A Ranking Approach for Intuitionistic Fuzzy Numbers and its Application

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ABSTRACT
To the best of our knowledge very few methods have been proposed in previous studies for comparing intuitionistic fuzzy (IF) numbers. In this paper, the limitations and the shortcomings of all these existing methods are pointed out. In order to overcome these limitations and shortcomings a new ranking approach—by modifying an existing ranking approach—is proposed for comparing IF numbers. Thus, with the help of proposed the ranking approach, a new method is proposed to find the optimal solution of such unbalanced minimum cost flow (MCF) problems in which all the parameters are represented by IF numbers.

Keywords: manufacturing system, replenishment lot size, discontinuous issuing policy, algebraic approach, imperfect EPQ model, rework.

1. Introduction

MCF problem, which is an important problem in combinatorial optimization and network flows, has many applications in practical problems such as transportation, communication, urban design and job scheduling models [1,3].

In its classical form the MCF problem minimizes the cost of transporting a product that is available at some sources and required at certain destinations. In many actual problems, the cost, capacities, supplies and demand parameters may be imprecise. To deal quantitatively with imprecise information the concept and techniques of probability can be employed. There are articles discussing the network flow problems where arc parameters are random variable [11, 21, 22], however, in order to construct, probability distributions require either a priori predictable regularity or a posteriori frequency distribution. Moreover, the premise that imprecision can be equated with randomness is still questionable. As an alternative, uncertain values can be represented by membership functions of the fuzzy set theory [26].

The main advantages of methodologies based on fuzzy theory are that they do not require prior predictable regularities or posterior frequency distributions and they can deal with imprecise input information containing feelings and emotions quantified based on the decision-maker's subjective judgment. From this point of view, Shih and Lee [23] proposed a fuzzy version of MCF problem using multilevel linear programming problem. But, they did not use the nice structure of network constraints.

Ghatee and Hashemi [7] proposed a method to find the fuzzy optimal solution of balanced fully fuzzy MCF problems. Ghatee and Hashemi [8, 9] and Ghatee et al. [10] applied the existing method [7] for solving real-life problems.

The most common concept used in all those studies is ranking of fuzzy numbers. Ranking of fuzzy numbers is an important issue in the study of fuzzy set theory. In order to rank fuzzy numbers, one fuzzy number needs to be compared with the others but it is difficult to determine clearly which of them is larger or smaller. Numerous methods have been proposed in previous studies to rank fuzzy numbers. There is not a unique method for comparing fuzzy numbers.

A membership function of a classical fuzzy set assigns to each element of the universe of discourse a number from the unit interval to indicate the degree of belongingness to the set under consideration. The degree of nondecision-maker's belongingness is just automatically the complement to one of the membership degree. Nonetheless, a human being who expresses the degree of membership of a given element in a fuzzy set very often does not express the corresponding degree of nonmembership as the complement to 1. This reflects a well-known psychological fact that the linguistic negation do not always identifies with logical negation. Thus, Atanassov [2] introduced the concept of an IF set which is characterized by two functions expressing the degree of belongingness and the degree of nonbelongingness respectively. This idea, which is a natural generalization of usual fuzzy set, seems to be useful when modeling many real-life situations.

Concerning ranking IF numbers some work has been reported in the literature. Grzegorzewski [12] defined two families of metrics in the space of IF numbers and proposed a method for comparing IF numbers based on these metrics. Mitchell [15] extended the natural ordering of real numbers to triangular intuitionistic fuzzy (TrIF) numbers by adopting a statistical view point and interpreting each IF number as ensemble of ordinary fuzzy numbers. Nayagam et al. [19] introduced TrIF numbers of special type and described a method to compare them. Although their ranking method appears to be attractive, the definition of TrIF number seems unrealistic. This is because the triangular nonmembership function is defined to geometrically behave in an identical manner as the membership function. Nan and Li [16] proposed a method for comparing TrIF number using lexicographic technique. Nehi [20] proposed a new method for comparing IF numbers in which two characteristic values for IF numbers are defined by the integral of the inverse fuzzy membership and nonmembership functions multiplied by the grade with powered parameter. Almost in parallel, Li [13] introduced a new definition of the TrIF number which has an appealing and logically reasonable interpretation. He defined two concepts of the value and the ambiguity of a TrIF number similar to those for a fuzzy number introduced by Delgado et al. [4]. Dubey and Mehra [5] defined a TrIF number which is more general than the one defined in [13, 16]. They extended the definitions of the value and the ambiguity index given by Li [13] to the newly defined TrIF numbers and proposed an approach to handle linear programming problems with data as IF numbers.

In this paper, the limitations of the existing methods [5, 12, 13, 15, 18, 19, 24, 25] as well as their shortcomings [13, 17, 20], are pointed out. Furthermore, to overcome such limitations and shortcomings, a new ranking approach—by modifying an existing ranking approach [20]—is proposed for comparing IF numbers. Also—with the help of the projected ranking approach—a new method is proposed to find the optimal solution of such unbalanced MCF problems in which all the parameters are represented by IF numbers.

This paper is organized as follows. In Section 2, some basic definitions and arithmetic operations of trapezoidal intuitionistic fuzzy (TIF) numbers are presented. In Sections 3 and 4, the limitations and shortcomings of the existing methods [5, 12, 13, 15, 18, 19, 24, 25] and [13, 17, 20] respectively, are pointed out. In Section 5, a ranking approach is proposed for comparing IF numbers is presented. In Section 6, a method for solving MCF problems in IF environment is proposed and to illustrate the proposed method, a numerical example is solved. In Section 7, the results are compared. Finally, in Section 8, the conclusion is discussed.

2. Preliminaries

In this section, some basic definitions and arithmetic operations are presented [20].

2.1 Basic definitions

Definition 2.1. An IF set $\widetilde{A}=\{x,\mu_{\widetilde{A}}(x),\upsilon_{\widetilde{A}}(x))\mid x\in X\}$ on the universal set X is characterized by a truth membership function $\mu_{\widetilde{A}},\mu_{\widetilde{A}},: X\rightarrow[0,1]$ and a false membership function $\upsilon_{\widetilde{A}},\upsilon_{\widetilde{A}},: X\rightarrow[0,1]$. The values $\mu_{\widetilde{A}}(x)$ and $\upsilon_{\widetilde{A}}(x)$ represent the degree of membership and the degree of nonmembership for $x\in X$ and always satisfies the condition $\mu_{\widetilde{A}}(x)+\upsilon_{\widetilde{A}}(x)\leq 1\ \forall\ x\in X$. The value $(1-\mu_{\widetilde{A}}(x)-\upsilon_{\widetilde{A}}(x))$ represents the degree of hesitation for $x\in X$.

Definition 2.2. Let $\widetilde{A}$ be an IF set. Then, $\widetilde{A}_\alpha=\{x\in X\mid\mu_{\widetilde{A}}(x)\geq\alpha,\upsilon_{\widetilde{A}}(x)\leq(1-\alpha)\}$ is said to be an $\alpha-$cut of $\widetilde{A}$.

Definition 2.3. An IF set $\widetilde{A}_\alpha = \{(x, \mu_{\widetilde{A}}(x), \upsilon_{\widetilde{A}}(x) \mid x \in X\}$ is called IF-normal, if there are at least two points $x_0, x_1 \in X$ such that $\mu_{\widetilde{A}}(x_0) = 1$, $\upsilon_{\widetilde{A}}(x_1) = 1$.

Definition 2.4. An IF set $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x), \upsilon_{\widetilde{A}}(x)) \mid x \in X\}$ is called IF-convex, if $\forall x_1, x_2 \in X, \lambda \in [0,1]$

$$\mu_{\widetilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_{\widetilde{A}}(x_1), \mu_{\widetilde{A}}(x_2))$$
$$\upsilon_{\widetilde{A}}(\lambda x_1 + (1-\lambda)x_2) \leq \max(\upsilon_{\widetilde{A}}(x_1), \upsilon_{\widetilde{A}}(x_2))$$

Definition 2.5. An IF set $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x), \upsilon_{\widetilde{A}}(x)) \mid x \in X\}$ of the real line is called IF number if

(a) $\widetilde{A}$ is IF-normal

(b) $\widetilde{A}$ is IF-convex

(c) $\mu_{\widetilde{A}}$ is upper semi continuous and $\upsilon_{\widetilde{A}}$ is lower semi continuous

(d) $\widetilde{A} = \{x \in X \mid \upsilon_{\widetilde{A}}(x) < 1\}$ is bounded

Definition 2.6. An IF number $\widetilde{A}$, defined on the universal set of real numbers R, denoted as $\widetilde{A} = \{a'_1, a_1, a'_2, a_2, a_3, a'_3, a_4, a'_4\}$, where $a'_1 \leq a_1 \leq a'_2 \leq a_2 \leq a_3 \leq a'_3 \leq a_4 \leq a'_4$ is said to be TIF number, if the degree of membership $\mu_{\widetilde{A}}(x)$ and the degree of non-membership $\upsilon_{\widetilde{A}}(x)$ are given by:

$$\mu_{\widetilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & \text{if } a_1 \leq x < a_2 \\ 1 & \text{if } a_2 \leq x \leq a_3 \\ \dfrac{x - a_1}{a_3 - a_4} & \text{if } a_3 < x \leq a_4 \\ 0 & \text{otherwise} \end{cases}$$

$$\upsilon_{\widetilde{A}}(x) = \begin{cases} \dfrac{a'_2 - x}{a'_2 - a'_1} & \text{if } a'_1 \leq x < a'_2 \\ 0 & \text{if } a'_2 \leq x \leq a'_3 \\ \dfrac{a'_3 - x}{a'_3 - a'_4} & \text{if } a'_3 < x \leq a'_4 \\ 1 & \text{otherwise} \end{cases}$$

If in a TIF number $\widetilde{A}$, let $a_2 = a_3$ (and hence $a'_2 = a'_3$), then it gives a TrIF number with parameters $a'_1 \leq a_1 \leq a'_2\,(a_2 = a_3 = a'_3) \leq a_4 \leq a'_4$ and is denoted by $\widetilde{A} = \{a'_1, a_1, a'_2, a_4, a'_4\}$.

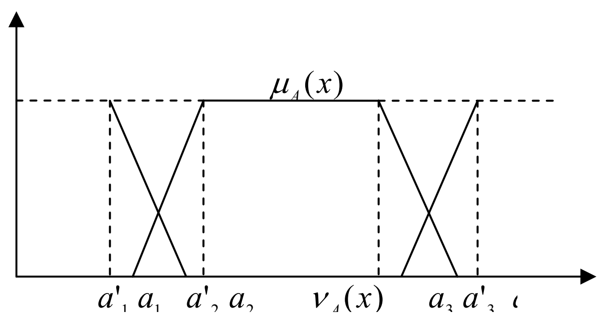

Figure 1. Trapezoidal intuitionistic fuzzy number.

Definition 2.7. Two TIF numbers $\widetilde{A} = \{a'_1, a_1, a'_2, a_2, a_3, a'_3, a_4, a'_4\}$, $\widetilde{B} = \{b'_1, b_1, b'_2, b_2, b_3, b'_3, b_4, b'_4\}$ are said to be equal i.e., $\widetilde{A} = \widetilde{B}$ if and only if $a'_1 = b'_1$, $a_1 = b_1$, $a'_2 = b'_2$, $a_2 = b_2$, $a_3 = b_3$, $a'_3 = b'_3$, $a_4 = b_4$ and $a'_4 = b'_4$.

Definition 2.8. A TIF number $\widetilde{A} = \{a'_1, a_1, a'_2, a_2, a_3, a'_3, a_4, a'_4\}$ is said to be non-negative TIF number if and only if $a'_1 \geq 0$.

Remark 1. In the existing methods [5] the TrIF number $\widetilde{A} = \{a'_1, a_1, a'_2, a_4, a'_4\}$ is represented by $\widetilde{A} = \{(a_1, a'_2, a_4; 1), (a'_1, a'_2, a'_4; 1)\}$.

Remark 2. In the existing methods [13] the TrIF number $\tilde{A} = \langle (a_1, a_2, a_3); 1, 1 \rangle$ is used which is obtained by the TrIF number $\tilde{A} = \{a'_1, a_1, a'_2, a_4, a'_4\}$ by assuming $a'_1 = a_1, a'_2, a_4 = a'_4$.

2.2 Arithmetic operations between TIF numbers

In this section, some arithmetic operations between TIF numbers, defined on a universal set of real numbers R, are presented.

(i) Let $\tilde{A} = \{a'_1, a_1, a'_2, a_2, a_3, a'_3, a_4, a'_4\}$ and $\tilde{B} = \{b'_1, b_1, b'_2, b_2, b_3, b'_3, b_4, b'_4\}$ be two TIF numbers. Then, $\tilde{A} \oplus \tilde{B} = (a'_1 + b'_1, a_1 + b_1, a'_2 + b'_2, a_2 + b_2, a_3 + b_3, a'_3 + b'_3, a_4 + b_4, a'_4 + b'_4\}$

(ii) Let $\tilde{A} = \{a'_1, a_1, a'_2, a_2, a_3, a'_3, a_4, a'_4\}$ and $\tilde{B} = \{b'_1, b_1, b'_2, b_2, b_3, b'_3, b_4, b'_4\}$ be two TIF numbers. Then, $\tilde{A} \ominus \tilde{B} = (a'_1 - b'_4, a_1 - b_4, a'_2 - b'_3, a_2 - b_3, a_3 - b_2, a'_3 - b'_2, a_4 - b_1, a'_4 - b'_1\}$

(iii) Let $\tilde{A} = \{a'_1, a_1, a'_2, a_2, a_3, a'_3, a_4, a'_4\}$ be any TIF number. Then,

$$\lambda\tilde{A} = \begin{cases} (\lambda a'_1, \lambda a_1, \lambda a'_2, \lambda a_2, \lambda a_3, \lambda a'_3, \lambda a_4, \lambda a'_4), & \lambda \geq 0 \\ (\lambda a'_4, \lambda a_4, \lambda a'_3, \lambda a_3, \lambda a_2, \lambda a'_2, \lambda a_1, \lambda a'_1), & \lambda \leq 0 \end{cases}$$

(iv) Let $\tilde{A} = \{a'_1, a_1, a'_2, a_2, a_3, a'_3, a_4, a'_4\}$ and $\tilde{B} = \{b'_1, b_1, b'_2, b_2, b_3, b'_3, b_4, b'_4\}$ be two non-negative TIF numbers. Then, $\tilde{A} \otimes \tilde{B} = (a'_1 b'_1, a_1 b_1, a'_2 b'_2, a_2 b_2, a_3 b_3, a'_3 b'_3, a_4 b_4, a'_4 b'_4)$

3. Limitations of the existing methods

In this section the limitations of the existing methods [5, 12, 13, 15, 18, 19, 24, 25] for comparing IF numbers are pointed out.

1. The existing methods [12, 15, 24, 25] can be used only for comparing IF set, however, none of the existing methods [12, 15, 24, 25] can be used for comparing IF numbers.
2. Nayagam and Sivaraman [18] pointed out the shortcomings of the existing method [19] and proposed a method for comparing such TrIF numbers $\{(a, b, c), (e, f, g)\}$ for which either the conditions $e \geq b$ and $f \geq c$ or the conditions $f \leq a$ and $g \leq b$ are satisfied, however, the existing method [18] cannot be used for comparing such TrIF numbers for which neither the conditions $e \geq b$ and $f \geq c$ nor the conditions $f \leq a$ and $g \leq b$ are satisfied.
3. Dubey and Mehra [5] pointed out the shortcomings of the existing methods [13] and proposed a method for comparing such TrIF sets $\tilde{a} = \{(\underline{a}^{\mu}, a, \overline{a}^{\mu}; w_{\tilde{a}}), (\underline{a}^{\upsilon}, a, \overline{a}^{\upsilon}; u_{\tilde{a}})\}$ for which the condition $V_{\mu}(\tilde{a}) \leq V_{\upsilon}(\tilde{a})$ is satisfied, where $V_{\mu}(\tilde{a}) = ((\underline{a}^{\mu} + 4a + \overline{a}^{\mu}) w_{\tilde{a}})/6$ and $V_{\upsilon}(\tilde{a}) = ((\underline{a}^{\upsilon} + 4a + \overline{a}^{\upsilon})(1 - u_{\tilde{a}}))/6$, however, the existing method [18] cannot be used for comparing such TrIF sets for which the condition $V_{\mu}(\tilde{a}) \leq V_{\upsilon}(\tilde{a})$ is not satisfied.

4. Shortcomings of the existing methods

In this section the shortcomings of the existing methods [13, 17, 19, 20] for comparing IF numbers are pointed out.

1. Li [13] proposed the following method for comparing TrIF sets. Let $\tilde{A} = \langle (a_1, a_2, a_3; w_1, u_1) \rangle$ and $\tilde{B} = \langle (a_1, a_2, a_3; w_2, u_2) \rangle$ be two TrIF sets.

(i) $\tilde{A} \precsim \tilde{B}$ if $L_T^{\lambda}(\tilde{A}) < L_T^{\lambda}(\tilde{B})$

(ii) $\tilde{A} \succsim \tilde{B}$ if $L_T^{\lambda}(\tilde{A}) > L_T^{\lambda}(\tilde{B})$

(iii) $\tilde{A} \cong \tilde{B}$ if $L_T^{\lambda}(\tilde{A}) = L_T^{\lambda}(\tilde{B})$

Where

$$L_T^{\lambda}(\tilde{A}) = \frac{V^{\lambda}(\tilde{A})}{1 + A^{\lambda}(\tilde{A})}, V^{\lambda}(\tilde{A}) = V_{\mu}(\tilde{A}) + \lambda(V_{\upsilon}(\tilde{A}) - V_{\mu}(\tilde{A})),$$

$$A^{\lambda}(\tilde{A}) = A_{\mu}(\tilde{A}) - \lambda(A_{\upsilon}(\tilde{A}) - A_{\mu}(\tilde{A})), V_{\mu}(\tilde{A}) = \frac{w_1}{6}(a_1 +$$

$2a_2+a_3)$, $A_\mu(\widetilde{A})=\frac{w_1}{3}(a_3-a_1)$, $A_\upsilon(\widetilde{A})=\frac{(1-u_1)}{3}(a_3-a_1)$

It is not genuine to apply this method due to the following reasons. It is obvious from the existing ranking approach [13] that if $(a_1+4a_2+a_3)\neq 0$ then the comparison of $\widetilde{A}$ and $\widetilde{B}$ will depend upon the values of w_1,u_1,w_2,u_2 and if $(a_1+4a_2+a_3)=0$ then $\widetilde{A}\cong\widetilde{B}$ for all values of w_1,u_1,w_2 and u_2 i.e., according to the existing approach [13], in the first case comparison of TrIF sets depends upon the degree of membership and the degree of nonmembership of IF sets whereas in the second case comparison of TrIF sets does not depend upon the degree of membership and the degree of nonmembership of IF sets which is a contradiction.

Example 4.1. Let $\widetilde{A}=\langle(1,1,1);w_1,u_1\rangle$ and $\widetilde{B}=\langle(1,1,1);w_2,u_2\rangle$ be two TrIF sets then according to the existing ranking approach [13] values of $L_T^\lambda(\widetilde{A})$ and $L_T^\lambda(\widetilde{B})$ will depend upon the values of w_1,u_1,w_2 and u_2 i.e., the ordering of $\widetilde{A}$ and $\widetilde{B}$ will depend upon the values of w_1,u_1,w_2 and u_2.

Example 4.2. Let $\widetilde{A}=\langle(-8,1,4);w_1,u_1\rangle$ and $\widetilde{B}=\langle(-8,1,4);w_2,u_2\rangle$ be two TrIF sets. Then, according to the existing ranking approach [13], $L_T^\lambda(\widetilde{A})=L_T^\lambda(\widetilde{B})=0\Rightarrow\widetilde{A}\cong\widetilde{B}$ i.e., in this case the ordering of $\widetilde{A}$ and $\widetilde{B}$ is independent from the values of w_1,u_1,w_2 and u_2.

2. Nayagam and Sivaraman [17] proposed the following method for comparing interval valued IF sets:

   Let $\widetilde{A}=\{x:[a_1,b_1],[c_1,d_1]\mid x\in X\}$ and $\widetilde{B}=\{x:[a_2,b_2],[c_2,d_2]\mid x\in X\}$ be two interval valued IF sets defined on a universal set X. Then

(i) $\widetilde{A}\precsim\widetilde{B}$ if $LG(\widetilde{A})<LG(\widetilde{B})$

(ii) $\widetilde{A}\succsim\widetilde{B}$ if $LG(\widetilde{A})>LG(\widetilde{B})$

(iii) $\widetilde{A}\cong\widetilde{B}$ if $LG(\widetilde{A})=LG(\widetilde{B})$

where $LG(\widetilde{A})=\frac{(a_1+b_1)(1-\delta)+\delta(2-(c_1+d_1)}{2}$

$LG(\widetilde{B})=\frac{(a_2+b_2)(1-\delta)+\delta(2-(c_2+d_2)}{2}$

and $\delta\in[0,1]$.

   Nayagam and Sivaraman [17] have used the same method for comparing TIF numbers

   $\widetilde{A}=\{(a_1,a_1,b_1,b_1),(c_1,c_1,d_1,d_1)\}$ and
   $\widetilde{B}=\{(a_2,a_2,b_2,b_2),(c_2,c_2,d_2,d_2)\}$.

   Nonetheless, it is not genuine to use this method for comparing TIF numbers due to the following reasons

   In the IF set $\widetilde{A}=\{x:[a_1,b_1],[c_1,d_1]\mid x\in X\}$ a_1,b_1 and c_1,d_1 represents the infimum and supremum values of membership degree and nonmembership degree corresponding to points X. Although, in the TIF number $\widetilde{A}=\{(a_1,a_1,b_1,b_1),(c_1,c_1,d_1,d_1)\}$ a_1 and b_1 represent those points of universal set X corresponding to which the membership degree is 1. Similarly, c_1 and d_1 represent those points of universal set X corresponding to which the nonmembership degree is 1.

3. Given that, IF numbers are the generalization of fuzzy numbers, hence, the approach which can be used for comparing IF numbers can also be used for comparing fuzzy numbers. To show the shortcomings of the existing approach [20], two fuzzy numbers are compared by using such approaches [6,14,20] and it was demonstrated that the ordering of fuzzy numbers obtained by using Nehi's approach [20] and Liou and Wang approach [14] is the same one and contradicts the ordering of fuzzy numbers obtained by using the Garcia and Lamata's approach[6], because, Garcia and Lamata's approach [6] pointed out that their approach is better than Liou and Wang's approach [14]. Thus, the ordering of IF numbers given by Nehi's is not genuine.

Example 4.3. Let us consider two trapezoidal fuzzy numbers $\tilde{A}=(1,4,4,5)$ and $\tilde{B}=(2,3,3,6)$. The intuitionistic representation of these trapezoidal fuzzy numbers is $\tilde{A}=(1,1,4,4,4,4,5,5)$ and $\tilde{B}=(2,2,3,3,3,3,6,6)$.

The ordering of these numbers obtained by using the existing ranking approaches [6, 14, 20] are shown in Table 1.

Approaches	Ordering
Liou and Wang [14]	$\tilde{A}\cong\tilde{B}$
Nehi [20]	$\tilde{A}\cong\tilde{B}$
Garcia and Lamata [6]	$\tilde{A}\succ\tilde{B}$

Table 1. Comparing results.

It is obvious from the results, shown in Table 1 that the ordering of fuzzy numbers $\tilde{A}$ and $\tilde{B}$, obtained by using the existing approach [20], is the same as the one obtained by the existing approach [14] whereas it is different from the ordering obtained by using the existing approach [6].

5. Proposed ranking approach for comparing two IF numbers

Garcia and Lamata [6] pointed out that although trapezoidal fuzzy number $\tilde{A}=(a_1,a_2,a_3,a_4)$, is defined by four points, in the existing formula [14], $\Re(\tilde{A})=(\lambda\int_0^1(a_1+(a_2-a_1)\alpha)+(1-\lambda)\int_0^1(a_4+(a_3-a_4)\alpha))$ where, $\lambda\in[0,1]$ the central points a_2 and a_3 are taken into account in an indirect way and proposed the following modified formula $\Re(\tilde{A})=(\beta[\int_0^1(\lambda a_2+(1-\lambda)a_3)d\alpha]+(1-\beta)[\lambda\int_0^1(a_1+(a_2-a_1)\alpha)d\alpha+(1-\lambda)\int_0^1(a_4+(a_3-a_4)\alpha)d\alpha])$. Because in the existing formula [20], $C_\mu^k(\tilde{A})=\frac{k+1}{2}\int_0^1 r^k[a_1+(a_2-a_1)r+a_4+(a_3-a_4)r]dr$, $C_\nu^k(\tilde{A})=\frac{k+1}{2}\int_0^1 r^k[a'_1+(a'_2-a'_1)r+a'_4+(a'_3-a'_4)r]dr$, where, $k\in[0,\infty)$ the central points a_2, a_3 and a'_2, a'_3 are also taken into account in an indirect way hence, to overcome the shortcomings of existing approach [20], pointed out in Section 4, a new ranking approach, by modifying the existing ranking approach [20], is proposed for comparing IF numbers $\tilde{A}=(a'_1,a_1,a'_2,a_2,a_3,a'_3,a_4,a'_4)$ and $\tilde{B}=(b'_1,b_1,b'_2,b_2,b_3,b'_3,b_4,b'_4)$.

The steps of the proposed ranking approach are as follows:

Step 1. Calculate, $M_\mu^{\beta,k}(\tilde{A})=(\beta[\frac{k+1}{2}\int_0^1 r^k(a_2+a_3)dr]+(1-\beta)[\frac{k+1}{2}\int_0^0 r^k[a_1+(a_2-a_1)r+a_4+(a_3-a_4)r]dr])$, $M_\mu^{\beta,k}(\tilde{B})=(\beta[\frac{k+1}{2}\int_0^1 r^k(b_2+b_3)dr]+(1-\beta)[\frac{k+1}{2}\int_0^0 r^k[b_1+(b_2-b_1)r+b_4+(b_3-b_4)r]dr])$, and check $M_\mu^{\beta,k}(\tilde{A})>M_\mu^{\beta,k}(\tilde{B})$ or $M_\mu^{\beta,k}(\tilde{A})<M_\mu^{\beta,k}(\tilde{B})$ or $M_\mu^{\beta,k}(\tilde{A})=M_\mu^{\beta,k}(\tilde{B})$.

Case (i) If $M_\mu^{\beta,k}(\tilde{A})>M_\mu^{\beta,k}(\tilde{B})$ then $\tilde{A}\succ\tilde{B}$ i.e., minimum$(\tilde{A},\tilde{B})=\tilde{B}$.

Case (ii) If $M_\mu^{\beta,k}(\tilde{A})<M_\mu^{\beta,k}(\tilde{B})$ then $\tilde{A}\prec\tilde{B}$ i.e., minimum$(\tilde{A},\tilde{B})=\tilde{A}$.

Case (iii) If $M_\mu^{\beta,k}(\tilde{A})=M_\mu^{\beta,k}(\tilde{B})$ then go to step 2.

Step 2. Calculate, $M_\upsilon^{\beta,k}(\tilde{A})=(\beta[\frac{k+1}{2}\int_0^1 r^k(a'_2+a'_3)dr]+(1-\beta)[\frac{k+1}{2}\int_0^0 r^k[a'_1+(a'_2-a'_1)r+a'_4+(a'_3-a'_4)$

$r]dr])$, $M_{\upsilon}^{\beta,k}(\widetilde{B}) = (\beta[\frac{k+1}{2}\int_0^1 r^k(b'_2+b'_3)\,dr] + (1-\beta)$ $[\frac{k+1}{2}\int_0^0 r^k[b'_1+(b'_2-b'_1)r+b'_4+(b'_3-b'_4)\,r]dr])$, and check $-M_{\upsilon}^{\beta,k}(\widetilde{A}) > -M_{\upsilon}^{\beta,k}(\widetilde{B})$ or $-M_{\upsilon}^{\beta,k}(\widetilde{A}) < -M_{\upsilon}^{\beta,k}(\widetilde{B})$ or $-M_{\upsilon}^{\beta,k}(\widetilde{A}) = -M_{\upsilon}^{\beta,k}(\widetilde{B})$.

Case (i) If $-M_{\upsilon}^{\beta,k}(\widetilde{A}) > -M_{\upsilon}^{\beta,k}(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e., minimum $(\widetilde{A},\widetilde{B}) = \widetilde{B}$.

Case (ii) If $-M_{\upsilon}^{\beta,k}(\widetilde{A}) < -M_{\upsilon}^{\beta,k}(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e., minimum $(\widetilde{A},\widetilde{B}) = \widetilde{A}$.

Case (iii) If $M_{\upsilon}^{\beta,k}(\widetilde{A}) = M_{\upsilon}^{\beta,k}(\widetilde{B})$ then $\widetilde{A} \cong \widetilde{B}$.

Remark 3. For $\beta = 1/3$ and $k = 0$ the index for membership and nonmembership functions $M_{\mu}^{1/3,0}(\widetilde{A}_i)$ and $M_{\upsilon}^{1/3,0}(\widetilde{A}_i)$, are as follows:

$$M_{\mu}^{1/3,0}(\widetilde{A}_i) = \frac{1}{3}\left(\frac{a_1+2a_2+2a_3+a_4}{2}\right) \text{ and}$$

$$M_{\upsilon}^{1/3,0}(\widetilde{A}_i) = \frac{1}{3}\left(\frac{a'_1+2a'_2+2a'_3+a'_4}{2}\right)$$

5.1. Advantages of the proposed ranking approach

In this section, the advantages of the proposed ranking approach, over the existing ranking approaches for comparing fuzzy numbers as well as IF numbers [5, 6, 13, 18-20] are discussed.

1. The existing ranking approach [6] can be used only for comparing fuzzy numbers but cannot be used for comparing IF numbers. Whereas the proposed ranking approach can be used for comparing fuzzy numbers as well as IF numbers and the ordering of fuzzy numbers, obtained by using the proposed ranking approach, is the same as that one obtained by using the existing ranking approach [6].
2. Although the existing ranking approaches [5, 13, 17-20] can be used for comparing fuzzy numbers as well as IF numbers, due to shortcomings pointed out in Section 4, it is not genuine to apply these approaches. Whereas by using the proposed ranking approach, all the shortcomings, occurring in the results due to the use of existing approaches [5, 13, 17-20], are resolved.

6. MCF problems in IF environment

The aim of the MCF problems is to find minimum cost for transporting the product from a given point to certain destinations with the assumption that there is at least one node, called intermediate node, at which the product can be stored in the case of excess availability of the product and the stored product can be supplied in the case of excess demand of the product.

Ghatee and Hashemi [7] modified the existing linear programming formulation of balanced MCF problems by representing all of its parameters as fuzzy numbers instead of crisp numbers and proposed a method to find the fuzzy optimal solution of such balanced MCF problems.

On the same direction, in this section the existing linear programming formulation of MCF problems is further generalized by representing all the parameters as TIF numbers and on the basis of the ranking approach, proposed in Section 5, a new method is proposed to find the IF optimal solution of such MCF problems.

6.1. Linear programming formulations of balanced MCF problems in crisp and IF environment

6.1.1. In this subsection, the linear programming formulation of balanced MCF problems in crisp environment is presented [1].

If the set of source nodes, destination nodes, such intermediate nodes from which some quantity of the product is supplied, such intermediate nodes at which some quantity of the product is stored and such intermediate nodes from which the product is neither supplied nor stored, of a directed and connected network, are represented by N_S, N_D, N_{IS}, N_{ID} and N_{IT} respectively. Then, the MCF problem can be formulated as

Minimize $\sum_{(i,j)\in A}(c_{ij}x_{ij})$

Subject to

$$\sum_{j:(i,j)\in A} x_{ij} = a_i \qquad \forall i \in N_S,$$
$$\sum_{j:(i,j)\in A} x_{ij} = \sum_{j:(j,i)\in A} x_{ji} + e_i \qquad \forall i \in N_{IS},$$
$$\sum_{i:(i,j)\in A} x_{ij} = b_j \qquad \forall j \in N_D, \qquad (P_1)$$
$$\sum_{i:(i,j)\in A} x_{ij} = \sum_{i:(j,i)\in A} x_{ji} + d_j \qquad \forall j \in N_{ID},$$
$$\sum_{j:(i,j)\in A} x_{ij} = \sum_{j:(j,i)\in A} x_{ji} \qquad \forall i \in N_{IT},$$
$$0 \le l_{ij} \le x_{ij} \le u_{ij} \qquad \forall (i,j)\in A$$

Where A : The set of arcs (i,j), x_{ij} : Decision variable denoting the transportation amount of the product from i^{th} node to j^{th} node, c_{ij} : Cost for transporting one unit quantity of the product from i^{th} node to j^{th} node, a_i : Supply of the product at i^{th} source node, e_i : Supply of the product at i^{th} intermediate source node, b_j : Demand of the product at j^{th} destination node, d_j : Demand of the product at j^{th} intermediate destination node, l_{ij} : Minimum quantity of the product that can be transported from i^{th} node to j^{th} node, u_{ij} : Maximum quantity of the product that can be transported from i^{th} node to j^{th} node.

6.1.2 Linear programming formulation of balanced MCF problems in IF environment

In this section the linear programming formulation of balanced MCF problems in IF environment is presented.

Suppose all the parameters $c_{ij}, x_{ij}, a_i, e_i, b_j, d_j, l_{ij}, u_{ij}$ are represented by non-negative TIF numbers $\tilde{c}_{ij}$, $\tilde{x}_{ij}, \tilde{a}_i, \tilde{e}_i, \tilde{b}_j, \tilde{d}_j, \tilde{l}_{ij}, \tilde{u}_{ij}$. Then (P_1) in IF environment is:

inimize $\sum_{(i,j)\in A}(\tilde{c}_{ij}\otimes \tilde{x}_{ij})$

Subject to

$$\sum_{j:(i,j)\in A} \tilde{x}_{ij} = \tilde{a}_i \qquad \forall i \in N_S,$$
$$\sum_{j:(i,j)\in A} \tilde{x}_{ij} = \sum_{j:(j,i)\in A} \tilde{x}_{ji} \oplus \tilde{e}_i \qquad \forall i \in N_{IS},$$
$$\sum_{i:(i,j)\in A} \tilde{x}_{ij} = \tilde{b}_j \qquad \forall j \in N_D, \qquad (P_2)$$
$$\sum_{i:(i,j)\in A} \tilde{x}_{ij} = \sum_{i:(j,i)\in A} \tilde{x}_{ji} \oplus \tilde{d}_j \qquad \forall j \in N_{ID},$$
$$\sum_{j:(i,j)\in A} \tilde{x}_{ij} = \sum_{j:(j,i)\in A} \tilde{x}_{ji} \qquad \forall i \in N_{IT},$$
$$\tilde{0} \lesssim \tilde{l}_{ij} \lesssim \tilde{x}_{ij} \lesssim \tilde{u}_{ij} \qquad \forall (i,j)\in A$$

6.2. Proposed method

In this section, a new method is proposed for finding the exact IF optimal solution of such MCF problems in which all the parameters are represented by IF numbers. If the supply of product at i^{th} source node and i^{th} intermediate source node are $\tilde{a}_i = (a'_{1_i}, a_{1_i}, a'_{2_i}, a_{2_i}, a_{3_i}, a'_{3_i}, a_{4_i}, a'_{4_i})$ and $\tilde{e}_i = (e'_{1_i}, e_{1_i}, e'_{2_i}, e_{2_i}, e_{3_i}, e'_{3_i}, e_{4_i}, e'_{4_i})$ respectively and the demand of the product at j^{th} destination node and j^{th} intermediate destination node are $\tilde{b}_j = (b'_{1_j}, b_{1_j}, b'_{2_j}, b_{2_j}, b_{3_j}, b'_{3_j}, b_{4_j}, b'_{4_j})$ and $\tilde{d}_j = (d'_{1_j}, d_{1_j}, d'_{2_j}, d_{2_j}, d_{3_j}, d'_{3_j}, d_{4_j}, d'_{4_j})$ respectively, then the exact IF optimal solution of IF MCF problems can be obtained by using the following steps

Step 1. Find the total fuzzy supply $\sum_{i\in N_S}\tilde{a}_i \oplus \sum_{i\in N_{IS}}\tilde{e}_i$ and the total IF demand $\sum_{j\in N_D}\tilde{b}_j \oplus \sum_{j\in N_{ID}}\tilde{d}_j$. Let $\sum_{i\in N_S}\tilde{a}_i \oplus \sum_{i\in N_{IS}}\tilde{e}_i = (m'_1, m_1, m'_2, m_2, m_3, m'_3, m_4, m'_4)$ and $\sum_{j\in N_D}\tilde{b}_j \oplus \sum_{j\in N_{ID}}\tilde{d}_j = (n'_1, n_1, n'_2, n_2, n_3, n'_3, n_4, n'_4)$.

Examine whether the problem is balanced or not, i.e.,

$$\sum_{i\in N_S}\tilde{a}_i \oplus \sum_{i\in N_{IS}}\tilde{e}_i = \sum_{j\in N_D}\tilde{b}_j \oplus \sum_{j\in N_{ID}}\tilde{d}_j \text{ or } \sum_{i\in N_S}\tilde{a}_i \oplus \sum_{i\in N_{IS}}\tilde{e}_i \neq \sum_{j\in N_D}\tilde{b}_j \oplus \sum_{j\in N_{ID}}\tilde{d}_j .$$

Case (1) If the problem is balanced i.e., $\sum_{i\in N_S}\tilde{a}_i \oplus \sum_{i\in N_{IS}}\tilde{e}_i = \sum_{j\in N_D}\tilde{b}_j \oplus \sum_{j\in N_{ID}}\tilde{d}_j$, then go to Step 2.

Case (2) If the problem is unbalanced i.e., $\sum_{i\in N_S}\tilde{a}_i \oplus \sum_{i\in N_{IS}}\tilde{e}_i \neq \sum_{j\in N_D}\tilde{b}_j \oplus \sum_{j\in N_{ID}}\tilde{d}_j$,then convert the unbalanced problem into balanced problem as follows

Case (2a) Introduce a dummy source node with IF supply $\big(\max\{0,(n'_1-m'_1)\}, \max\{0,(n'_1-m'_1\}+\max\{0,(n_1-n'_1)-(m_1-m'_1)\}, \max\{0,(n'_1-m'_1)\}+\max\{0,(n_1-n'_1)-(m_1-m'_1)\}+\max\{0,(n'_2-n_1)-(m'_2-m_1)\}, \max\{0,(n'_1-m'_1)\}+\max\{0,(n_1-n'_1)-(m_1-m'_1)\}+\max\{0,(n'_2-n_1)-(m'_2-m_1)\}+\max\{0,(n_2-n'_2)-(m_2-m'_2)\}, \max\{0,(n'_1-m'_1)\}+\max\{0,(n_1-n'_1)-(m_1-m'_1)\}+\max\{0,(n'_2-n_1)-(m'_2-m_1)\}+\max\{0,(n_2-n'_2)-(m_2-m'_2)\}+\{0,(n_3-n_2)-(m_3-m_2)\}, \max\{0,(n'_1-m'_1)\}+\max\{0,(n_1-n'_1)-(m_1-m'_1)\}+\max\{0,(n'_2-n_1)-(m'_2-m_1)\}+\max\{0,(n_2-n'_2)-(m_2-m'_2)\}+\max\{0,(n_3-n_2)-(m_3-m_2)\}+\max\{0,(n'_3-n_3)-(m'_3-m_3)\}, \max\{0,(n'_1-m'_1)\}+\max\{0,(n_1-n'_1)-(m_1-m'_1)\}+\max\{0,(n'_2-n'_1)-(m'_2-m_1)\}+\max\{0,(n_2-n'_2)-(m_2-m'_2)\}+\max\{0,(n_3-n_2)-(m_3-m_2)\}+\max\{0,(n'_3-n_3)-(m'_3-m_3)\}+\max\{0,(n_4-n'_3)-(m_4-m'_3)\}, \max\{0,(n'_1-m'_1)\}+\max\{0,(n_1-n'_1)-(m_1-m'_1)\}+\max\{0,(n'_2-n'_1)-(m'_2-m_1)\}+\max\{0,(n_2-n'_2)-(m_2-m'_2)\}+\max\{0,(n_3-n_2)-(m_3-m_2)\}+\max\{0,(n'_3-n_3)-(m'_3-m_3)\}+\max\{0,(n_4-n'_3)-(m_4-m'_3)\}+\max\{0,(n'_4-n_4)-(m'_4-m_4)\}\big)$ and also introduced a dummy destination node with IF demand $\big(\max\{0,(m'_1-n'_1)\}, \max\{0,(m'_1-n'_1\}+\max\{0,(m_1-m'_1)-(n_1-n'_1)\}, \max\{0,(m'_1-n'_1)\}+\max\{0,(m_1-m'_1)-(n_1-n'_1)\}+\max\{0,(m'_2-m_1)-(n'_2-n_1)\}, \max\{0,(m'_1-n'_1)\}+\max\{0,(m_1-m'_1)-(n_1-n'_1)\}+\max\{0,(m'_2-m_1)-(n'_2-n_1)\}+\max\{0,(m_2-m'_2)-(n_2-n'_2)\}, \max\{0,(m'_1-n'_1)\}+\max\{0,(m_1-m'_1)-(n_1-n'_1)\}+\max\{0,(m'_2-m_1)-(n'_2-n_1)\}+\max\{0,(m_2-m'_2)-(n_2-n'_2)+\max\{0,(m_3-m_2)-(n_3-n_2)\}, \max\{0,(m'_1-n'_1)\}+\max\{0,(m_1-m'_1)-(n_1-n'_1)\}+\max\{0,(m'_2-m_1)-(n'_2-n_1)\}+\max\{0,(m_2-m'_2)-(n_2-n'_2)\}+\max\{0,(m_3-m_2)-(n_3-n_2)\}+\max\{0,(m'_3-m_3)-(n'_3-n_3)\}, \max\{0,(m'_1-n'_1)\}+\max\{0,(m_1-m'_1)-(n_1-n'_1)\}+\max\{0,(m'_2-m'_1)-(n'_2-n_1)\}+\max\{0,(m_2-m'_2)-(n_2-n'_2)\}+\max\{0,(m_3-m_2)-(n_3-n_2)\}+\max\{0,(m'_3-m_3)-(n'_3-n_3)\}+\max\{0,(m_4-m'_3)-(n_4-n'_3)\}, \max\{0,(m'_1-n'_1)\}+\max\{0,(m_1-m'_1)-(n_1-n'_1)\}+\max\{0,(m'_2-m'_1)-(n'_2-n_1)\}+\max\{0,(m_2-m'_2)-(n_2-n'_2)\}+\max\{0,(m_3-m_2)-(n_3-n_2)\}+\max\{0,(m'_3-m_3)-(n'_3-n_3)\}+\max\{0,(m_4-m'_3)-(n_4-n'_3)\}+\max\{0,(m'_4-m_4)-(n'_4-n_4)\}\big)$. Assume the IF cost for transporting one unit quantity of the product from the introduced dummy source node to all intermediate nodes, existing destination nodes and introduced dummy destination node as zero IF number. Similarly, assume the IF cost for transporting one unit quantity of the product from all intermediate nodes, existing source nodes and introduced dummy source node to the introduced dummy destination node as zero IF number and go to Step 2.

Step 2 Assuming $\tilde{c}_{ij}=(c'_{1_{ij}},c_{1_{ij}},c'_{2_{ij}},c_{2_{ij}},c_{3_{ij}},c'_{3_{ij}},c_{4_{ij}},c'_{4_{ij}})$, $\tilde{x}_{ij}=(x'_{1_{ij}},x_{1_{ij}},x'_{2_{ij}},x_{2_{ij}},x_{3_{ij}},x'_{3_{ij}},x_{4_{ij}},x'_{4_{ij}})$, $\tilde{a}_i=(a'_{1_i},a_{1_i},a'_{2_i},a_{2_i},a_{3_i},a'_{3_i},a_{4_i},a'_{4_i})$, $\tilde{e}_i=(e'_{1_i},e_{1_i},e'_{2_i},e_{2_i},e_{3_i},e'_{3_i},e_{4_i},e'_{4_i})$, $\tilde{b}_j=(b'_{1_j},b_{1_j},b'_{2_j},b_{2_j},b_{3_j},b'_{3_j},b_{4_j},b'_{4_j})$, $\tilde{d}_j=(d'_{1_j},d_{1_j},d'_{2_j},d_{2_j},d_{3_j},d'_{3_j},d_{4_j},d'_{4_j})$, $\tilde{l}_{ij}=(l'_{1_{ij}},l_{1_{ij}},l'_{2_{ij}},l_{2_{ij}},l_{3_{ij}},l'_{3_{ij}},l_{4_{ij}},l'_{4_{ij}})$ $\tilde{u}_{ij}=(u'_{1_{ij}},u_{1_{ij}},u'_{2_{ij}},u_{2_{ij}},u_{3_{ij}},u'_{3_{ij}},u_{4_{ij}},u'_{4_{ij}})$ using the arithmetic operations of IF numbers, defined in Subsection 2.2 and using the IF linear programming formulation (P_2) of the balanced IF MCF problem, the IF linear programming formulation of balanced IF MCF problem obtained in Step 1 can be written as

Minimize $\sum_{(i,j)\in A}(c'_{1_{ij}}\,x'_{1_{ij}},c_{1_{ij}}x_{1_{ij}},c'_{2_{ij}}\,x'_{2_{ij}},c_{2_{ij}}x_{2_{ij}},c_{3_{ij}}x_{3_{ij}},c'_{3_{ij}}\,x'_{3_{ij}},c_{4_{ij}}x_{4_{ij}},c'_{4_{ij}}\,x'_{4_{ij}})$

Subject to

$$\Big(\sum_{j:(i,j)\in A} x'_{1_{ij}}, \sum_{j:(i,j)\in A} x_{1_{ij}}, \sum_{j:(i,j)\in A} x'_{2_{ij}}, \sum_{j:(i,j)\in A} x_{2_{ij}}, \sum_{j:(i,j)\in A} x_{3_{ij}}, \sum_{j:(i,j)\in A} x'_{3_{ij}}, \sum_{j:(i,j)\in A} x_{4_{ij}}, \sum_{j:(i,j)\in A} x'_{4_{ij}}\Big) = (a'_{1_i}, a_{1_i}, a'_{2_i}, a_{2_i}, a_{3_i}, a'_{3_i}, a_{4_i}, a'_{4_i}), \quad \forall i \in N_S$$

$$\Big(\sum_{j:(i,j)\in A} x'_{1_{ij}}, \sum_{j:(i,j)\in A} x_{1_{ij}}, \sum_{j:(i,j)\in A} x'_{2_{ij}}, \sum_{j:(i,j)\in A} x_{2_{ij}}, \sum_{j:(i,j)\in A} x_{3_{ij}}, \sum_{j:(i,j)\in A} x'_{3_{ij}}, \sum_{j:(i,j)\in A} x_{4_{ij}}, \sum_{j:(i,j)\in A} x'_{4_{ij}}\Big) = \Big(\sum_{j:(j,i)\in A} x'_{1_{ji}} + e'_{1_i}, \sum_{j:(j,i)\in A} x_{1_{ji}} + e_{1_i}, \sum_{j:(j,i)\in A} x'_{2_{ji}} + e'_{2_i}, \sum_{j:(j,i)\in A} x_{2_{ji}} + e_{2_i}, \sum_{j:(j,i)\in A} x_{3_{ji}} + e_{3_i}, \sum_{j:(j,i)\in A} x'_{3_{ji}} + e'_{3_i}, \sum_{j:(j,i)\in A} x_{4_{ji}} + e_{4_i}, \sum_{j:(j,i)\in A} x'_{4_{ji}} + e'_{4_i}\Big), \quad \forall i \in N_{IS}$$

$$\Big(\sum_{j:(i,j)\in A} x'_{1_{ij}}, \sum_{j:(i,j)\in A} x_{1_{ij}}, \sum_{j:(i,j)\in A} x'_{2_{ij}}, \sum_{j:(i,j)\in A} x_{2_{ij}}, \sum_{j:(i,j)\in A} x_{3_{ij}}, \sum_{j:(i,j)\in A} x'_{3_{ij}}, \sum_{j:(i,j)\in A} x_{4_{ij}}, \sum_{j:(i,j)\in A} x'_{4_{ij}}\Big) = (b'_{1_j}, b_{1_j}, b'_{2_j}, b_{2_j}, b_{3_j}, b'_{3_j}, b_{4_j}, b'_{4_j}), \quad \forall j \in N_D \qquad (P_3)$$

$$\Big(\sum_{i:(i,j)\in A} x'_{1_{ij}}, \sum_{i:(i,j)\in A} x_{1_{ij}}, \sum_{i:(i,j)\in A} x'_{2_{ij}}, \sum_{i:(i,j)\in A} x_{2_{ij}}, \sum_{i:(i,j)\in A} x_{3_{ij}}, \sum_{i:(i,j)\in A} x'_{3_{ij}}, \sum_{i:(i,j)\in A} x_{4_{ij}}, \sum_{i:(i,j)\in A} x'_{4_{ij}}\Big) = \Big(\sum_{i:(j,i)\in A} x'_{1_{ji}} + d'_{1_j}, \sum_{i:(j,i)\in A} x_{1_{ji}} + d_{1_j}, \sum_{i:(j,i)\in A} x'_{2_{ji}} + d'_{2_j}, \sum_{i:(j,i)\in A} x_{2_{ji}} + d_{2_j}, \sum_{i:(j,i)\in A} x_{3_{ji}} + d_{3_j}, \sum_{i:(j,i)\in A} x'_{3_{ji}} + d'_{3_j}, \sum_{i:(j,i)\in A} x_{4_{ji}} + d_{4_j}, \sum_{i:(j,i)\in A} x'_{4_{ji}} + d'_{4_j}\Big), \quad \forall j \in N_{ID}$$

$$\Big(\sum_{j:(i,j)\in A} x'_{1_{ij}}, \sum_{j:(i,j)\in A} x_{1_{ij}}, \sum_{j:(i,j)\in A} x'_{2_{ij}}, \sum_{j:(i,j)\in A} x_{2_{ij}}, \sum_{j:(i,j)\in A} x_{3_{ij}}, \sum_{j:(i,j)\in A} x'_{3_{ij}}, \sum_{j:(i,j)\in A} x_{4_{ij}}, \sum_{j:(i,j)\in A} x'_{4_{ij}}\Big) = \Big(\sum_{j:(j,i)\in A} x'_{1_{ji}}, \sum_{j:(j,i)\in A} x_{1_{ji}}, \sum_{j:(j,i)\in A} x'_{2_{ji}}, \sum_{j:(j,i)\in A} x_{2_{ji}}, \sum_{j:(j,i)\in A} x_{3_{ji}}, \sum_{j:(j,i)\in A} x'_{3_{ji}}, \sum_{j:(j,i)\in A} x_{4_{ji}}, \sum_{j:(j,i)\in A} x'_{4_{ji}}\Big), \forall i \in N_{IT}$$

$(l'_{1_{ij}}, l_{1_{ij}}, l'_{2_{ij}}, l_{2_{ij}}, l_{3_{ij}}, l'_{3_{ij}}, l_{4_{ij}}, l'_{4_{ij}}) \tilde{\leq} (x'_{1_{ij}}, x_{1_{ij}}, x'_{2_{ij}}, x_{2_{ij}}, x_{3_{ij}}, x'_{3_{ij}}, x_{4_{ij}}, x'_{4_{ij}}) \tilde{\leq} (u'_{1_{ij}}, u_{1_{ij}}, u'_{2_{ij}}, u_{2_{ij}}, u_{3_{ij}}, u'_{3_{ij}}, u_{4_{ij}}, u'_{4_{ij}})$ $(x'_{1_{ij}}, x_{1_{ij}}, x'_{2_{ij}}, x_{2_{ij}}, x_{3_{ij}}, x'_{3_{ij}}, x_{4_{ij}}, x'_{4_{ij}})$ is a non-negative IF number $\forall (i,j) \in A$

Step 3. Using Definition 2.7 and Definition 2.8, the IF linear programming problem (P_3), can be written as

$$\text{Minimize} \sum_{(i,j)\in A} (c'_{1_{ij}}\, x'_{1_{ij}}, c_{1_{ij}} x_{1_{ij}}, c'_{2_{ij}}\, x'_{2_{ij}}, c_{2_{ij}} x_{2_{ij}}, c_{3_{ij}} x_{3_{ij}}, c'_{3_{ij}}\, x'_{3_{ij}}, c_{4_{ij}} x_{4_{ij}}, c'_{4_{ij}}\, x'_{4_{ij}})$$

Subject to

$$\sum_{j:(i,j)\in A} x'_{1_{ij}} = a'_{1_i}, \sum_{j:(i,j)\in A} x_{1_{ij}} = a_{1_i}, \sum_{j:(i,j)\in A} x'_{2_{ij}} = a'_{2_i}, \sum_{j:(i,j)\in A} x_{2_{ij}} = a_{2_i}, \sum_{j:(i,j)\in A} x_{3_{ij}} = a_{3_i}, \sum_{j:(i,j)\in A} x'_{3_{ij}} = a'_{3_i}, \sum_{j:(i,j)\in A} x_{4_{ij}} = a_{4_i}, \sum_{j:(i,j)\in A} x'_{4_{ij}} = a'_{4_i}; \quad \forall i \in N_S$$

$$\sum_{j:(i,j)\in A} x'_{1_{ij}} = \sum_{j:(j,i)\in A} x'_{1_{ji}} + e'_{1_i}, \quad \sum_{j:(i,j)\in A} x_{1_{ij}} = \sum_{j:(j,i)\in A} x_{1_{ji}} + e_{1_i},$$
$$\sum_{j:(i,j)\in A} x'_{2_{ij}} = \sum_{j:(j,i)\in A} x'_{2_{ji}} + e'_{2_i}, \quad \sum_{j:(i,j)\in A} x_{2_{ij}} = \sum_{j:(j,i)\in A} x_{2_{ji}} + e_{2_i},$$
$$\sum_{j:(i,j)\in A} x_{3_{ij}} = \sum_{j:(j,i)\in A} x_{3_{ji}} + e_{3_i}, \quad \sum_{j:(i,j)\in A} x'_{3_{ij}} = \sum_{j:(j,i)\in A} x'_{3_{ji}} + e'_{3_i},$$
$$\sum_{j:(i,j)\in A} x_{4_{ij}} = \sum_{j:(j,i)\in A} x_{4_{ji}} + e_{4_i}, \quad \sum_{j:(i,j)\in A} x'_{4_{ij}} = \sum_{j:(j,i)\in A} x'_{4_{ji}} + e'_{4_i}, \quad \forall i \in N_{IS}$$

$$\sum_{j:(i,j)\in A} x'_{1_{ij}} = b'_{1_j}, \sum_{j:(i,j)\in A} x_{1_{ij}} = b_{1_j}, \sum_{j:(i,j)\in A} x'_{2_{ij}} = b'_{2_j}, \sum_{j:(i,j)\in A} x_{2_{ij}} = b_{2_j}, \sum_{j:(i,j)\in A} x_{3_{ij}} = b_{3_j}, \sum_{j:(i,j)\in A} x'_{3_{ij}} = b'_{3_j}, \sum_{j:(i,j)\in A} x_{4_{ij}} = b_{4_j}, \sum_{j:(i,j)\in A} x'_{4_{ij}} = b'_{4_j}, \quad \forall j \in N_D \qquad (P_4)$$

$$\sum_{i:(i,j)\in A} x'_{1_{ij}} = \sum_{i:(j,i)\in A} x'_{1_{ji}} + d'_{1_j}, \quad \sum_{i:(i,j)\in A} x_{1_{ij}} = \sum_{i:(j,i)\in A} x_{1_{ji}} + d_{1_j},$$
$$\sum_{i:(i,j)\in A} x'_{2_{ij}} = \sum_{i:(j,i)\in A} x'_{2_{ji}} + d'_{2_j}, \quad \sum_{i:(i,j)\in A} x_{2_{ij}} = \sum_{i:(j,i)\in A} x_{2_{ji}} + d_{2_j},$$
$$\sum_{i:(i,j)\in A} x_{3_{ij}} = \sum_{i:(j,i)\in A} x_{3_{ji}} + d_{3_j}, \quad \sum_{i:(i,j)\in A} x'_{3_{ij}} = \sum_{i:(j,i)\in A} x'_{3_{ji}} + d'_{3_j},$$
$$\sum_{i:(i,j)\in A} x_{4_{ij}} = \sum_{i:(j,i)\in A} x_{4_{ji}} + d_{4_j}, \quad \sum_{i:(i,j)\in A} x'_{4_{ij}} = \sum_{i:(j,i)\in A} x'_{4_{ji}} + d'_{4_j} \quad \forall j \in N_{ID}$$

$$\sum_{j:(i,j)\in A} x'_{1_{ij}} = \sum_{j:(j,i)\in A} x'_{1_{ji}}, \sum_{j:(i,j)\in A} x_{1_{ij}} = \sum_{j:(j,i)\in A} x_{1_{ji}}, \sum_{j:(i,j)\in A} x'_{2_{ij}} = \sum_{j:(j,i)\in A} x'_{2_{ji}}, \sum_{j:(i,j)\in A} x_{2_{ij}} = \sum_{j:(j,i)\in A} x_{2_{ji}}, \sum_{j:(i,j)\in A} x_{3_{ij}} = \sum_{j:(j,i)\in A} x_{3_{ji}},$$
$$\sum_{j:(i,j)\in A} x'_{3_{ij}} = \sum_{j:(j,i)\in A} x'_{3_{ji}}, \sum_{j:(i,j)\in A} x_{4_{ij}} = \sum_{j:(j,i)\in A} x_{4_{ji}}, \sum_{j:(i,j)\in A} x'_{4_{ij}}$$

$$= \sum_{j:(j,i)\in A} x'_{4_{ji}} \qquad \forall i \in N_{IT}$$

$$x_{1_{ij}} - x'_{1_{ij}} \geq 0, x'_{2_{ij}} - x_{1_{ij}} \geq 0, x_{2_{ij}} - x'_{2_{ij}} \geq 0, x_{3_{ij}} - x_{2_{ij}} \geq 0, x'_{3_{ij}} - x_{3_{ij}} \geq 0, x_{4_{ij}} - x'_{3_{ij}} \geq 0, x'_{4_{ij}} - x_{4_{ij}} \geq 0, x'_{1_{ij}} \geq 0$$

$$\left.\begin{aligned} &(l'_{1_{ij}}, l_{1_{ij}}, l'_{2_{ij}}, l_{2_{ij}}, l_{3_{ij}}, l'_{3_{ij}}, l_{4_{ij}}, l'_{4_{ij}}) \precsim (x'_{1_{ij}}, x_{1_{ij}}, \\ &x'_{2_{ij}}, x_{2_{ij}}, x_{3_{ij}}, x'_{3_{ij}}, x_{4_{ij}}, x'_{4_{ij}}) \precsim (u'_{1_{ij}}, u_{1_{ij}}, u'_{2_{ij}}, \\ &u_{2_{ij}}, u_{3_{ij}}, u'_{3_{ij}}, u_{4_{ij}}, u'_{4_{ij}}) \end{aligned}\right\} (C_1)$$

Step 4. Suppose the IF linear programming problem (P_4) have f basic feasible solutions and $\{((x'_{1_{ij}})^w, (x_{1_{ij}})^w, (x'_{2_{ij}})^w, (x_{2_{ij}})^w, (x_{3_{ij}})^w, (x'_{3_{ij}})^w, (x_{4_{ij}})^w (x'_{4_{ij}})^w\}$ be the w^{th} basic feasible solution then the goal is to find such a basic feasible solution corresponding to which the value of the objective function is minimum i.e.,

$$\underset{1\leq w\leq f}{\text{minimum}}\Big(\sum_{(i,j)\in A}(c'_{1_{ij}}(x'_{1_{ij}})^w, c_{1_{ij}}(x_{1_{ij}})^w, c'_{2_{ij}}(x'_{2_{ij}})^w, c_{2_{ij}}(x_{2_{ij}})^w, c_{3_{ij}}(x_{3_{ij}})^w, c'_{3_{ij}}(x'_{3_{ij}})^w, c_{4_{ij}}(x_{4_{ij}})^w, c'_{4_{ij}}(x'_{4_{ij}})^w)$$

which can be obtained by using the ranking approach proposed in Section 5 i.e., the IF optimal solution of the IF linear programming problem, (P_4), can be obtained by solving the following crisp linear programming problem

$$\text{Minimize} \sum_{(i,j)\in A} M_\mu^{\beta,k}(c'_{1_{ij}}x'_{1_{ij}}, c_{1_{ij}}x_{1_{ij}}, c'_{2_{ij}}x'_{2_{ij}}, c_{2_{ij}}x_{2_{ij}}, c_{3_{ij}}x_{4_{ij}}, c'_{3_{ij}}x'_{3_{ij}}, c_{4_{ij}}x_{4_{ij}}, c'_{4_{ij}}x'_{4_{ij}})$$

Subject to (P_5)

$$M_\mu^{\beta,k}(l'_{1_{ij}}, l_{1_{ij}}, l'_{2_{ij}}, l_{2_{ij}}, l_{3_{ij}}, l'_{3_{ij}}, l_{4_{ij}}, l'_{4_{ij}}) \leq M_\mu^{\beta,k}(x'_{1_{ij}}, x_{1_{ij}}, x'_{2_{ij}}, x_{2_{ij}}, x_{3_{ij}}, x'_{3_{ij}}, x_{4_{ij}}, x'_{4_{ij}}) \leq M_\mu^{\beta,k}(u'_{1_{ij}}, u_{1_{ij}}, u'_{2_{ij}}, u_{2_{ij}}, u_{3_{ij}}, u'_{3_{ij}}, u_{4_{ij}}, u'_{4_{ij}})$$

As well as all the constraints of problem (P_4) except (C_1)

Case (i) If there does not exist any alternative optimal solution then put the values of $x'_{1_{ij}}, x_{1_{ij}}, x'_{2_{ij}}, x_{2_{ij}}, x_{3_{ij}}, x'_{3_{ij}}, x_{4_{ij}}$ and $x'_{4_{ij}}$ in $\tilde{x}_{ij} = (x'_{1_{ij}}, x_{1_{ij}}, x'_{2_{ij}}, x_{2_{ij}}, x_{3_{ij}}, x'_{3_{ij}}, x_{4_{ij}}, x'_{4_{ij}})$ to find the IF optimal solution $\{\tilde{x}_{ij}\}$ and find the IF optimal value $\sum_{(i,j)\in A}(\tilde{c}_{ij} \otimes \tilde{x}_{ij})$ by putting the value of $\tilde{x}_{ij}$.

Case (ii) If alternative solution exist then go to Step 5.

Step 5. Solve the crisp linear programming problem (P_6) to find the optimal solution $\{x'_{1_{ij}}, x_{1_{ij}}, x'_{2_{ij}}, x_{2_{ij}}, x_{3_{ij}}, x'_{3_{ij}}, x_{4_{ij}}, x'_{4_{ij}}\}$.

$$\text{Minimize} \sum_{(i,j)\in A} M_\upsilon^{\beta,k}(c'_{1_{ij}}x'_{1_{ij}}, c_{1_{ij}}x_{1_{ij}}, c'_{2_{ij}}x'_{2_{ij}}, c_{2_{ij}}x_{2_{ij}}, c_{3_{ij}}x_{4_{ij}}, c'_{3_{ij}}x'_{3_{ij}}, c_{4_{ij}}x_{4_{ij}}, c'_{4_{ij}}x'_{4_{ij}})$$

Subject to (P_6)

$$M_\upsilon^{\beta,k}(l'_{1_{ij}}, l_{1_{ij}}, l'_{2_{ij}}, l_{2_{ij}}, l_{3_{ij}}, l'_{3_{ij}}, l_{4_{ij}}, l'_{4_{ij}}) \leq M_\upsilon^{\beta,k}(x'_{1_{ij}}, x_{1_{ij}}, x'_{2_{ij}}, x_{2_{ij}}, x_{3_{ij}}, x'_{3_{ij}}, x_{4_{ij}}, x'_{4_{ij}}) \leq M_\upsilon^{\beta,k}(u'_{1_{ij}}, u_{1_{ij}}, u'_{2_{ij}}, u_{2_{ij}}, u_{3_{ij}}, u'_{3_{ij}}, u_{4_{ij}}, u'_{4_{ij}})$$

$$\sum_{(i,j)\in A} M_\mu^{\beta,k}(c'_{1_{ij}}x'_{1_{ij}}, c_{1_{ij}}x_{1_{ij}}, c'_{2_{ij}}x'_{2_{ij}}, c_{2_{ij}}x_{2_{ij}}, c_{3_{ij}}x_{4_{ij}}, c'_{3_{ij}}x'_{3_{ij}}, c_{4_{ij}}x_{4_{ij}}, c'_{4_{ij}}x'_{4_{ij}}) = a$$

As well as all the constraints of problem (P_4) except (C_1) where, a is the optimal value of the crisp linear programming problem (P_5).

Step 6. Put the values of $x'_{1_{ij}}, x_{1_{ij}}, x'_{2_{ij}}, x_{2_{ij}}, x_{3_{ij}}, x'_{3_{ij}}, x_{4_{ij}}$ and $x'_{4_{ij}}$ in $\tilde{x}_{ij} = (x'_{1_{ij}}, x_{1_{ij}}, x'_{2_{ij}}, x_{2_{ij}}, x_{3_{ij}}, x'_{3_{ij}}, x_{4_{ij}}, x'_{4_{ij}})$ to find the IF optimal solution $\{\tilde{x}_{ij}\}$.

Step 7. Put the values of $\tilde{x}_{ij}$, obtained from step 6, in $\sum_{(i,j)\in A}(\tilde{c}_{ij} \otimes \tilde{x}_{ij})$, to find the minimum total IF transportation cost.

6.3. Advantages of the proposed method

The existing method [7] can be used only to find the optimal solution of such MCF problems in which the parameters are represented by fuzzy numbers, however, the existing method [7] cannot be used to find the optimal solution for the same type of problems in which the parameters are represented by IF numbers, whereas the proposed method can be used to find the optimal solution for both types of problems.

6.4. Illustrative example

In this section, to illustrate the proposed method the IF MCF problem, chosen in Example 6.1, cannot be solved by the existing method [7], it can be solved by using the proposed method.

Example 6.1. Find the minimum IF transportation cost for the IF MCF problem with three nodes and three arcs as shown in Figure 2. The IF data for the chosen IF MCF problem is summarized in Table 2.

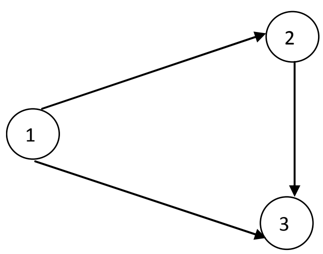

Figure 2. A network with three nodes and three arcs.

Node no	Intuitionistic fuzzy supply	Intuitionistic fuzzy demand	$\tilde{c}_{ij}$	$\tilde{u}_{ij}$
1	(50, 100, 150, 200, 250, 270, 300, 350)	–	$\tilde{c}_{12} = (5, 10, 50, 100, 1000, 5500, 10000, 15000)$	$\tilde{u}_{12} = (20, 50, 100, 150, 220, 240, 260, 300)$
2	–	(0, 20, 60, 100, 150, 170, 200, 250)	$\tilde{c}_{13} = (20, 40, 200, 400, 4000, 22000, 40000, 60000)$	$\tilde{u}_{13} = (4, 8, 13, 50, 55, 60, 65, 71)$
3	–	(0, 30, 40, 50, 100, 100, 100, 150)	$\tilde{c}_{23} = (15, 30, 150, 300, 3000, 16500, 30000, 45000)$	$\tilde{u}_{23} = (15, 30, 60, 65, 70, 75, 81, 85)$

Table 2. Parameters for IF MCF problem (3 nodes and 3 arcs).

Solution. The optimal solution of IF MCF problem, chosen in Example 6.1, was obtained as follows

Step 1. Total IF supply $=$ (50,100,150,200,250,270, 300,350) and total IF demand $=$ (0,50,100,150,250, 270,300,400). Given that the total IF supply $\neq$ total IF demand, hence, it is an unbalanced IF MCF problem. Now, as described in the proposed method (using Case (2a) of Step 1 of the proposed method), the unbalanced IF MCF problem can be converted into a balanced IF MCF problem, by introducing a dummy source node 4 with IF supply (0,0,0,0,50 ,50,50,100) and a dummy destination node 5 with IF demand (50,50,50,50,50,50,50,50) as shown in Figure 3, hence, that total IF supply = total IF demand i.e., (50,100,150,200,250,270,300,350) $\oplus$ (0,0,0,0, 50,50,50,100) $=$ (0,50,100,150,250,270,300,400) $\oplus$ (50,50,50,50,50,50,50,50).

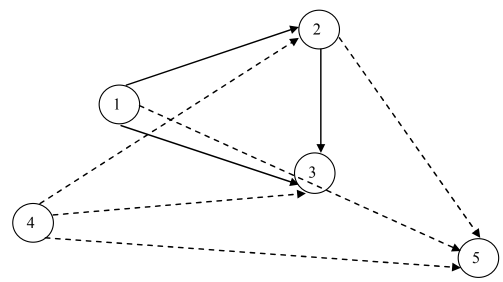

Figure 3. A modified network with a dummy source and a dummy destination.

Assuming the IF cost for transporting one unit quantity of the product from the introduced dummy source node 4 to the intermediate nodes 2 and existing destination node 3 and introduced dummy destination node 5 as zero IF number. Similarly, assuming the IF cost for transporting one unit quantity of the product from intermediate nodes 2 and existing source node 1 and introduced dummy source node 4 to the introduced dummy destination node 5 as zero IF number i.e., $\tilde{c}_{42} = \tilde{c}_{43} = \tilde{c}_{45} = \tilde{c}_{15} = \tilde{c}_{25} = (0,0,0,0,0,0,0,0)$ the IF linear programming formulation of balanced IF MCF problem, can be written as

$$\text{Minimize}((5,10,50,100,1000,5500,10000,15000) \otimes \tilde{x}_{12} \oplus (20,40,200,400,4000,22000,40000,60000) \otimes \tilde{x}_{13} \oplus (15,30,150,300,3000,16500,30000,45000) \otimes \tilde{x}_{23} \oplus (0,0,0,0,0,0,0,0) \otimes \tilde{x}_{42} \oplus (0,0,0,0,0,0,0,0) \otimes \tilde{x}_{43} \oplus (0,0,0,0,0,$$

$0,0,0)\otimes\tilde{x}_{45}\oplus(0,0,0,0,0,0,0,0)\otimes\tilde{x}_{15}\oplus(0,0,0,0,0,0,0,0)$
$\otimes\tilde{x}_{25})$

Subject to

$\tilde{x}_{12}\oplus\tilde{x}_{13}\oplus\tilde{x}_{15}=(50,100,150,200,250,270,300,350)$

$\tilde{x}_{12}\oplus\tilde{x}_{42}=\tilde{x}_{23}\oplus\tilde{x}_{25}\oplus(0,20,60,100,150,170,200,250)$

$\tilde{x}_{13}\oplus\tilde{x}_{23}\oplus\tilde{x}_{43}=(0,30,40,50,100,100,100,150)$ (P_7)

$\tilde{x}_{42}\oplus\tilde{x}_{43}\oplus\tilde{x}_{45}=(0,0,0,0,50,50,50,100)$

$\tilde{x}_{15}\oplus\tilde{x}_{25}\oplus\tilde{x}_{45}=(50,50,50,50,50,50,50,50)$

$(0,0,0,0,0,0,0,0)\,\tilde{\leq}\,\tilde{x}_{12}\,\tilde{\leq}\,(20,50,100,150,220,240,260,300)$

$(0,0,0,0,0,0,0,0)\,\tilde{\leq}\,\tilde{x}_{13}\,\tilde{\leq}\,(4,8,13,50,55,60,65,71)$

$(0,0,0,0,0,0,0,0)\,\tilde{\leq}\,\tilde{x}_{23}\,\tilde{\leq}\,(15,30,60,65,70,75,81,85)$

$\tilde{x}_{12},\tilde{x}_{13},\tilde{x}_{15},\tilde{x}_{23},\tilde{x}_{25},\tilde{x}_{42},\tilde{x}_{43},\tilde{x}_{45}$ are non-negative IF numbers.

Step 2. Assuming $\tilde{x}_{ij}=(x'_{1_{ij}},x_{1_{ij}},x'_{2_{ij}},x_{2_{ij}},x_{3_{ij}},x'_{3_{ij}},x_{4_{ij}},x'_{4_{ij}})$ and using the arithmetic operations, the IF linear programming problem (P_7), can be written as

Minimize$(5x'_{1_{12}}+20x'_{1_{13}}+15x'_{1_{23}},10x_{1_{12}}+40x_{1_{13}}+30x_{1_{23}},50x'_{2_{12}}+200x'_{2_{23}}+150x'_{2_{23}},100x_{2_{12}}+400x_{2_{23}}+300x_{2_{23}},1000x_{3_{12}}+4000x_{3_{13}}+3000x_{3_{23}},5500x'_{3_{12}}+22000x'_{3_{13}}+16500x'_{3_{23}},10000x_{4_{12}}+40000x_{4_{13}}+30000x_{4_{23}},15000x'_{4_{12}}+60000x'_{4_{13}}+45000x'_{4_{23}})$

Subject to

$(x'_{1_{12}}+x'_{1_{13}}+x'_{1_{15}},x_{1_{12}}+x_{1_{13}}+x_{1_{15}},x'_{2_{12}}+x'_{2_{13}}+x'_{2_{15}},x_{2_{12}}+x_{2_{13}}+x_{2_{15}},x_{3_{12}}+x_{3_{13}}+x_{3_{15}},x'_{3_{12}}+x'_{3_{13}}+x'_{3_{15}},x_{4_{12}}+x_{4_{13}}+x_{4_{15}},x'_{4_{12}}+x'_{4_{13}}+x'_{4_{15}})=(50,100,150,200,250,270,300,350)$

$(x'_{1_{12}}+x'_{1_{42}},x_{1_{12}}+x_{1_{42}},x'_{2_{12}}+x'_{2_{42}},x_{2_{12}}+x_{2_{42}},x_{3_{12}}+x_{3_{42}},x'_{3_{12}}+x'_{3_{42}},x_{4_{12}}+x_{4_{42}},x'_{4_{12}}+x'_{4_{42}})=(x'_{1_{23}}+x'_{1_{25}}+0,x_{1_{23}}+x_{1_{25}}+20,x'_{2_{23}}+x'_{2_{25}}+60,x_{2_{23}}+x_{2_{25}}+100,x_{3_{23}}+x_{3_{25}}+150,x'_{3_{23}}+x'_{3_{25}}+170,x_{4_{23}}+x_{4_{25}}+200,x'_{4_{23}}+x'_{4_{25}}+250)$

$(x'_{1_{13}}+x'_{1_{23}}+x'_{1_{43}},x_{1_{13}}+x_{1_{23}}+x_{1_{43}},x'_{2_{13}}+x'_{2_{23}}+x'_{2_{43}},x_{2_{13}}+x_{2_{23}}+x_{2_{43}},x_{3_{13}}+x_{3_{23}}+x_{3_{43}},x'_{3_{13}}+x'_{3_{23}}+x'_{3_{43}},x_{4_{13}}+x_{4_{23}}+x_{4_{43}},x'_{4_{13}}+x'_{4_{23}}+x'_{4_{43}})=(0,30,40,50,100,100,100,150)$

(P_8)

$(x'_{1_{42}}+x'_{1_{43}}+x'_{1_{45}},x_{1_{42}}+x_{1_{43}}+x_{1_{45}},x'_{2_{42}}+x'_{2_{43}}+x'_{2_{45}},x_{2_{42}}+x_{2_{43}}+x_{2_{45}},x_{3_{42}}+x_{3_{43}}+x_{3_{45}},x'_{3_{42}}+x'_{3_{43}}+x'_{3_{45}},x_{4_{42}}+x_{4_{43}}+x_{4_{45}},x'_{4_{42}}+x'_{4_{43}}+x'_{4_{45}})=(0,0,0,0,50,50,50,100)$

$(x'_{1_{15}}+x'_{1_{25}}+x'_{1_{45}},x_{1_{15}}+x_{1_{25}}+x_{1_{45}},x'_{2_{15}}+x'_{2_{25}}+x'_{2_{45}},x_{2_{15}}+x_{2_{25}}+x_{2_{45}},x_{3_{15}}+x_{3_{25}}+x_{3_{45}},x'_{3_{15}}+x'_{3_{25}}+x'_{3_{45}},x_{4_{15}}+x_{4_{25}}+x_{4_{45}},x'_{4_{15}}+x'_{4_{25}}+x'_{4_{45}})=(50,50,50,50\ 50,50,50,50)$

$(0,0,0,0,0,0,0,0)\,\tilde{\leq}\,(x'_{1_{12}},x_{1_{12}},x'_{2_{12}},x_{2_{12}},x_{3_{12}},x_{4_{12}},x'_{4_{12}})\,\tilde{\leq}\,(20,50,100,150,220,240,260,300)$

$(0,0,0,0,0,0,0,0)\,\tilde{\leq}\,(x'_{1_{13}},x_{1_{13}},x'_{2_{13}},x_{2_{13}},x_{3_{13}},x'_{3_{13}},x_{4_{13}},x'_{4_{13}})\,\tilde{\leq}\,(4,8,13,50,55,60,65,71)$

$(0,0,0,0,0,0,0,0)\,\tilde{\leq}\,(x'_{1_{23}},x_{1_{23}},x'_{2_{23}},x_{2_{23}},x_{3_{23}},x'_{3_{23}},x_{4_{23}},x'_{4_{23}})\,\tilde{\leq}\,(15,30,60,65,70,75,81,85)$

$\tilde{x}_{12},\tilde{x}_{13},\tilde{x}_{15},\tilde{x}_{23},\tilde{x}_{25},\tilde{x}_{42},\tilde{x}_{43},\tilde{x}_{45}$ are non-negative IF numbers.

Step 3. Using the Definition 2.7 and Definition 2.8, the IF linear programming problem (P_8), can be written as

Minimize$(5x'_{1_{12}}+20x'_{1_{13}}+15x'_{1_{23}},10x_{1_{12}}+40x_{1_{13}}+30$

$x_{1_{23}}, 50x'_{2_{12}} + 200x'_{2_{23}} + 150x'_{2_{23}}, 100x_{2_{12}} + 400x_{2_{23}} + 300x_{2_{23}}, 1000x_{3_{12}} + 4000x_{3_{13}} + 3000x_{3_{23}},\ 5500x'_{3_{12}} + 22000\ x'_{3_{13}} + 16500\ x'_{3_{23}}, 10000\ x_{4_{12}} + 40000\ x_{4_{13}} + 30000\ x_{4_{23}}, 15000\ x'_{4_{12}} + 60000\ x'_{4_{13}} + 45000\ x'_{4_{23}})$

Subject to

$x'_{1_{12}} + x'_{1_{13}} + x'_{1_{15}} = 50,\ x_{1_{12}} + x_{1_{13}} + x_{1_{15}} = 100,\ x'_{2_{12}} + x'_{2_{13}} + x'_{2_{15}} = 150,\ x_{2_{12}} + x_{2_{13}} + x_{2_{15}} = 200,\ x_{3_{12}} + x_{3_{13}} + x_{3_{15}} = 250,\ x'_{3_{12}} + x'_{3_{13}} + x'_{3_{15}} = 270,\ x_{4_{12}} + x_{4_{13}} + x_{4_{15}} = 300,\ x'_{4_{12}} + x'_{4_{13}} + x'_{4_{15}} = 350$

$x'_{1_{12}} + x'_{1_{42}} - x'_{1_{23}} - x'_{1_{25}} = 0, x_{1_{12}} + x_{1_{42}} - x_{1_{23}} - x_{1_{25}} = 20, x'_{2_{12}} + x'_{2_{42}} - x'_{2_{23}} - x'_{2_{25}} = 60,\ x_{2_{12}} + x_{2_{42}} - x_{2_{23}} - x_{2_{25}} = 100,\ x_{3_{12}} + x_{3_{42}} - x_{3_{23}} - x_{3_{25}} = 150,\ x'_{3_{12}} + x'_{3_{42}} - x'_{3_{23}} - x'_{3_{25}} = 170, x_{4_{12}} + x_{4_{42}} - x_{4_{23}} - x_{4_{25}} = 200, x'_{4_{12}} + x'_{4_{42}} - x'_{4_{23}} - x'_{4_{25}} = 250$

$x'_{1_{13}} + x'_{1_{23}} + x'_{1_{43}} = 0,\ x_{1_{13}} + x_{1_{23}} + x_{1_{43}} = 30,\ x'_{2_{13}} + x'_{2_{23}} + x'_{2_{43}} = 40, x_{2_{13}} + x_{2_{23}} + x_{2_{43}} = 50, x_{3_{13}} + x_{3_{23}} + x_{3_{43}} = 100, x'_{3_{13}} + x'_{3_{23}} + x'_{3_{43}} = 100, x_{4_{13}} + x_{4_{23}} + x_{4_{43}} = 100,\ x'_{4_{13}} + x'_{4_{23}} + x'_{4_{43}} = 150$

(P_9)

$x'_{1_{42}} + x'_{1_{43}} + x'_{1_{45}} = 0,\ x_{1_{42}} + x_{1_{43}} + x_{1_{45}} = 0,\ x'_{2_{42}} + x'_{2_{43}} + x'_{2_{45}} = 0,\ x_{2_{42}} + x_{2_{43}} + x_{2_{45}} = 0,\ x_{3_{42}} + x_{3_{43}} + x_{3_{45}} = 50, x'_{3_{42}} + x'_{3_{43}} + x'_{3_{45}} = 50, x_{4_{42}} + x_{4_{43}} + x_{4_{45}} = 50,\ x'_{4_{42}} + x'_{4_{43}} + x'_{4_{45}} = 100$

$x'_{1_{15}} + x'_{1_{25}} + x'_{1_{45}} = 50, x_{1_{15}} + x_{1_{25}} + x_{1_{45}} = 50, x'_{2_{15}} + x'_{2_{25}} + x'_{2_{45}} = 50, x_{2_{15}} + x_{2_{25}} + x_{2_{45}} = 50, x_{3_{15}} + x_{3_{25}} + x_{3_{45}} = 50, x'_{3_{15}} + x'_{3_{25}} + x'_{3_{45}} = 50, x_{4_{15}} + x_{4_{25}} + x_{4_{45}} = 50, x'_{4_{15}} + x'_{4_{25}} + x'_{4_{45}} = 50$

$x_{1_{ij}} - x'_{1_{ij}} \geq 0, x'_{2_{ij}} - x_{1_{ij}} \geq 0, x_{2_{ij}} - x'_{2_{ij}} \geq 0, x_{3_{ij}} - x_{2_{ij}} \geq 0, x'_{3_{ij}} - x_{3_{ij}} \geq 0, x_{4_{ij}} - x'_{3_{ij}} \geq 0, x'_{4_{ij}} - x_{4_{ij}} \geq 0, x'_{1_{ij}} \geq 0; \quad \forall (i,j) \in A$

$$\left.\begin{array}{l}(0,0,0,0,0,0,0,0) \tilde{\leq} (x'_{1_{12}}, x_{1_{12}}, x'_{2_{12}}, x_{2_{12}}, x_{3_{12}}, \\ x_{4_{12}}, x'_{4_{12}}) \tilde{\leq} (20,50,100,150,220,240,260,300) \\ (0,0,0,0,0,0,0,0) \tilde{\leq} (x'_{1_{13}}, x_{1_{13}}, x'_{2_{13}}, x_{2_{13}}, x_{3_{13}}, \\ x'_{3_{13}}, x_{4_{13}}, x'_{4_{13}}) \tilde{\leq} (4,8,13,50,55,60,65,71) \\ (0,0,0,0,0,0,0,0) \tilde{\leq} (x'_{1_{23}}, x_{1_{23}}, x'_{2_{23}}, x_{2_{23}}, x_{3_{23}}, \\ x'_{3_{23}}, x_{4_{23}}, x'_{4_{23}}) \tilde{\leq} (15,30,60,65,70,75,81,85)\end{array}\right\}(C_2)$$

Step 4. Using Step 4 of the proposed method and assuming $\beta = 1/3, k = 0,$ the IF optimal solution of the IF linear programming problem (P_9), can be obtained by solving the following crisp linear programming problem

Minimize $1/6(10x_{1_{12}} + 40x_{1_{13}} + 30x_{1_{23}} + 200x_{2_{12}} + 800\ x_{2_{13}} + 600x_{2_{23}} + 2000x_{3_{12}} + 8000x_{3_{13}} + 6000x_{3_{23}} + 10000x_{4_{12}} + 40000x_{4_{13}} + 30000x_{4_{23}})$

Subject to

(P_{10})

$x_{1_{12}} + 2x_{2_{12}} + 2x_{3_{12}} + x_{4_{12}} \geq 0$

$x_{1_{12}} + 2x_{2_{12}} + 2x_{3_{12}} + x_{4_{12}} \leq 1050$

$x_{1_{13}} + 2x_{2_{13}} + 2x_{3_{13}} + x_{4_{13}} \geq 0$

$x_{1_{13}} + 2x_{2_{13}} + 2x_{3_{13}} + x_{4_{13}} \leq 283$

$x_{1_{23}} + 2x_{2_{23}} + 2x_{3_{23}} + x_{4_{23}} \geq 0$

$x_{1_{23}} + 2x_{2_{23}} + 2x_{3_{23}} + x_{4_{23}} \leq 381$

As well as all the constraints of the problem (P_9) except (C_2)

Given that on solving the crisp linear programming problem (P_{10}), alternative optimal solutions exist i.e., case (ii) of Step 4 of the proposed method is satisfied and the optimal value of the crisp linear programming problem (P_{10}) is $4761400/6$ hence, by using Step 5 of the proposed method the IF optimal solution of the chosen IF MCF problem can be obtained by solving the following crisp linear programming problem:

Minimize $1/6(5x'_{1_{12}} + 20x'_{1_{13}} + 15x'_{1_{23}} + 100x'_{2_{12}} + 400\ x'_{2_{13}} + 300x'_{2_{23}} + 11000x'_{3_{12}} + 44000x'_{3_{13}} + 33000$

$x'_{3_{23}} + 15000x'_{4_{12}} + 60000x'_{4_{13}} + 45000x'_{4_{23}})$

Subject to

(P_{11})

$x'_{1_{12}} + 2x'_{2_{12}} + 2x'_{3_{12}} + x'_{4_{12}} \geq 0$

$x'_{1_{12}} + 2x'_{2_{12}} + 2x'_{3_{12}} + x'_{4_{12}} \leq 1000$

$x'_{1_{13}} + 2x'_{2_{13}} + 2x'_{3_{13}} + x'_{4_{13}} \geq 0$

$x'_{1_{13}} + 2x'_{2_{13}} + 2x'_{3_{13}} + x'_{4_{13}} \leq 221$

$x'_{1_{23}} + 2x'_{2_{23}} + 2x'_{3_{23}} + x'_{4_{23}} \geq 0$

$x'_{1_{23}} + 2x'_{2_{23}} + 2x'_{3_{23}} + x'_{4_{23}} \leq 370$

$10x_{1_{12}} + 40x_{1_{13}} + 30x_{1_{23}} + 200x_{2_{12}} + 800x_{2_{13}} + 600$
$x_{2_{23}} + 2000x_{3_{12}} + 8000x_{3_{13}} + 6000x_{3_{23}} + 10000$
$x_{4_{12}} + 40000x_{4_{13}} + 30000x_{4_{23}} = 4761400$

As well as all the constraints of problem (P_9) except (C_2)

Step 5. Solving the crisp linear programming problem, obtained in Step 4, the optimal values of $x_{1_{12}}$, $x'_{2_{12}}, x_{2_{12}}, x_{3_{12}}, x'_{3_{12}}, x_{4_{12}}, x'_{4_{12}}, x'_{2_{13}}, x_{2_{13}}$, $x_{3_{13}}$, $x'_{3_{13}}$, $x_{4_{13}}, x'_{4_{13}}, x'_{1_{15}}, x_{1_{15}}, x'_{2_{15}}, x_{2_{15}}, x'_{3_{15}}, x_{4_{15}}, x'_{4_{15}}$, $x_{1_{23}}$, $x'_{2_{23}}, x_{2_{23}}, x_{3_{23}}, x'_{3_{23}}, x_{4_{23}}, x'_{4_{23}}, x_{3_{43}}, x'_{3_{43}}, x_{4_{43}}$, and $x'_{4_{43}}$ are 50,90,130,180,200,230,280,10,20, 20,20,20,20,50,50,50,50,50,50,50,50,30,30,30, 30,30, 30,30,50,50,50 and 100 respectively.

Step 6. Put the values of $x'_{1_{ij}}, x_{1_{ij}}$, $x'_{2_{ij}}, x_{2_{ij}}$, $x_{3_{ij}}$ $, x'_{3_{ij}}, x_{4_{ij}}$ and $x'_{4_{ij}}$ in $\tilde{x}_{ij} = (x'_{1_{ij}}, x_{1_{ij}}, x'_{2_{ij}}, x_{2_{ij}}, x_{3_{ij}}, x'_{2_{ij}}, x_{2_{ij}}, x_{3_{ij}}, x'_{3_{ij}}, x_{4_{ij}}, x'_{4_{ij}})$ the IF optimal solution is $\tilde{x}_{12} = (0,50,90,130,180,200,230,280)$, $\tilde{x}_{13} = (0,0,10,20,20,20,20,20)$, $\tilde{x}_{15} = (50,50,50,50,50,50,50,50,50)$, $\tilde{x}_{23} = (0,30,30,30,30,30,30,30)$, $\tilde{x}_{43} = (0,0,0,0,50,50,50,100)$ and the remaining $\tilde{x}_{ij}$ are (0,0,0,0,0,0,0,0).

Step 7. Putting the values of $\tilde{x}_{ij} = (x'_{1_{ij}}, x_{1_{ij}}, x'_{2_{ij}}$, $x_{2_{ij}}, x_{3_{ij}}$, $x'_{2_{ij}}, x_{2_{ij}}, x_{3_{ij}}, x'_{3_{ij}}, x_{4_{ij}}, x'_{4_{ij}})$ in ((5,10,50, 100,1000,5500,10000,15000) $\otimes \tilde{x}_{12} \oplus$(20,40,200,400, 4000,22000,40000,60000) $\otimes \tilde{x}_{13} \oplus$ (15,30,150,300, 3000,16500,30000,45000) $\otimes \tilde{x}_{23} \oplus$(0,0,0,0,0,0,0,0) $\otimes$ $\tilde{x}_{42} \oplus$(0,0,0,0,0,0,0,0) $\otimes \tilde{x}_{43} \oplus$(0,0,0,0,0,0,0,0) $\otimes \tilde{x}_{45} \oplus$ (0,0,0,0,0,0,0,0) $\otimes$ $\tilde{x}_{15} \oplus$ (0,0,0,0,0,0,0,0) $\otimes$ $\tilde{x}_{25}$) the minimum total IF transportation cost is (0,1400, 11000,20000,350000,2035000,4000000,6750000).

7. Comparative study

The results of the existing fuzzy MCF problem (Example 3.5 [8], pp. 2498) and the IF MCF problem, chosen in Example 6.1, obtained by using the existing method and the method proposed in Subsection 6.2, are shown in Table 3.

Example	Existing method [8]	Proposed method
3.5 [8, pp. 2498]	$(1924000, 1903300, 7299800)_{LR}$	$(1924000, 1903300, 7299800)_{LR}$
6.1	Not applicable	$(0, 1400, 11000, 20000, 350000, 2035000, 4000000, 6750000)$

Table 3. Comparative study.

Given that in the existing MCF problem (Example 3.5 [8], pp. 2498) all the parameters are represented by fuzzy numbers hence, as discussed in Subsection 6.3, it can be solved by using the existing method [8] as well as by the proposed method, whereas in the MCF problem, chosen in Example 6.1, all the parameters are represented by IF numbers. Thus, as discussed in Subsection 6.3, it can be solved by the proposed method but cannot be solved by using the existing method [8].

8. Conclusion

Based on the proposed study, it can be concluded that it is better to use the proposed ranking approach for comparing fuzzy and IF numbers as compared to existing ranking approaches. Also, It can be concluded that it is not possible to find the IF optimal solution of IF MCF problems by using any of the existing methods. Only the proposed method can be used for the same problems.

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