposed in Theorem 3.1. Namely, if we omit one of conditions (i) and (ii), then the conclusion of the theorem may be false.

Example 3.4. Let $T = [0, 1]$ and $K := \mathbb{R}^2_+$. Consider $f \in QC_K[\mathbb{R}, \mathbb{R}^2]$ and $g, g_k \in QC[\mathbb{R} \times T, \mathbb{R}]$, $k \geqslant 1$, which are given as follows

$$f(x)=(x,2x)\quad \forall x\in\mathbb{R},\qquad g(x,t)=-tx-t\quad \forall x\in\mathbb{R},\ \forall t\in[0,1],$$
$$g_k(x,t)=\begin{cases}-tx & \text{if } t\in[0,\frac{1}{k+1}],\\ -tx-\frac{k+1}{k}t+\frac{1}{k} & \text{if } t\in[\frac{1}{k+1},1],\end{cases}\quad \forall x\in\mathbb{R}.$$

Put $p:=(f,g)$, $p_k:=(f,g_k)\in P:=QC_K[\mathbb{R},\mathbb{R}^2]\times QC[\mathbb{R}\times T,\mathbb{R}]$ for all $k\geqslant 1$. It is easy to see that $p_k\to p$. We have

$$C(p)=[-1,+\infty),\qquad \mathcal{S}(p)=\{-1\},\qquad C(p_k)=[0,+\infty),\qquad \mathcal{S}(p_k)=\{0\}\quad \forall k\geqslant 1.$$

For a fixed point $e\in\operatorname{int}K$ and for $\bar{x}:=-1\in\mathcal{S}(p)$, we can verify that

$$\operatorname{argmin}\{h_{e,f(\bar{x})}\circ f(x)\mid x\in C(p)\}=\{\bar{x}\}.$$

This means that assumption (ii) of Theorem 3.1 is fulfilled. Meantime, C is not lsc at p. In other words, assumption (i) of Theorem 3.1 fails to hold. Observe that $\mathcal{S}$ is not lsc at p.

Example 3.5. Let $T=[0,1]\cup\{2\}$ and $K:=[0,+\infty)$. Consider $f\in QC_K[\mathbb{R}^2,\mathbb{R}]$ and $g,g_k\in QC[\mathbb{R}^2\times T,\mathbb{R}]$, $k\geqslant 2$, which are given as follows

$$f(x)=x_1\quad \forall x=(x_1,x_2)\in\mathbb{R}^2,$$
$$g(x,t)=\begin{cases}(t-1)x_1+tx_2-t & \text{if } t\in[0,1],\\ (t-1)x_1+(1-t)x_2 & \text{if } t=2,\end{cases}\quad \forall x=(x_1,x_2)\in\mathbb{R}^2,$$
$$g_k(x,t)=\begin{cases}(t-1)x_1+tx_2 & \text{if } t\in[0,\frac{1}{k+1}],\\ (t-1)x_1+tx_2-\frac{k+1}{k}t+\frac{1}{k} & \text{if } t\in[\frac{1}{k+1},1],\\ (t-1)x_1+(1-t)x_2 & \text{if } t=2,\end{cases}\quad \forall x=(x_1,x_2)\in\mathbb{R}^2.$$

Put $p:=(f,g)$, $p_k:=(f,g_k)\in P:=QC_K[\mathbb{R}^2,\mathbb{R}]\times QC[\mathbb{R}^2\times T,\mathbb{R}]$ for all $k\geqslant 2$. It is easy to see that $p_k\to p$. We have

$$C(p)=\{(x_1,x_2)\in\mathbb{R}^2\mid 0\leqslant x_1\leqslant 1,\ x_1\leqslant x_2\leqslant 1\},$$
$$C(p_k)=\left\{(x_1,x_2)\in\mathbb{R}^2\;\middle|\;0\leqslant x_1\leqslant\frac{1}{k},\ x_1\leqslant x_2\leqslant kx_1\right\}\cup\left\{\frac{1}{k}\leqslant x_1\leqslant 1,\ x_1\leqslant x_2\leqslant 1\right\},$$
$$\mathcal{S}(p)=\{(0,x_2)\mid 0\leqslant x_2\leqslant 1\},\qquad \mathcal{S}(p_k)=\{(0,0)\}\quad \forall k\geqslant 2.$$

Taking $\hat{x}=(\frac{1}{2},\frac{3}{4})\in\mathbb{R}^2$, we have

$$g(\hat{x},t)\leqslant -\frac{1}{4}\quad \forall t\in T.$$

This means that $C(p)$ satisfies the strong Slater condition. In addition, $G(\cdot):=\sup_{t\in T}g(\cdot,t)$ is convex. So, C is lsc at p by Lemma 2.3(b). In other words, assumption (i) of Theorem 3.1 is satisfied. Choosing $e:=1\in\operatorname{int}K$, we get by Remark 2.5 that for each $\bar{x}\in\mathcal{S}(p)$, $h_{e,f(\bar{x})}(y)=y$ for every $y\in\mathbb{R}$. Hence,

$$\operatorname{argmin}\{h_{e,f(\bar{x})}\circ f(x)\mid x\in C(p)\}=\mathcal{S}(p)\neq\{\bar{x}\}\quad \forall\bar{x}\in\mathcal{S}(p),$$

which shows that assumption (ii) of Theorem 3.1 is violated. Actually, $\mathcal{S}$ is not lsc at p.

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