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The median function on median graphs and semilattices ☆

F.R. McMorris[a], H.M. Mulder[b], R.C. Powers[a,*]

[a] Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616 USA

[b] Econometrisch Instituut, Erasmus Universiteit, P.O. Box 1738, 3000 DR Rotterdam, Netherlands

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Abstract

A median of a k-tuple $\pi=(x_1,\dots,x_k)$ of vertices of a finite connected graph G is a vertex x for which $\sum_{i=1}^{k} d(x,x_i)$ is minimum, where d is the geodesic metric on G. The function M with domain the set of all k-tuples with $k>0$ and defined by $M(\pi)=\{x \mid x \text{ is a median of } \pi\}$ is called the median function on G. In this paper a new characterization of the median function is given for G a median graph. This is used to give a characterization of the median function on median semilattices.

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1. Introduction

Let (X,d) be a finite metric space and $\pi=(x_1,\dots,x_k)\in X^k$, a *profile*. A *median* for π is an element $x\in X$ for which $\sum_{i=1}^{k} d(x,x_i)$ is minimum and the *median function* on (X,d) is the function that returns the set of all medians of a profile π. Letting $X^*=\bigcup_{k>0}X^k$ and M denote the median function we then have $M: X^*\to 2^X-\{\emptyset\}$ defined by $M(\pi)=\{x \mid x \text{ is a median of } \pi\}$, for all $\pi\in X^*$. Since medians for π can be interperted as "closest" elements to π, the median function has a rich history involving consensus and location (see [5,10]). The reader should be alerted to the fact that for historical reasons, M is often called the *median procedure*.

When the metric space is arbitrary, conditions characterizing M are illusive so additional structure is usually imposed on the space by graph or order theoretic conditions. For example in [7] the median function is characterized on median graphs while in [4,8], M is studied on various partially ordered sets. In the present paper we give a different characterization of M on median graphs in which the conditions allow for a

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* Corresponding author.

E-mail address: rcpowe01@athena.louisville.edu (R.C. Powers)

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fairly direct translation over to the order theoretic situation enabling us to characterize M on median semilattices.

2. Graph and order theoretic preliminaries

In this section, we will give basic results and introduce terminology necessary for Sections 3 and 4. We first consider the graph theoretic preliminaries. All graphs will be finite, and we use the standard notation $G=(V,E)$ to denote a graph with vertex set V and edge set E. We will often write only G and leave V and E understood. Also, we will not distinguish between a subset W of V and the subgraph induced by W. Recall that in a connected graph, the *distance* $d(x,y)$ between two vertices x and y is the length of a shortest x,y-path, or an *x,y-geodesic*. A *median graph* G is a connected graph such that for every three vertices x,y,z of G, there is a unique vertex w on a geodesic between each pair of x,y,z. Note that this vertex w is the unique median for the profile $\pi=(x,y,z)$. It follows easily from the definition that median graphs are bipartite. The *interval* between the vertices x and y is the set $I(x,y)$ of all vertices on x,y-geodesics, i.e., $I(x,y)=\{w\in V \mid d(x,w)+d(w,y)=d(x,y)\}$. It is an easy observation that a graph G is a median graph if and only if $|I(x,y)\cap I(x,z)\cap I(y,z)|=1$ for all vertices x,y,z of G. Median graphs were first studied in 1961 by Avann [1, 2], and independently introduced by Nebeský [16] and Mulder and Schrijver [15]. The simplest examples of median graphs are trees and n-cubes.

A set W of vertices of a graph G is *convex* if $I(x,y)\subseteq W$, for every $x,y\in W$, and a *convex subgraph* of G is a subgraph induced by a convex set of vertices of G. Clearly a convex subgraph of a connected graph is also connected. Moreover, the intersection of convex sets (subgraphs) is convex. In median graphs, convex sets can be viewed in a very useful way through the notion of a gate. For $W\subseteq V$ and $x\in V$, the vertex $z\in W$ is a *gate* for x in W if $z\in I(x,w)$ for all $w\in W$. Note that a vertex x has at most one gate in any set W, and if x has a gate z in W, then z is the unique nearest vertex to x in W. The set W is *gated* if every vertex has a gate in W and a *gated subgraph* is a subgraph induced by a gated set [6]. It is not difficult to see that in any graph, a gated set is convex and that in a median graph a set is gated if and only if it is convex.

Recall that for two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, the *union* $G_1\cup G_2$ is the graph with vertex set $V_1\cup V_2$ and edge set $E_1\cup E_2$, and the *intersection* $G_1\cap G_2$ is the graph with vertex set $V_1\cap V_2$ and edge set $E_1\cap E_2$. We write $G_1\cap G_2=\emptyset$ ($\neq\emptyset$) when $V_1\cap V_2=\emptyset$ ($\neq\emptyset$). A *proper cover* of G consists of two convex subgraphs G_1 and G_2 of G such that $G=G_1\cup G_2$ and $G_1\cap G_2\neq\emptyset$. Note that this implies that there are no edges between $G_1\backslash G_2$ and $G_2\backslash G_1$. Clearly, every graph G admits the *trivial proper cover* G_1,G_2 with $G_1=G_2=G$. On the other hand, a cycle does not have a proper cover with two proper convex subgraphs.

We are now able to give the definition of the operation which helps yield a characterization of median graphs. Let $G'=(V',E')$ be properly covered by the convex

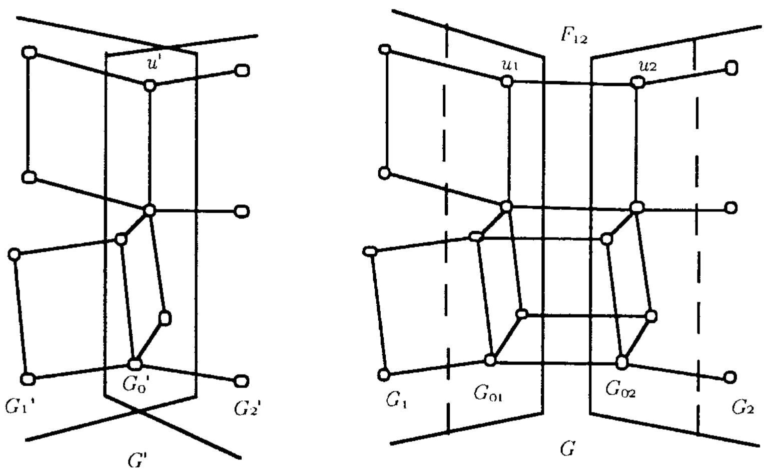

Fig. 1. The expansion procedure.

subgraphs $G_1'=(V_1',E_1')$ and $G_2'=(V_2',E_2')$ and set $G_0'=G_1'\cap G_2'$. For $i=1,2$, let G_i be an isomorphic copy of G_i', and let λ_i be an isomorphism from G_i' onto G_i. We set $G_{0i}=\lambda_i[G_0']$ and $\lambda_i(u')=u_i$, for u' in G_0'. The *expansion* of G' with respect to the proper cover G_1',G_2' is the graph G obtained from the disjoint union of G_1 and G_2 by inserting an edge between u_1 in G_{01} and u_2 in G_{02}, for each u' in G_0'. This is illustrated in Fig. 1. The following fundamental result on median graphs was first proved in [12,13].

Theorem 1. *A graph G is a median graph if and only if G can be obtained by successive expansions from the one vertex graph K_1.*

Using this theorem, trees can be obtained from K_1 by restricting the expansions to those of the following type: G_1 is always the whole graph G and G_2 is a single vertex. Expansion with respect to such a cover amounts to adding a new vertex adjacent to the one in G_2. The n-cubes can be obtained from K_1 by using only trivial proper covers. An important feature which follows from the proof of Theorem 1 is that in obtaining a graph G from a median graph H by a succession of expansions, the expansions can be applied in any order.

For an arbitrary edge v_1v_2 in a median graph G, let G_1 be the subgraph of G induced by all vertices nearer to v_1 than to v_2, and let G_2 be the subgraph induced by all vertices nearer to v_2 than to v_1. Since G is bipartite, it follows that G_1,G_2 partitions G. We call such a partition a *split*. Let F_{12} be the set of edges between G_1 and G_2, and let G_{0i} be the subgraph induced by the endvertices in G_i of the edges in F_{12}, for $i=1,2$. The proofs of the following facts can be found in [12,13]; these facts were established as steps in the proof of Theorem 1 above:

(i) F_{12} is a matching as well as a cutset (i.e. a minimal disconnecting set of edges).

(ii) The subgraphs G_1, G_2, G_{01}, G_{02} are convex subgraphs of G.
(iii) The obvious mapping of G_{01} onto G_{02} defined by the edges in F_{12} ($u_1 \to u_2$, for any edge u_1u_2 in F_{12} with u_i in G_{0i}, for $i=1,2$) is an isomorphism.
(iv) For every edge u_1u_2 of F_{12} with u_i in G_{0i} ($i=1,2$), the subgraph G_1 consists of all the vertices of G nearer to u_1 than to u_2, so that u_1 is the gate in G_1 for u_2. A similar statement holds for G_2.

Now the *contraction* G' of G with respect to the split G_1, G_2 is obtained from G by contracting the edges of F_{12} to single vertices. To illustrate this in Fig. 1, move from right to left. Clearly, expansion and contraction are inverse operations. The *contraction map* κ, of G onto G', associated with F_{12} is thus defined by $\kappa|_{G_i} = \lambda^{-1}$, for $i=1,2$. If $\pi = (x_1,\dots,x_k)$ is a profile on G, then π is contracted to a profile $\pi_i' = (x_1',\dots,x_k')$ on G', where $x_i' = \kappa(x_i)$ is the contraction of x_i, for $i=1,\dots,k$.

Let G be a median graph and G_1, G_2 a split. If π is a profile on G, then let π_i denote the subprofile of π (in the same order) consisting of all those vertices of π lying in G_i. If H is any subgraph of G let $\pi(H)$ be the subprofile of vertices in H. We now have $\pi' = \kappa(\pi_i)$ and $\pi_i = \lambda_i(\pi_i')$ where κ and λ_i are applied component-wise. For the profile $\pi = (x_1,\dots,x_k)$ in G, let $|\pi| = k$, the *length* of π. We call G_1, G_2 an *unequal split* (with respect to π) if $|\pi_1| \neq |\pi_2|$, otherwise G_1, G_2 is an *equal split*.

A corollary, or more precisely, a step in the proof of Theorem 3 in [7] is the following result, which will be very important in what follows.

Theorem 2. *Let $G=(V,E)$ be a median graph, and let π be a profile of G. Let G_1, G_2 be an equal split of G. Then x is in $M(\pi)$ if and only if y is in $M(\pi)$, for any edge xy between G_1 and G_2.*

We now turn to the order theoretic preliminaries. As before, all sets are finite. A partially ordered set is a nonempty set V together with a reflexive, antisymmetric, transitive relation $\leqslant$ defined on V. If V is a partially ordered set and $x, y \in V$, then y *covers* x if $x \leqslant y$ and $x \leqslant z < y$ implies that $x=z$. The *covering graph* of V is the graph $G=(V,E)$ where $xy \in E$ iff x covers y or y covers x. The partially ordered set $(V,\leqslant)$ is a *meet semilattice* if and only if every two element set $\{x,y\}$ has an infimum, denoted $x \wedge y$, and is a *join semilattice* if and only if $\{x,y\}$ has a supremum, $x \vee y$. An element s in the meet semilattice V is *join irreducible* if $s = x \vee y$ implies that either $s=x$ or $s=y$. An *atom* of the meet semilattice V is an element that covers the universal lower bound of V. A *lattice* is a partially ordered set V for which $x \wedge y$ and $x \vee y$ exist for all $x, y \in V$. The lattice $(V,\leqslant)$ is *distributive* when $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ for all $x, y, z \in V$.

Our main concern is with the following ordered version of median graphs. A meet semilattice $(V,\leqslant)$ is a *median semilattice* if and only if, for every $x \in V$, the set $\{t \mid t \leqslant x\}$ is a distributive lattice, and any three elements of V have an upper bound whenever each pair of them have an upper bound. The relationship between median graphs and median semilattices is well-known (see [2,3,13]). Indeed if $G=(V,E)$ is a median graph and $z \in V$, then $(V,\leqslant_z)$ is a median semilattice where $\leqslant_z$ is defined by $x \leqslant_z y$ if and only if $x \in I(z,y)$. Conversely the covering graph of a median semilattice is a median graph. Note that different median semilattices may have the same median graph as their covering graph.

For $(V, \leqslant)$ a median semilattice and $x \in V$, let $h(x)$ denote the length of a shortest path from x to the universal lower bound of V, in the covering graph of $(V, \leqslant)$. Finally we recall that the usual lattice metric $d_{\leqslant}$ on $(V, \leqslant)$ defined by $d_{\leqslant}(u,v) = h(u) + h(v) - 2h(u \wedge v)$ coincides with the geodesic metric on the covering graph of $(V, \leqslant)$ (see [11,9]).

3. The main result

Before proving our main result, we need to extend the theory of median graphs developed in [7,12,14]. The next theorem is implicitly contained in the proof of Lemma 9 of Mulder [14]. Here we state the result explicitly and provide its own proof.

Theorem 3. *Let $G = (V,E)$ be a median graph. Let G_1, G_2 be a split of G, and let x be a vertex of G_2. Then there exists a split H_1, H_2 with x in H_{02} and $G_1 \subseteq H_1$ and $H_2 \subseteq G_2$.*

Proof. Note that $G_1 \cup G_{02}$ is convex, and so is gated. Let y be the gate of x in G_{02}, so that y is also the gate of x in $G_1 \cup G_{02}$. We use induction on $k = d(x,y)$. If $k = 0$, then take $H_1 = G_1$ and $H_2 = G_2$, whence $H_{02} = G_{02}$.

Assume $k \geqslant 1$, and let z be the neighbor of y on some y,x-geodesic. Then y is the gate of z in G_{02} as well as $G_1 \cup G_{02}$. So all vertices of $G_1 \cup G_{02}$ are nearer to y than to z. Let T_1, T_2 be the split of edge yz with y in T_1 and z in T_2. Note that x is nearer to z than to y. So x is in T_2. Clearly, we have $G_1 \cup G_{02} \subseteq T_1$, whence $T_2 \subset G_2$. By induction, there is a split H_1, H_2 of G with x in H_{02} and $T_1 \subseteq H_1$ and $H_2 \subseteq T_2$. Hence $G_1 \subseteq H_1$ and $H_2 \subseteq G_2$, and we are done. $\square$

In some cases it is convenient to consider the contraction G' of a median graph G with respect to the split G_1, G_2 from a slightly different perspective: we may obtain G' by "contracting G_2 into G_1". That is, each vertex of G_{02} is identified with its neighbor in G_{01}, so that G_1 is "fully retained" in G'. We use this perspective in the proof of the next theorem.

Theorem 4. *Let $G = (V,E)$ be a median graph, and let π be a profile G. Then*

$$M(\pi) = \cap\{G_1 | G_1, G_2 \textit{ split with } |\pi_1| > |\pi_2|\}.$$

Proof. If every split of G with respect to the profile π is an equal split, then, by convention and Theorem 5 of Mc Morris et al. [7], $M(\pi) = \cap\emptyset = V$. So we can assume that there is at least one unequal split of G.

By Theorem 3 of McMorris et al. [7], we have $M(\pi) \subseteq G_1$, for each unequal split G_1, G_2 of G with $|\pi_1| > |\pi_2|$. Now we contract with respect to all unequal splits G_1, G_2

of G, where we contract G_2 into G_1 whenever $|\pi_1| > |\pi_2|$, thus obtaining the median graph G'. Then G' has only equal splits with respect to π', viz., those splits corresponding to equal splits in G. So $M(\pi') = G'$. Let G_1, G_2 be any unequal split of G with $|\pi_1| > |\pi_2|$. By Theorem 3, we know that, in obtaining G', all vertices of G_2 are, eventually, contracted into G_1. This implies that $G' = \cap\{G_1 | G_1, G_2 \text{ split with } |\pi_1| > |\pi_2|\}$, and we are done. □

Let $G=(V,E)$ be a median graph. A *consensus function* f on G is just a function $f : V^* \to 2^V - \{\emptyset\}$ returning a nonempty subset of vertices for each profile on G. We call f *faithful* if $f((x))=\{x\}$ for all x in V. Next, f is *consistent* if $f(\pi\rho)=f(\pi)\cap f(\rho)$ whenever $f(\pi)\cap f(\rho)\neq\emptyset$, for profiles π and ρ on G, where $\pi\rho$ is the concatenation of π and ρ. A consensus function is said to be $\frac{1}{2}$-*condorcet* if, for each profile π on G and for each split G_1, G_2 of G with $|\pi_1| = |\pi_2|$, and each edge u_1u_2 in F_{12} with u_i in G_i $(i=1,2)$, we have: u_1 is in $f(\pi)$ if and only if u_2 is in $f(\pi)$. Note that the first two properties, faithfulness and consistency, make sense in arbitrary graphs, whereas $\frac{1}{2}$-condorcet presupposes that we can define splits in some way in G. Note also that faithfulness together with consistency implies *unanimity*, i.e., $f((x,\dots,x))=\{x\}$ for all x in V. It is not hard to show that the median function M on any finite metric space is a faithful and consistent consensus function [4,7]. It is an immediate consequence of Theorem 2 that on median graphs the median function is $\frac{1}{2}$-condorcet. Our main result states that the three conditions of faithfulness, consistency, and $\frac{1}{2}$-condorcet, suffice to characterize the median function among the consensus functions on a median graph.

Theorem 5. *Let $G = (V,E)$ be a median graph, and let $f : V^* \to 2^V - \{\emptyset\}$ be a consensus function on G. Then $f = M$ if and only if f is faithful, consistent, and $\frac{1}{2}$-condorcet.*

Proof. As observed above, the median function on a median graph is faithful, consistent, and $\frac{1}{2}$-condorcet.

Conversely, let f be a function on G which is faithful, consistent, and $\frac{1}{2}$-condorcet. First we prove that, for any unequal split G_1, G_2 with $|\pi_1| > |\pi_2|$, we have $f(\pi)\subseteq G_1$. Assume the contrary, and let x be a vertex in $f(\pi)\cap G_2$. Let H_1, H_2 be a split as in Theorem 3 with x in H_{02}, $G_1 \subseteq H_1$, and $H_2 \subseteq G_2$, and let y be the neighbor of x in H_{01}. Then we have

$$|\pi(H_1)| \geqslant |\pi_1| > |\pi_2| \geqslant |\pi(H_2)|.$$

So H_1, H_2 is an unequal split with respect to π. Set $|\pi| = k$, and $|\pi(H_1)| = p$, so that $p > k - p$. Then $2p - k > 0$. Let $\rho = \pi(x)_{2p-k}$ be the concatenation of π and the profile consisting of $2p - k$ copies of x, whence $|\rho(H_1)| = |\rho(H_2)|$, so that H_1, H_2 is an equal split for ρ. By the choice of x, we have x in $f(\pi)$, and, by unanimity, $f((x,\dots,x)) = \{x\}$. So consistency gives us $f(\rho) = \{x\}$. On the other hand, f being $\frac{1}{2}$-condorcet implies that x is in $f(\rho)$ if and only if y is in $f(\rho)$. This contradiction shows that $f(\pi)\subseteq G_1$.

By Theorem 4, we deduce that $f(\pi) \subseteq M(\pi)$. Now $M(\pi)$, being the intersection of convex subgraphs, is itself convex, and thus induces a connected subgraph. Let xy be any edge in $M(\pi)$, and let H_1, H_2 be the split associated with xy. It follows from Theorem 4 above and Theorem 5 of McMorris et al. [7] that H_1, H_2 is an equal split of G with respect to π. Then, f being $\frac{1}{2}$-condorcet, we have that x is in $f(\pi)$ if and only if y is in $f(\pi)$. Since $\emptyset \neq f(\pi) \subseteq M(\pi)$, we infer from the connectivity of $M(\pi)$ that $f(\pi) = M(\pi)$. □

4. Translation to order

In this section, we use the one-to-one correspondence between median semilattices $(V, \leqslant)$ and pairs (G, z) where $G = (V, E)$ is a median graph, and z is a vertex of G, to translate Theorem 5 into a result on consensus functions on median semilattices.

Lemma 6. *Let $G = (V, E)$ be a median graph, and let G_1, G_2 be a split of G. Let z be a vertex of G_1, and let s be the gate of z in G_2. Then the neighbor w_s of s in G_1 is the gate of z in G_{01} as well as $G_{01} \cup G_2$.*

Proof. Recall that s, being the gate of z in G_2, is the nearest vertex in G_2 from z. Hence s is in G_{02} and s has a unique neighbor w_s in G_{01}. Clearly, w_s is on a z,s-geodesic. If w_s were not the gate of z in G_{01}, then, by convexity of G_{01}, there would be a neighbor y_1 of w_s in G_{01} with $d(z, y_1) = d(z, w_s) - 1$. Let y_2 be the neighbor of y_1 in G_{02}. Then we have

$$d(z, y_2) \leqslant d(z, y_1) + 1 = d(z, w_s) = d(z, s) - 1,$$

contrary to the fact that s is the nearest vertex from z in G_2. So w_s is the gate of z in G_{01}, and, by convexity of $G_{01} \cup G_2$, also in $G_{01} \cup G_2$. □

Theorem 7. *Let $G = (V, E)$ be a median graph and let z be any vertex of G. For any split G_1, G_2 of G with z in G_1, the gate s of z in G_2 is the unique join-irreducible in G_{02} in the median semilattice $(V, \leqslant_z)$.*

Proof. Let w_s be the unique neighbor of s in G_1. By Lemma 6, w_s is the gate of z in $G_{01} \cup G_2$, so that w_s lies in $I(z, s)$. Hence s covers w_s in $(V, \leqslant_z)$. Let y be any other neighbor of s. Then y is in G_2, and s, being the gate of z in G_2, is on a geodesic between z and y, so that $s \leqslant_z y$. Hence w_s is the unique vertex covered by s in $(V, \leqslant_z)$, so that s is join irreducible.

Take any y in $G_{02} - s$. Since s is the gate of z in G_{02}, there is a z, y-geodesic P passing through s. Because of convexity of G_{02}, the neighbor x of y on P is in G_{02}. By definition, y covers x in $(V, \leqslant_z)$. On the other hand, let w_y be the unique neighbor of y in G_{01}. Since w_y is the gate of y in G_1, there is a z, y-geodesic passing through w_y. So, by definition, y also covers w_y in $(V, \leqslant_z)$, whence y is not join-irreducible. □

Recall that the median semilattices with the median graph G as covering graphs are precisely the posets $(V, \leqslant_z)$, where z is any vertex of G. We can state this also in order-theoretic language as follows (see [13]). Let $(V, \leqslant)$ be a median semilattice with universal lower bound z, and let b be any other element of V. Define the partial order $\leqslant_b$ on V by $u \leqslant_b v$ whenever $u \geqslant b \wedge v$ and $u = (b \wedge u) \vee (u \wedge v)$. Then $(V, \leqslant_b)$ is a median semilattice as well. Actually, if $G = (V, E)$ is the covering graph of $(V, \leqslant)$, then $(V, \leqslant)$ is precisely the median semilattice $(V, \leqslant_z)$ obtained from G, and $(V, \leqslant_b)$ is just obtainable from G in the usual way. This gives an order-theoretic procedure to obtain from $(V, \leqslant)$ all median semilattices with G as their covering graph.

The next result, which is not so obvious from the order-theoretic point of view, is now ready at hand using the theory of median graphs.

Corollary 8. *Let $G = (V, E)$ be a median graph. Then all median semilattices $(V, \leqslant)$ having G as covering graph have the same number of join-irreducibles.*

Proof. Theorem 7 provides us with a one-to-one correspondence between the join-irreducibles of $(V, \leqslant_z)$ and the splits of G, for any vertex z of G. So the number of join-irreducibles of $(V, \leqslant)$ equals the number of splits of G. □

In [4], Barthélemy and Janowitz introduced the notion of t-condorcet for consensus functions on distributive semilattices. As was already observed in a note added in proof in [8] the Barthélemy–Janowitz notion needs to be modified to make their theorems on t-condorcet consensus functions true. Next, we present an example that shows the need of modification and then present an alternative.

Let $(V, \leqslant)$ be a distributive meet semilattice, and let $\pi = (x_1, \ldots, x_k)$ be a profile on V. For any element u of V, we define the *index* of u with respect to π to be

$$\gamma(u, \pi) = \frac{|\{i \mid u \leqslant x_i\}|}{k}.$$

A consensus function $f : V^* \to 2^V - \{\emptyset\}$ on $(V, \leqslant)$ is *t-condorcet* if the following holds for any profile π on V: if s is join-irreducible in $(V, \leqslant)$ covering w_s and $\gamma(s, \pi) = t$, then $x \vee s$ is in $f(\pi)$ if and only if $x \vee w_s$ is in $f(\pi)$, provided $x \vee s$ exists. In the case of median semilattices with $t = \frac{1}{2}$, this amounts precisely to the order theoretic equivalent of our $\frac{1}{2}$-condorcet axiom for consensus functions on graphs. The Barthélemy–Janowitz axiom in case $t = \frac{1}{2}$ reads as follows: if s is join-irreducible with $\gamma(s, \pi) = \frac{1}{2}$, then $x \vee s$ is in $f(\pi)$ if and only if x is in $f(\pi)$, provided $x \vee s$ exists. The example of Fig. 2 shows that the median function M is not $\frac{1}{2}$-condorcet in this sense. Here we take $\pi = (x \vee w_s,\ x \vee s)$, so that $M(\pi) = \{x \vee w_s,\ x \vee s\}$.

Note that, if all join-irreducibles are atoms in $(V, \leqslant)$, the Barthélemy–Janowitz axiom and our new axiom are identical. Therefore, it sufficed to add this condition in proof in [8] to make all results in [8], that were based on results in [4], correct. The final result of this paper is a generalization of Theorem 4 in [8] to cases where there are join-irreducibles that are not atoms.

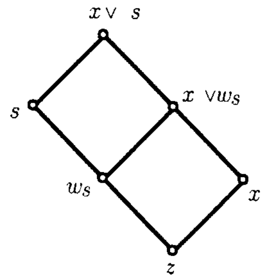

Fig. 2. The Hasse diagram of a median semilattice.

Theorem 9. *Let $(V, \leqslant)$ be a median semilattice, and let $f : V^* \to 2^V - \{\emptyset\}$ be a consensus function. Then f is the median function on $(V, \leqslant)$ if and only if f is faithful, consistent, and $\frac{1}{2}$-condorcet.*

Proof. Let z be the universal lower bound of $(V, \leqslant)$. Let $G=(V,E)$ be the covering graph of $(V, \leqslant)$. Since the distance functions of $(V, \leqslant)$ and $G=(V,E)$ coincide, the median function $M_{\leqslant}$ of $(V, \leqslant)$ and the median function M_G of G are identical consensus functions on V. We thus have that $M_{\leqslant}$ is faithful and consistent. Now we prove that $M_{\leqslant}$ is $\frac{1}{2}$-condorcet in the order-theoretic sense.

Let π be a profile on $(V, \leqslant)$. Let s be any join-irreducible in $(V, \leqslant)$ covering w_s with $\gamma(s,\pi)=\frac{1}{2}$. Let G_1, G_2 be the split of G defined by the edge sw_s with w_s in G_1 and s in G_2. Then, by Theorem 7, s is the gate of z in G_2. So G_2 consists precisely of the vertices y of G with $s \leqslant y$ in $(V, \leqslant)$. Therefore, we have $|\pi_2| = |\pi|\gamma(s,\pi) = \frac{1}{2}|\pi|$, whence $|\pi_1|=|\pi_2|$ in G. For any edge y_1y_2 between G_1 and G_2, we have, by Theorem 2, that y_1 is in $M_G(\pi)$ if and only if y_2 is in $M_G(\pi)$. Take any element x of $(V, \leqslant)$ for which $x \vee s$ exists. If x is in G_2, then $x = x \vee s = x \vee w_s$, and there is nothing to prove. So take x in G_1. Note that $x \vee s$ is in G_2, and x is in $I(z, x \vee s)$, whence $I(x, x \vee s) \subseteq I(z, x \vee s)$. Let y_2 be the gate of x in G_2. Then, by definition, y_2 is in $I(x, x \vee s)$. So $x \leqslant y_2 \leqslant x \vee s$. On the other hand, we also have $s \leqslant y_2 \leqslant x \vee s$. Therefore, $y_2 = x \vee s$. In a similar way we deduce that $y_1 = x \vee w_s$. So we may conclude that $x \vee s$ is in $M_{\leqslant}(\pi)$ if and only if $x \vee w_s$ is in $M_{\leqslant}(\pi)$.

Conversely, let $f : V^* \to 2^V - \{\emptyset\}$ be a consensus function that is faithful, consistent, and $\frac{1}{2}$-condorcet on $(V, \leqslant)$. Then f is faithful and consistent on G. Let π be a profile on V, and let G_1, G_2 be an equal split of G. Assume z is in G_1. Then, by Theorem 7, the unique join-irreducible s in G_{02} is the gate z in G_2. Let s cover w_s, which is then, by Lemma 6, the gate of z in G_{01} as well as $G_{01} \cup G_2$. Let xy be an edge between G_1 and G_2 with x in G_1. Then we have $x = x \vee w_s$ and $y = x \vee s$. So, f being $\frac{1}{2}$-condorcet on $(V, \leqslant)$, we have that x is in $f(\pi)$ if and only if y is in $f(\pi)$. Thus we have shown that f is $\frac{1}{2}$-condorcet on G. By Theorem 5, we conclude that $f = M_G = M_{\leqslant}$, and we are done. □

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