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Symmetric Gobos in a Finite Projective Plane

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ABSTRACT

A gobo G in any incidence structure K is a (perhaps degenerate) tactical configuration having the property that no three points in G are collinear and no three lines in G are concurrent. General results are obtained where K is a finite projective plane of order n and G has k points and k lines such that each point (line) lies on r lines (points) of G. Particular attention is called to the contrast between the case $r = 1$ and the case $r \neq 1$ when $k = n$.

1. INTRODUCTION

In [2] the author defined a *gobo* to be a substructure $\mathfrak{G}$ of any incidence structure π such that $\mathfrak{G}$ is a tactical configuration (perhaps degenerate) having the additional property that no three points of $\mathfrak{G}$ are collinear and no three lines of $\mathfrak{G}$ are concurrent. We shall restrict ourselves to the case π is a finite projective plane of order n. A set of k points in π such that no three are collinear is called a *k-arc*. Thus, a (p_0, b_0, p_1, b_1)-gobo $\mathfrak{G} = (\mathfrak{P}, \mathfrak{L})$ in π is the union of a p_0-arc $\mathfrak{P}$ and the dual $\mathfrak{L}$ of a b_0-arc such that every line in $\mathfrak{L}$ lies on exactly p_1 points of $\mathfrak{P}$ and every point in $\mathfrak{P}$ lies on exactly b_1 lines of $\mathfrak{L}$. It follows that $p_0 b_1 = p_1 b_0$, $0 < p_0 \leqslant n + 2$, $0 < b_0 \leqslant n + 2$, $0 \leqslant p_1 \leqslant 2$, and $0 \leqslant b_1 \leqslant 2$. A (k, k, r, r)-gobo is said to be *symmetric* and, for simplification, is called a (k, r)-gobo.

A k-arc $\mathfrak{P}$ in π can be *r-flagged* if there exists an $\mathfrak{L}$ such that $(\mathfrak{P}, \mathfrak{L})$ is a (k, r)-gobo. A (k, r)-gobo is *r-complete* if it is not contained in a $(k + 1, r)$-gobo. A k-arc is *complete* if it is not contained in a $(k + 1)$-arc. An $(n + 1)$-arc is called an *oval*. In [2] the study of $(n, 1)$-gobos leads to a characterization of complete n-arcs in π when n is odd: An n-arc in a projective plane of odd order n is complete if and only if it can not be 1-flagged. In examining the statements that are analogous to the theorems that give this characterization, we find that each fails for $r \neq 1$:

582a/11/3-1

THEOREM A. *Each of the following statements is true for* $r = 1$. *For the remaining cases,* $r = 0$ *and* $r = 2$, *each statement fails.*

I. *An* n*-arc contained in an oval in a projective plane of order* $n > 4$ *can not be* r*-flagged in more than one way.*

II. *The arc of every* (n, r)*-gobo in a projective plane of odd order* n *is contained in an oval.*

III. *There does not exist an* r*-complete* (n, r)*-gobo in a projective plane of odd order* n.

Before proving Theorem A, we shall prove several theorems concerning arcs and gobos. These theorems, which are of interest in their own right, and examples they provide will be used to prove Theorem A.

2. ARCS AND GOBOS

We now assume the definitions and results of [1] and [2]. Let $\mathfrak{P}_1$ and $\mathfrak{P}_2$ be disjoint arcs in π. We say $\mathfrak{P}_1$ is *interior* or *exterior* to $\mathfrak{P}_2$ if every point of $\mathfrak{P}_1$ is, respectively, an interior or an exterior point of $\mathfrak{P}_2$. (Of course it may happen that $\mathfrak{P}_1$ is neither interior nor exterior to $\mathfrak{P}_2$.) Dually, if $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are disjoint duals of arcs, $\mathfrak{L}_1$ is *interior* or *exterior* to $\mathfrak{L}_2$ if every line of $\mathfrak{L}_1$ is, respectively, an interior or an exterior line of $\mathfrak{L}_2$.

Let $\mathfrak{P}$ be an arc and $\mathfrak{L}$ the dual of an arc in π. We say $\mathfrak{P}$ is *external, internal,* or *interior* to $\mathfrak{L}$ if every point of $\mathfrak{P}$ is an external, an internal, or an interior point of $\mathfrak{L}$, respectively. Dually, $\mathfrak{L}$ is *internal, external,* or *exterior* to $\mathfrak{P}$ if every line of $\mathfrak{L}$ is an internal (secant), an external (tangent), or an exterior line of $\mathfrak{P}$, respectively.

Let $\mathfrak{P}$ be a k-arc and $\mathfrak{L}$ the dual of a k-arc. Then, from the definitions, we have that $(\mathfrak{P}, \mathfrak{L})$ is a $(k, 1)$-gobo if and only if (a) $\mathfrak{P}$ is internal to $\mathfrak{L}$, and (b) $\mathfrak{L}$ is external to $\mathfrak{P}$. Note that (a) and (b) are not equivalent. However, here and in the following theorem, (a) and (b) are dual statements.

THEOREM 1. *Let* $\mathfrak{P}$ *be a* k*-arc and* $\mathfrak{L}$ *the dual of a* k*-arc in a finite projective plane. Then the following are equivalent:*

(a) $\mathfrak{P}$ *is interior to* $\mathfrak{L}$.
(b) $\mathfrak{L}$ *is exterior to* $\mathfrak{P}$.
(c) $(\mathfrak{P}, \mathfrak{L})$ *is a* $(k, 0)$*-gobo.*

Also, the following are equivalent to each other:

(a) $\mathfrak{P}$ *is external to* $\mathfrak{L}$.
(b) $\mathfrak{L}$ *is internal to* $\mathfrak{P}$.
(c) $(\mathfrak{P}, \mathfrak{L})$ *is a* $(k, 2)$*-gobo.*

Proof. For each case, the definition of (c) is equivalent to (a) and (b). However, since k is finite, (a) and (b) are seen to be equivalent.

THEOREM 2. *Let* $(\mathfrak{P}, \mathfrak{L})$ *be a* $(k, 1)$*-gobo in a finite projective plane with* k *even. If* $\mathfrak{Q}$ *is a* k*-arc interior to* $\mathfrak{P}$, *then* $(\mathfrak{Q}, \mathfrak{L})$ *is a* $(k, 0)$*-gobo.*

Proof. By hypothesis, k is even and less than $n + 2$. Thus, every interior point of $\mathfrak{P}$ is an interior point of $\mathfrak{L}$. The result follows from the first part of Theorem 1.

Let $(\mathfrak{P}, \mathfrak{L})$ be a $(k, 0)$-gobo in π. If $\mathfrak{Q}$ is any h-arc contained in $\mathfrak{P}$ and $\mathfrak{M}$ is any subset of h lines from $\mathfrak{L}$, then, clearly, $(\mathfrak{Q}, \mathfrak{M})$ is an $(h, 0)$-gobo.
A $(k, 2)$-gobo $\mathfrak{G}$ in π is *irreducible* if it is not the union of a $(k_1, 2)$-gobo $\mathfrak{G}_1$ and a $(k_2, 2)$-gobo $\mathfrak{G}_2$ where $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are disjoint. It is easily seen that a $(k, 2)$-gobo $\mathfrak{G}$ is the union of u mutually disjoint, irreducible $(k_i, 2)$-gobos $\mathfrak{G}_i$ where $k_1 + k_2 + \cdots + k_u = k$. Further, except for order, the $\mathfrak{G}_i$ are unique. We may take $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_u$ and say $\mathfrak{G}$ is of *kind* $\langle k_1, k_2, ..., k_u \rangle$. Since the smallest $(k, 2)$-gobo is a $(3, 2)$-gobo (a *triangle*), we have $1 \leqslant u \leqslant k/3$. Also, it follows that an $(n + 2, 2)$-gobo, an $(n + 1, 2)$-gobo, or an $(n, 2)$-gobo is necessarily complete in a plane of order n; if n is odd, then an $(n - 1, 2)$-gobo is also necessarily complete.

THEOREM 3. *Let* $(\mathfrak{P}, \mathfrak{L})$ *and* $(\mathfrak{Q}, \mathfrak{M})$ *be* $(n + 1, 1)$*-gobos in a projective plane of odd order* n. *Then,* $(\mathfrak{P}, \mathfrak{M})$ *is an* $(n + 1, 0)$*-gobo if and only if* $\mathfrak{P}$ *is interior to* $\mathfrak{Q}$. *Also,* $(\mathfrak{Q}, \mathfrak{L})$ *is an* $(n + 1, 2)$*-gobo if and only if* $\mathfrak{Q}$ *is exterior to* $\mathfrak{P}$.

Proof. With $k = n + 1$ even, the interior points of $\mathfrak{Q}$ are exactly the interior points of $\mathfrak{M}$, and the exterior points of $\mathfrak{P}$ are exactly the external points of $\mathfrak{L}$. The result follows from Theorem 1.

If $(\mathfrak{P}, \mathfrak{L})$ is a $(k, 1)$-gobo and $(\mathfrak{Q}, \mathfrak{L})$ is a $(k, 2)$-gobo, then, trivally, $\mathfrak{Q}$ is exterior to $\mathfrak{P}$. However, it does not follow that $\mathfrak{P}$ is necessarily interior to $\mathfrak{Q}$, even in the case $k = n + 1$ and the plane is Desarguesian of odd order. In fact, as Theorem 3 and Example 1 below demonstrate, $\mathfrak{P}$ may also be exterior to $\mathfrak{Q}$. Further, if $(\mathfrak{P}, \mathfrak{L})$ and $(\mathfrak{Q}, \mathfrak{M})$ are both $(n + 1, 1)$-gobos and $(\mathfrak{P}, \mathfrak{M})$ is an $(n + 1, 0)$-gobo then it does not follow that $(\mathfrak{Q}, \mathfrak{L})$ is an $(n + 1, 2)$-gobo, even with n odd. As Theorem 3 and Example 1 below demonstrate, $(\mathfrak{Q}, \mathfrak{L})$ may also be an $(n + 1, 0)$-gobo since each of $\mathfrak{P}$ and $\mathfrak{Q}$ may be interior to the other.

EXAMPLE 1. Using the standard notation, let $\mathfrak{O}_i$ be the oval having equation $x^2 + y^2 = iz^2$, $i \neq 0$, in the projective plane over the Galois field of order 7. $\mathfrak{O}_1$ and $\mathfrak{O}_5$ are exterior to each other, while $\mathfrak{O}_2$ and $\mathfrak{O}_5$ are interior to each other. We note that $\mathfrak{O}_1$ is interior to $\mathfrak{O}_2$ and that $\mathfrak{O}_2$ is interior to $\mathfrak{O}_5$, but $\mathfrak{O}_1$ is exterior to $\mathfrak{O}_5$. Also $\mathfrak{O}_2$ is exterior to $\mathfrak{O}_1$ and $\mathfrak{O}_1$ is exterior to $\mathfrak{O}_4$, but $\mathfrak{O}_2$ is interior to $\mathfrak{O}_4$.

Let $\mathfrak{L}_i$ be the set of tangents to $\mathfrak{O}_i$. Since $\mathfrak{O}_1$ is interior to both $\mathfrak{O}_2$ and $\mathfrak{O}_6$, $(\mathfrak{O}_1, \mathfrak{L}_2)$ and $(\mathfrak{O}_1, \mathfrak{L}_6)$ are $(n+1, 0)$-gobos. Since $\mathfrak{O}_1$ is exterior to both $\mathfrak{O}_3$ and $\mathfrak{O}_4$, $\mathfrak{G}_1 = (\mathfrak{O}_1, \mathfrak{L}_3)$ and $\mathfrak{G}_2 = (\mathfrak{O}_1, \mathfrak{L}_4)$ are both $(n+1, 2)$-gobos. $\mathfrak{G}_1$ is of kind $\langle 8 \rangle$ and $\mathfrak{G}_2$ is of kind $\langle 4, 4 \rangle$. $\mathfrak{O}_1$ is an oval that can be both 0-flagged and 2-flagged in more than one way.

Let $\mathfrak{P}$ and $\mathfrak{Q}$ be two intersecting arcs in π. We say $\mathfrak{P}$ is *interiorly tangent* or *exteriorly tangent* to $\mathfrak{Q}$ if every point of $\mathfrak{P} \backslash \mathfrak{Q}$ is, respectively, an interior point or an exterior point of $\mathfrak{Q}$.

EXAMPLE 2. We shall give an example of a 0-complete $(n, 0)$-gobo $(\mathfrak{P}, \mathfrak{L})$ in a Desarguesian plane of odd order n such that neither $\mathfrak{P}$ nor $\mathfrak{L}$ is complete. In the same plane as Example 1, let $\mathfrak{H}_i$ be the set of all points (x, y, z) satisfying $xy = iz^2$, $i \neq 0$. The oval $\mathfrak{H}_2$ is interiorly tangent to the oval $\mathfrak{H}_1$. The only points $\mathfrak{H}_1$ and $\mathfrak{H}_2$ have in common are (010) and (100). The only common tangents are the lines with equations $x = 0$ and $y = 0$. Let $\mathfrak{P}$ be the n-arc obtained by deleting the point (010) from $\mathfrak{H}_2$. Let $\mathfrak{L}$ be the dual of an n-arc obtained by deleting the line with equation $y = 0$ from the set of tangents to $\mathfrak{H}_1$. $\mathfrak{P}$ has only the one completion point and it lies on a line of $\mathfrak{L}$; $\mathfrak{L}$ has only the one completion line and it lies on a point of $\mathfrak{P}$. Thus, $(\mathfrak{P}, \mathfrak{L})$ has the desired properties.

THEOREM 4. *Let $\mathfrak{P}$ be a k-arc exterior to oval $\mathfrak{Q}$ in a projective plane of odd order. If k is even and $\mathfrak{Q}$ is interior to $\mathfrak{P}$, then there exists $\mathfrak{L}$ such that $(\mathfrak{P}, \mathfrak{L})$ is a $(k, 2)$-gobo.*

Proof. Since every point of $\mathfrak{P}$ lies on exactly two tangents of $\mathfrak{Q}$ and every tangent of $\mathfrak{Q}$ that intersects $\mathfrak{P}$ is a secant of $\mathfrak{P}$, $\mathfrak{L}$ may be constructed as follows. Let p_1 be an arbitrary point of $\mathfrak{P}$, and let q_1 be a point of $\mathfrak{Q}$ such that p_1q_1 is a tangent of $\mathfrak{Q}$. Then p_1q_1 intersects $\mathfrak{P}$ in a unique point p_2 with $p_2 \neq p_1$. Let q_2 be the unique point of $\mathfrak{Q}$ such that $q_2 \neq q_1$ and p_2q_2 is a tangent of $\mathfrak{Q}$. Let p_2q_2 intersect $\mathfrak{P}$ in the unique point p_3 with $p_3 \neq p_2$. As p_3 was uniquely determined by p_2, we continue constructing the points $p_4, p_5, \ldots, p_{r+1}$ until $p_{r+1} = p_1$. This is possible since k is finite. At this point we have an irreducible $(r, 2)$-gobo $\mathfrak{G}_1 = (\mathfrak{P}_1, \mathfrak{L}_1)$ where $\mathfrak{P}_1 = \{p_1, p_2, \ldots, p_r\}$ and

$$\mathfrak{L}_1 = \{p_1p_2, p_2p_3, \ldots, p_{r-1}p_r, p_rp_1\}. \quad 3 \leqslant r \leqslant k.$$

If $r = k$, we are done. If $r < k$, pick a new point $p_1{}^2$ from $\mathfrak{P}$ but not in

$\mathfrak{P}_1$ and repeat the process. Continue in this fashion until every point of $\mathfrak{P}$ occurs exactly once in the list

$$p_1, p_2, \ldots, p_r; p_1^2, p_2^2, \ldots, p_s^2; \ldots; p_1^u, p_2^u, \ldots, p_t^u.$$

For $i > 1$, $\mathfrak{G}_i = (\mathfrak{P}_i, \mathfrak{L}_i)$, where $\mathfrak{P}_i = \{p_1^i, p_2^i, \ldots, p_j^i\}$ and

$$L_i = \{p_1^i p_2^i, p_2^i p_3^i, \ldots, p_j^i p_1^i\},$$

is an irreducible $(j, 2)$-gobo. The union of the u $\mathfrak{P}_i$ is $\mathfrak{P}$. The $\mathfrak{L}_i$ are pairwise disjoint and each is contained in the set of tangents to the oval $\mathfrak{Q}$. Let $\mathfrak{L}$ be the union of the u $\mathfrak{L}_i$. $\mathfrak{L}$ is the dual of a k-arc, and $\mathfrak{G} = (\mathfrak{P}, \mathfrak{L})$ is the desired $(k, 2)$-gobo.

Besides having proved the theorem, we have a schema

$$p_1^1, p_2^1, \ldots, p_r^1; p_1^2, p_2^2, \ldots, p_s^2; \ldots; p_1^u, p_2^u, \ldots, p_t^u$$

that completely describes a $(k, 2)$-gobo of kind $\langle r, s, \ldots, t\rangle$.

EXAMPLE 3. $\mathfrak{P} = \{(001), (011), (101), (111), (561), (651)\}$ is a complete $(n - 1)$-arc in the projective plane of order 7. $\mathfrak{P}$ can be 1-flagged in more than one way, [2]. We can not use Theorem 4 to 2-flag $\mathfrak{P}$, since $\mathfrak{P}$ has only six interior points. In fact, the set of interior points of $\mathfrak{P}$ is a complete 6-arc whose set of interior points is $\mathfrak{P}$. However, $\mathfrak{P}$ may be 2-flagged to obtain complete $(6, 2)$-gobos of kind $\langle 6\rangle$ and $\langle 3, 3\rangle$ respectively:

(011), (001), (101), (651), (561), (111).

(011), (001), (101); (651), (561), (111).

The part of Theorem 3 referring to $(k, 2)$-gobos is a special case of the construction in Theorem 4. Both Theorem 3 and Theorem 4 require that k be even. The following theorem is relevant to 2-flagging a k-arc with k odd. Although the hypotheses are rather strict, the theorem will be useful in 2-flagging n-arcs when n is odd.

THEOREM 5. *Suppose $\mathfrak{P}$ is a k-arc properly contained in an oval $\mathfrak{O}$ in a projective plane of odd order and $\mathfrak{Q}$ is an oval containing $\mathfrak{O}\backslash\mathfrak{P}$ but disjoint from $\mathfrak{P}$. If $\mathfrak{O}$ is exteriorly tangent to $\mathfrak{Q}$ and $\mathfrak{Q}$ is interiorly tangent to $\mathfrak{O}$, then there exists $\mathfrak{L}$ such that $(\mathfrak{P}, \mathfrak{L})$ is a $(k, 2)$-gobo.*

Proof. First observe that, if point t is common to $\mathfrak{Q}$ and $\mathfrak{O}$, then the tangent of $\mathfrak{Q}$ at t and the tangent of $\mathfrak{O}$ at t coincide, since $\mathfrak{Q}$ is interiorly tangent to $\mathfrak{O}$. Now, since $\mathfrak{O}$ is exteriorly tangent to $\mathfrak{Q}$, each point of

$\mathfrak{P} = \mathfrak{O}\backslash\mathfrak{Q}$ lies on exactly two tangents of $\mathfrak{Q}$. Such a tangent is off $\mathfrak{O}\backslash\mathfrak{P}$ and, since $\mathfrak{Q}$ is interiorly tangent to $\mathfrak{O}$, is a secant of $\mathfrak{P}$. We now observe that we have satisfied the conditions that were necessary for the construction of the $\mathfrak{L}$ in the proof of Theorem 4. By this same construction, we obtain an $\mathfrak{L}$ that gives us our result here.

EXAMPLE 4. In the projective plane of order 7, let $\mathfrak{T}_i$ be the oval having equation $x^2 - yz + iz^2 = 0$. $\mathfrak{T}_1$, $\mathfrak{T}_2$, and $\mathfrak{T}_4$ are each interiorly tangent to $\mathfrak{T}_0$. $\mathfrak{T}_0$ is exteriorly tangent to each of $\mathfrak{T}_1$, $\mathfrak{T}_2$, and $\mathfrak{T}_4$. Thus, the n-arc $\mathfrak{P}$ consisting of all points $(x, x^2, 1)$ can be 2-flagged by the construction of Theorem 5, taking $\mathfrak{O} = \mathfrak{T}_0$ and $\mathfrak{Q}$ to be $\mathfrak{T}_1$, $\mathfrak{T}_2$, or $\mathfrak{T}_4$. This results in three distinct (7, 2)-gobos, each of kind $\langle 7 \rangle$:

(001), (241), (421), (611), (111), (321), (541).

(001), (111), (241), (321), (421), (541), (611).

(001), (421), (111), (541), (241), (611), (321).

EXAMPLE 5. Let $\mathfrak{H}_i$ be as in Example 2. $\mathfrak{H}_1$ is exteriorly tangent to $\mathfrak{H}_2$, and $\mathfrak{H}_2$ is interiorly tangent to $\mathfrak{H}_1$. $\mathfrak{P} = \mathfrak{H}_1\backslash\mathfrak{H}_2$ is a 6-arc. Application of Theorem 5 to the $(n - 1)$-arc $\mathfrak{P}$, contained in an oval, yields a necessarily complete (6, 2)-gobo of kind $\langle 3, 3 \rangle$:

(111), (241), (421); (351), (661), (531).

$\mathfrak{P}$ may also be 2-flagged to obtain a (6, 2)-gobo of kind $\langle 6 \rangle$:

(111), (351), (531), (661), (421), (241).

EXAMPLE 6. Finally, we give an example of a complete $(n - 2, 2)$-gobo $\mathfrak{G} = (\mathfrak{P}, \mathfrak{L})$ in the Desarguesian plane of order $n = 7$. $\mathfrak{G}$ must be of kind $\langle 5 \rangle$:

(011), (601), (061), (101), (221).

$\mathfrak{G}$ can be properly contained in a $(k, 2)$-gobo only when $k = n + 1$. The only oval containing $\mathfrak{P}$ is $\mathfrak{O}_1$ in Example 1. Since three of the external points of $\mathfrak{L}$ lie on the triangle determined by $\mathfrak{O}_1\backslash\mathfrak{P}$, it follows that $\mathfrak{G}$ is complete.

3. PROOF OF THEOREM A

For the case $r = 1$, the statements I, II, and III are Theorems 4, 5, and 6 of [2], respectively.

For the case $r = 0$:

(I) The oval $\mathfrak{O}_1$ in Example 1 can be 0-flagged in more than one way. Hence, by the first remark following Theorem 2, any n-arc contained in $\mathfrak{O}_1$ can be 0-flagged in several ways.

(II) Since it is well known that any n-arc in a Desarguesian plane of odd order is never complete, we must turn to a non-Desarguesian plane for a counterexample. Let

$$\mathfrak{N}_1 = \{A_1, B_0, C_9, D_8, D_9, E_2, F_2, G_5, G_8\}$$

be the complete n-arc in the Hughes plane of order 9, given on page 329 of [1]. (Professor R. H. F. Denniston has kindly pointed out to the author that the complete n-arc $\mathfrak{N}_2$ given on the same page of [1] *is* projectively equivalent to $\mathfrak{N}_1$.) Then

$$\mathfrak{L} = \{A_3A_4, A_4A_5, A_2B_3, A_5C_6, A_7C_8, A_3B_{10}, A_9C_3, A_6B_9, A_9B_{12}\}$$

is the dual of an n-arc and is complete. $(\mathfrak{N}_1, \mathfrak{L})$ is a 0-complete $(n, 0)$-gobo for $n = 9$.

(III) The example in II proves III. However, it is not necessary to go to a non-Desarguesian plane as Example 2 shows.

For the case $r = 2$:

(I) Follows from Example 4.

(II) The n-arc $\mathfrak{N}_1$, above, in the Hughes plane of order 9 can be 2-flagged to give a complete $(n, 2)$-gobo of kind $\langle 9 \rangle$:

$$A_1, B_0, G_5, C_9, F_2, G_8, E_2, D_9, D_8.$$

(III) Since an $(n, 2)$-gobo is always complete, III follows from either I or II.

References

1. G. E. Martin, On arcs in a finite projective plane, *Canad. J. Math.* **19** (1967), 376–393.
2. G. E. Martin, Gobos in a finite projective plane, *J. Combinatorial Theory* **10** (1971), 92–96.