On singular, functional, nonsmooth and implicit phi-Laplacian initial and boundary value problems

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Abstract

In this paper we apply fixed point results for mappings in partially ordered spaces to derive existence results for extremal solutions of phi-Laplacian initial and boundary value problems. The considered problems can be singular, functional, discontinuous, nonlocal and implicit. Concrete examples are also solved.

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1. Introduction

In this paper we apply fixed point results presented in [4,6] for mappings in partially ordered spaces to derive existence results for first- and second-order differential equations. The considered problems include many kinds of special types. For instance,

– differential equations and initial/boundary conditions may be implicit;

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– differential operators of differential equations may be singular;
– differential equations and initial or boundary conditions may depend functionally on the unknown function and its derivative;
– differential equations and initial or boundary conditions may contain discontinuous nonlinearities;
– problems on infinite intervals are also included.

Concrete examples are also presented and solved to illustrate the obtained results.

2. Existence results for first-order implicit initial value problems

In this section we study the first-order implicit initial value problem (IVP)
\[
\begin{cases}
Lu(t) := \frac{d}{dt}(p(t)\phi(u(t))) = f(t, u, Lu) \\
\text{for almost every (a.e.) } t \in J := (a, b), \\
\lim_{t \to a^+} p(t)\phi(u(t)) = c(u, Lu),
\end{cases}
\] (2.1)

where \(-\infty \leq a < b \leq \infty\), \(p \in L^1_{\text{loc}}(J)\), \(\phi : I \to \mathbb{R}\), \(f : J \times L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J) \to \mathbb{R}\) and \(c : L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J) \to \mathbb{R}\).

We are looking for solutions of (2.1) from the set
\[
S := \{u \in L^1_{\text{loc}}(J) \mid p \cdot (\phi \circ u) \text{ is locally absolutely continuous}\}. \quad (2.2)
\]

Denote
\[
X := \left\{ h \in L^1_{\text{loc}}(J) \mid \int_{a^+}^{b} h(t)\,dt = \lim_{r \uparrow a} \int_{r}^{s} h(t)\,dt \text{ is finite for some } s \in J \right\}. \quad (2.3)
\]

Assuming that \(L^1_{\text{loc}}(J)\), \(X\) and \(S\) are ordered a.e. pointwise, we shall show that the IVP (2.1) has extremal solutions in \(S\) if the functions \(p\), \(\phi\), \(f\) and \(c\) satisfy the following hypotheses:

**(φ)** \(\phi\) is an increasing homeomorphism from an open interval \(I\) of \(\mathbb{R}\) onto \(\mathbb{R}\).

**(pφ)** \(p\) is a.e. positive-valued and \(\phi^{-1}\left(\frac{K}{p(t)}\right) \in L^1_{\text{loc}}(J)\) for all \(K \in \mathbb{R}\).

**(fa)** \(f(\cdot, u, v)\) is Lebesgue measurable and \(h_- \leq f(\cdot, u, v) \leq h_+\) for all \(u, v \in L^1_{\text{loc}}(J)\) and for some \(h_\pm \in X\).

**(fb)** There exists a \(\lambda \geq 0\) such that \(f(\cdot, u_1, v_1) + \lambda v_1 \leq f(\cdot, u_2, v_2) + \lambda v_2\) whenever \(u_i, v_i \in L^1_{\text{loc}}(J), i = 1, 2, u_1 \leq u_2\) and \(v_1 \leq v_2\).

**(c)** \(c_\pm \in \mathbb{R}\) and \(c_- \leq c(u_1, v_1) \leq c(u_2, v_2) \leq c_+\) whenever \(u_i, v_i \in L^1_{\text{loc}}(J), i = 1, 2, u_1 \leq u_2\) and \(v_1 \leq v_2\).

We shall first convert the IVP (2.1) to a system of two equations.
Lemma 2.1. Assume that the hypotheses \((\phi)\) and \((p\phi)\) hold, and that \(f(., u, v) \in X\) for all \(u, v \in L^1_{\text{loc}}(J)\). Then \(u\) is a solution of the IVP (2.1) in \(S\) if and only if \((u, Lu) = (u, v)\), where \((u, v)\) is a solution of the system

\[
\begin{align*}
\left\{ \begin{array}{l}
\mathbf{u}(t) = \phi^{-1}(\frac{1}{p(t)} [c(u, v) + \int_{a+}^{t} v(s) \, ds]), & t \in J, \\
\mathbf{v}(t) = f(t, u, v) & \text{for a.e. } t \in J,
\end{array} \right.
\end{align*}
\]  

(2.4)

in \(L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J)\).

Proof. Assume that \(u\) is a solution of (2.1) in \(S\). Denoting

\[
\mathbf{v}(t) = Lu(t) = \frac{d}{dt} \left( p(t)\phi(u(t)) \right), \quad t \in J,
\]

the definition (2.2) of \(S\) and (2.5) ensure that

\[
\int_{a}^{s} \mathbf{v}(t) \, dt = \int_{r}^{s} \frac{d}{dt} \left( p(t)\phi(u(t)) \right) \, dt = p(s)\phi(u(s)) - p(r)\phi(u(r)), \quad a < r \leq s < b.
\]

This result and the initial condition of (2.1) imply that the first equation of (2.4) holds. The validity of the second equation of (2.4) is a consequence of the differential equation of (2.1) and the definition (2.5) of \(v\).

Conversely, let \((u, v)\) be a solution of the system (2.4) in \(L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J)\). According to (2.4) we have

\[
p(t)\phi(u(t)) = c(u, v) + \int_{a+}^{t} v(s) \, ds, \quad t \in J.
\]

(2.6)

This equation implies that \(u \in S\), and by differentiation we obtain from (2.6) that

\[
\mathbf{v}(t) = \frac{d}{dt} p(t)\phi(u(t)) = Lu(t) \quad \text{for a.e. } t \in J.
\]

This result, Eq. (2.6) and the second equation of (2.4) imply that \(u\) is a solution of the IVP (2.1). \(\Box\)

The following fixed point result is a consequence of [4, Theorem A.2.1], or [6, Theorem 1.2.1 and Proposition 1.2.1].

Lemma 2.2. Given a partially ordered set \(P = (P, \leq)\) and its order interval \([x_-, x_+] = \{x \in P \mid x_- \leq x \leq x_+\}\), assume that \(G : [x_-, x_+] \to [x_-, x_+]\) is increasing, i.e., \(Gx \leq Gy\) whenever \(x_- \leq x \leq y \leq x_+\), and that each well-ordered chain of the range \(\text{ran } G\) of \(G\) has a supremum in \(P\) and each inversely well-ordered chain of \(\text{ran } G\) has an infimum in \(P\). Then \(G\) has least and greatest fixed points, and they are increasing with respect to \(G\).

In the application of Lemma 2.2 to the IVP (2.1) we use Lemma 2.1 and the following result.
Lemma 2.3. Assume that $W$ is a nonempty subset of $L^1_{\text{loc}}(J)$, $J = (a, b)$, $-\infty \leq a < b \leq \infty$, and that there exist functions $u_{\pm} \in L^1_{\text{loc}}(J)$, $i = 1, 2$, such that

$$u_-(t) \leq u(t) \leq u_+(t) \quad \text{for all } u \in W \text{ and for a.e. } t \in J.$$  

(2.7)  

(a) If $W$ is well-ordered, it contains an increasing sequence which converges a.e. pointwise to $\sup W$.

(b) If $W$ is inversely well-ordered, it contains a decreasing sequence which converges a.e. pointwise to $\inf W$.

Proof. (a) Assume that $W$ is well-ordered. Choose a sequence of finite and closed subintervals $J_n$, $n \in \mathbb{N}$, of $J$ such that $J = \bigcup_{n=0}^{\infty} J_n$, and that $J_n \subset J_{n+1}$ for each $n \in \mathbb{N}$. The given assumptions ensure that for each $n \in \mathbb{N}$ the restrictions $u_{|J_n}$, $u \in W$, form a well-ordered and order-bounded chain $W_n$ in $L^1(J_n)$, ordered a.e. pointwise. Consequently, by the proof of [5, Lemma 4.2], for each $n \in \mathbb{N}$,

$$v_n = \sup W_n$$

exists in $L^1(J_n)$, and there exists an increasing sequence $(u^k_n)_{k=0}^{\infty}$ of $W$ and a null-set $Z_n \subset J_n$ such that

$$v_n(t) = \lim_{k \to \infty} u^k_n(t) = \sup_{k \in \mathbb{N}} u^k_n(t) \quad \text{for each } t \in J_n \setminus Z_n.$$  

(2.8)  

Defining $v_n(t) = 0$ for $t \in J \setminus J_n$, we obtain a sequence of Lebesgue measurable functions $v_n : J \to \mathbb{R}$. The sequence $(v_n)$ is also increasing since $J_n \subset J_{n+1}$, $n \in \mathbb{N}$. It is also a.e. pointwise bounded by (2.7) and (2.8), whence

$$u^*(t) = \lim_{n \to \infty} v_n(t) = \sup_{n \in \mathbb{N}} v_n(t)$$  

(2.9)  

exists for a.e. $t \in J$. Defining $u^*(t) = 0$ for the remaining $t \in J$, we obtain a Lebesgue measurable function $u^* : J \to \mathbb{R}$. Denoting

$$u_n = \max\{u^j_n \mid 0 \leq j \leq n\}, \quad n \in \mathbb{N},$$

we obtain an increasing sequence $(u_n)$ of $W$ which satisfies

$$u^k_n(t) \leq u_n(t) \leq u^*(t)$$

for each $k = 0, \ldots, n$ and $t \in J_n \setminus Z_n$. Moreover, by (2.7) the sets $Z_n$ can be so chosen that $(u_n(t))_{n=0}^{\infty}$ is bounded and increasing for each $t \in J \setminus Z$, where $Z = \bigcup_{n=0}^{\infty} Z_n$. Thus

$$u(t) = \lim_{n \to \infty} u_n(t) = \sup_{n \in \mathbb{N}} u_n(t)$$

exists for each $t \in J \setminus Z$. The definitions of $v_n$ and $u$ imply that

$$v_n(t) \leq u(t) \leq u^*(t) \quad \text{for each } t \in J_n \setminus Z_n.$$  

Thus

$$u^*(t) = \lim_{n \to \infty} v_n(t) \leq u(t) \leq u^*(t)$$
for a.e. \( t \in J \). This result implies that \( u = u^* \), whence \( u_n(t) \to u^*(t) \) for a.e. \( t \in J \). Since \( (u_n)_{n=0}^{\infty} \) is a sequence of \( W \), it follows from (2.7) that

\[
\inf_{w} (t) \leq u^*(t) \leq \sup_{w} (t) \quad \text{for a.e. } t \in J.
\]

This result and Lebesgue measurability of \( u^* \) imply that \( u^* \in L^1_{\text{loc}}(J) \).

It remains to prove that \( u^* = \sup W \). If \( w \in W \), then \( w|_{J_n} \leq u_n \), whence

\[
w(t) \leq v_n(t) \leq u^*(t) \quad \text{for a.e. } t \in J_n \text{ and for each } n \in \mathbb{N}.
\]

Thus \( w \leq u^* \) for each \( w \in W \), so that \( u^* \) is an upper bound of \( W \). If \( v \in L^1_{\text{loc}}(J) \) is another upper bound of \( W \), then for each \( w \in W \)

\[
w(t) \leq v(t) \quad \text{for a.e. } t \in J_n \text{ and for each } n \in \mathbb{N}.
\]

Thus \( w|_{J_n} \leq v|_{J_n} \) for all \( n \in \mathbb{N} \) and \( w \in W \), whence

\[
v_n(t) \leq v(t) \quad \text{for a.e. } t \in J_n \text{ and for each } n \in \mathbb{N}.
\]

This result and the definition (2.9) of \( u^* \) imply that \( u^* \leq v \). Consequently, \( u^* = \sup W \) in \( L^1_{\text{loc}}(J) \).

(b) If \( W \) is inversely well-ordered, then \(-W\), satisfies the hypotheses imposed on \( W \) in (a). Thus there exists an increasing sequence \( (u_n) \in -W \) such that \( u_n \to u = \sup(-W) \) a.e. pointwise on \( J \). Denoting \( u_n = -v_n, n \in \mathbb{N} \), we obtain a decreasing sequence of \( W \) which converges a.e. pointwise to \(-u = \inf W \). \( \Box \)

Now we are ready to prove our main existence result for the IVP (2.1).

**Theorem 2.1.** Assume that the hypotheses \((\phi), (p\phi), (fa), (fb) \) and \(c\) hold. Then the IVP (2.1) has least and greatest solutions in \( S \), and they are increasing with respect to \( f \) and \( c \).

**Proof.** Assume that \( P = L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J) \) is ordered componentwise. The relations

\[
x_{\pm}(t) := \left( \phi^{-1} \left( \frac{1}{p(t)} \left[ c_{\pm} + \int_{a^+}^{b^+} h_{\pm}(s) \, ds \right] \right), h_{\pm}(t) \right), \quad t \in J, \quad (2.10)
\]

define functions \( x_{\pm} \in P \). If \((u, v) \in [x_-, x_+] \), then \( v \in L^1_{\text{loc}}(J) \) and \( h_- \leq v \leq h_+ \). Hence it is easy to show that \( v \in X \), because \( h_{\pm} \in X \). Thus, by applying the given hypotheses, we see that the relations

\[
G_1(u, v)(t) = \phi^{-1} \left( \frac{1}{p(t)} \left[ c(u, v) + \int_{a^+}^{b^+} v(s) \, ds \right] \right),
\]

\[
G_2(u, v)(t) = f(t, u, v) + \lambda v(t) \quad \frac{1}{1 + \lambda}, \quad (2.11)
\]

define an increasing mapping \( G = (G_1, G_2) : [x_-, x_+] \to [x_-, x_+] \).

Let \( W \) be a well-ordered chain in ran \( G \). The sets \( W_1 = \{ u \mid (u, v) \in W \} \) and \( W_2 = \{ v \mid (u, v) \in W \} \) are well-ordered and order-bounded chains in \( L^1_{\text{loc}}(J) \). It then follows from Lemma 2.3 that sup \( W_1 \) and sup \( W_2 \) exist in \( L^1_{\text{loc}}(J) \). Obviously, (sup \( W_1 \), sup \( W_2 \)) is
a supremum of $W$ in $P$. Similarly one can show that each inversely well-ordered chain of ran $G$ has an infimum in $P$.

The above proof shows that the operator $G = (G_1, G_2)$ defined by (2.11) satisfies the hypotheses of Lemma 2.2, whence $G$ has a least fixed point $x_\ast = (u_\ast, v_\ast)$ and a greatest fixed point $x^* = (u^*, v^*)$. It follows from (2.11) that $(u_\ast, v_\ast)$ and $(u^*, v^*)$ are solutions of the system (2.4). According to Lemma 2.1 $u_\ast$ and $u^*$ belong to $S$ and are solutions of the IVP (2.1).

To prove that $u_\ast$ and $u^*$ are least and greatest of all solutions of (2.1) in $S$, let $u \in S$ be a solution of (2.1). In view of Lemma 2.1, $(u, v) = (u, Lu)$ is a solution of the system (2.4). Applying the hypotheses (fa) and (c) it is easy to show that $x = (u, v)$ is a fixed point of $G = (G_1, G_2): [x_-, x_+] \to [x_-, x_+]$, defined by (2.11). Because $x_\ast = (u_\ast, v_\ast)$ and $x^* = (u^*, v^*)$ are least and greatest fixed points of $G$, then $(u_\ast, v_\ast) \leq (u, v) \leq (u^*, v^*)$. In particular, $u_\ast \leq u \leq u^*$, whence $u_\ast$ and $u^*$ are least and greatest of all solutions of the IVP (2.1).

The last assertion is an easy consequence of the last conclusion of Lemma 2.2 and the definition of $G$. □

As a special case, we obtain an existence result for the IVP
\begin{equation}
\begin{align*}
\frac{d}{dt}(p(t)\phi(u(t))) &= g(t, u(t), \frac{d}{dt}(p(t)\phi(u(t)))) \quad \text{for a.e. } t \in J, \\
\lim_{t \to a^+} p(t)\phi(u(t)) &= c.
\end{align*}
\end{equation}

**Proposition 2.1.** Let the hypotheses $(\phi)$ and $(\phi\phi)$ hold, and let $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the following hypotheses:

\begin{enumerate}
\item[(ga)] $g(\cdot, u(\cdot), v(\cdot))$ is Lebesgue measurable and $h_- \leq g(\cdot, u(\cdot), v(\cdot)) \leq h_+$ for all $u, v \in L^1_{\text{loc}}(J)$ and for some $h_\pm \in X$.
\item[(gb)] $g(t, x, z) \leq g(t, y, w)$ for a.e. $t \in J$ and whenever $x \leq y$ and $z \leq w$ in $\mathbb{R}$.
\end{enumerate}

Then the IVP (2.12) has for each choice of $c \in \mathbb{R}$ least and greatest solutions in $S$. Moreover, these solutions are increasing with respect to $g$ and $c$.

**Proof.** If $c \in \mathbb{R}$, the IVP (2.12) is reduced to (2.1) when we define
\begin{equation}
\begin{align*}
f(t, u, v) &= g(t, u(t), v(t)), \\
c(u, v) &\equiv c,
\end{align*}
\end{equation}
where $t \in J$, $u, v \in L^1_{\text{loc}}(J)$. The hypotheses (ga) and (gb) imply that $f$ satisfies the hypotheses (fa) and (fb). The hypothesis (c) is also valid, whence (2.1), with $f$ and $c$ defined by (2.13), and hence also (2.12), has by Theorem 2.1 least and greatest solutions. The last assertion follows from the last assertion of Theorem 2.1. □

If we replace the hypothesis (fa) by the following hypothesis:
(fc) \( f(\cdot, u, v) \in X \) for all \( u, v \in L^1_{\text{loc}}(J) \), and there exist \( x_\pm = (u_\pm, v_\pm) \in L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J) \) such that \( x_- \leq x_+, x_- \leq Gx_- \) and \( Gx_+ \leq x_+ \), where \( G = (G_1, G_2) \) is defined by (2.11),

we get the following result.

**Corollary 2.1.** Assume that the hypotheses (\( \phi \)), (\( p\phi \)), (fb), (fc) and (c) hold. Then the IVP (2.1) has least and greatest solutions in \( \{ u \in S \mid u_- \leq u \leq u_+ \} \).

**Remarks 2.1.** If \( \lim_{t \to a^+} p(t) = 0 \), the differential operator \( \frac{d}{dt}(p(t)\phi(u(t))) \) in (2.1) is singular. An example of a function \( \phi \) with property (\( \phi \)) is

\[
\phi(x) = \frac{x}{\sqrt{1 - x^2}}, \quad x \in (-1, 1),
\]

arising in relativistic dynamics. In this case the operator \( G = (G_1, G_2) \) given by (2.11) can be rewritten as

\[
G_1(u, v)(t) = \frac{c(u, v) + \int_{a^+}^t v(s)ds}{\sqrt{p^2(t) + (c(u, v) + \int_{a^+}^t v(s)ds)^2}},
\]

\[
G_2(u, v)(t) = \frac{f(t, u, v) + \lambda v(t)}{1 + \lambda}.
\]

(2.14)

This formula shows that \(-1 \leq G_1(u, v)(t) \leq 1\) whenever \( G_1(u, v)(t) \) is defined. Thus we have the following result.

**Proposition 2.2.** Assume that \( p \in L^1_{\text{loc}}(J) \) is positive-valued, that the hypotheses (fa) and (fb) hold, and that \( c(u_1, v_1) \leq c(u_2, v_2) \) in \( \mathbb{R} \) whenever \( u_1 \leq v_1 \) in \( L^1_{\text{loc}}(J), i = 1, 2 \). Then the IVP

\[
\begin{cases}
Lu := \frac{d}{dt}\left(\frac{p(t)u(t)}{\sqrt{1 - u(t)^2}}\right) = f(t, u, Lu) \quad \text{a.e. in } J = (a, b), \\
\lim_{t \to a^+} \frac{u(t)}{\sqrt{1 - u(t)^2}} = c(u, Lu)
\end{cases}
\]

(2.15)

has least and greatest solutions in \( S \).

As a consequence of Proposition 2.2, we obtain an existence result also for a periodic boundary value problem.

**Corollary 2.2.** Let \( p \) and \( f \) satisfy the hypotheses of Proposition 2.2. Then for each choice of \( t_1, t_2 \in J, t_1 < t_2 \), the periodic boundary value problem

\[
Lu := \frac{d}{dt}\left(\frac{p(t)u(t)}{\sqrt{1 - u(t)^2}}\right) = f(t, u, Lu) \quad \text{a.e. in } [t_1, t_2], \quad u(t_1) = u(t_2)
\]

(2.16)

has least and greatest solutions.
Proof. The asserted result follows from Proposition 2.2 when we replace $a$ by $t_1$ and $c(u, Lu)$ by $p(t_1) \frac{u(t_1)}{\sqrt{1-u(t_1)^2}}$ in (2.15). \hfill \square

Example 2.1. Choose $J = (0, \infty)$, and consider the IVP

$$\begin{cases}
    Lu(t) := \frac{d}{dt} \left( \frac{p(t)u(t)}{\sqrt{1-u(t)^2}} \right) \\
    = h(t) + \left[ K \tanh \left( q(t) \int_{1}^{4} (u(s) + Lu(s)) \, ds \right) \right] / K \quad \text{a.e. in } J,
  \\
    \lim_{t \to 0^+} \frac{p(t)u(t)}{\sqrt{1-u(t)^2}} = c \cdot \frac{p(1)u(1)}{\sqrt{1-u(1)^2}},
\end{cases}$$

where $p \in L^1_{\text{loc}}(J)$, $p(t) > 0$ for $t \in J$, $q \in L^1_{+}(J)$, $h \in X$, $c \geq 0$, $K > 0$ and $[z]$ denotes the greatest integer $\leq z$. Problem (2.17) is of the form (2.15) with

$$c(u) = c \cdot \frac{p(1)u(1)}{\sqrt{1-u(1)^2}} \quad \text{and}$$

$$f(t, u, v) = h(t) + \left[ K \tanh \left( q(t) \int_{1}^{4} (u(s) + v(s)) \, ds \right) \right] / K, \quad t \in J. \quad (2.18)$$

It is easy to see that the hypotheses of Corollary 2.2 hold, whence (2.12) has least and greatest solutions.

Remark 2.2. If $h(t) = \frac{1}{t} \sin \frac{1}{t}$, $t \in J = (0, \infty)$, then $h$ and the function $f(\cdot, u, v)$ defined by (2.18) belong to $X$, but not to $L^1((0, T))$ for any $T > 0$.

Example 2.2. The singular IVP

$$\begin{cases}
    Lu(t) := \frac{d}{dt} \left( \frac{t u(t)}{\sqrt{1-u(t)^2}} \right) \\
    = \frac{100(t-1)+[1000 \tan h(\int_{1}^{4} (u(s)+Lu(s)/4) \, ds)]}{1000} \quad \text{a.e. in } (0, \infty),
  \\
    \lim_{t \to 0^+} \frac{t u(t)}{\sqrt{1-u(t)^2}} = 0,
\end{cases}$$

is a special case of (2.17) when $p(t) = t$, $q(t) = \frac{1}{4}$, $h(t) = \frac{t-1}{10}$, $c = 0$ and $K = 1000$. Thus (2.19) has extremal solutions. To determine them, notice first that we can choose $c_{\pm} = 0$ and $h_{\pm}(t) = \frac{-1}{10} \pm 1$ in (2.10). Thus the functions $x_{\pm}$ defined by (2.10) can be calculated, and one obtains

$$x_-(t) = \left( -\frac{22}{\sqrt{t^2 - 44t + 887}} \cdot \frac{t}{10} - \frac{11}{10} \right),$$

$$x_+(t) = \left( \frac{18}{\sqrt{t^2 + 36t + 724}} \cdot \frac{t}{10} - \frac{9}{10} \right).$$
In this case the chains needed in the proof of Corollary 2.2 are reduced to ordinary iteration sequences \((G^n x_\pm)\), where \(G = (G_1, G_2)\) is defined by (2.14). Calculating the iterations 
\(G^n x_-\), or equivalently, the successive approximations

\[
\begin{align*}
  u_n(t) &= \frac{\int_{a}^{t} v_{n-1}(s) \, ds}{\sqrt{t^2 + \left( \int_{a}^{t} v_{n-1}(s) \, ds \right)^2}}, \quad t \in J, \\
  v_n(t) &= \frac{100(t-1) + \tanh(1000 \int_{a}^{t} (u_{n-1}(s) + v_{n-1}(s)) \, ds)}{1000} \quad \text{a.e. in } (0, \infty), \\
  u_0(t) &= \frac{t - 22}{\sqrt{t^2 - 44t + 884}}, \quad v_0(t) = \frac{461}{500},
\end{align*}
\]

it turns out that \(G^5 x_- = G^6 x_-\). Thus \(u_\ast = G^5 x_-\) is a least solution of (2.19). Similarly, one can show that \(u^\ast = G^4 x_+\) is an greatest solution of (2.19). The exact expressions of these solutions are

\[
\begin{align*}
  u_\ast(t) &= \frac{25t - 373}{\sqrt{625t^2 - 18650t + 389129}}, \quad u^\ast(t) = \frac{25t + 372}{\sqrt{625t^2 + 18600t + 388384}}.
\end{align*}
\]

### 3. Existence results for second-order implicit initial value problems

Next we study the second-order implicit initial value problem (IVP),

\[
\begin{align*}
  &Lu(t) := \frac{d}{dt} \{ p(t)(\phi(u'(t))) \} = f(t, u, u', Lu) \quad \text{for a.e. } t \in J := (a, b), \\
  &\lim_{\tau \to a^+} p(t)(\phi(u'(t))) = c(u, u', Lu), \quad \lim_{\tau \to a^+} u(t) = d(u, u', Lu),
\end{align*}
\]

where \(-\infty \leq a < b \leq \infty\), \(p \in L^1_{\text{loc}}(J)\), \(\phi : I \to \mathbb{R}\), \(f : J \times L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J) \to \mathbb{R}\)
and \(c, d : L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J) \to \mathbb{R}\).

We are now looking for solutions of (3.1) from the set

\[
Y := \left\{ u : J \to \mathbb{R} \mid u \text{ and } p \cdot (\phi \circ u') \text{ are locally absolutely continuous} \right\}.
\]

Denote, as in Section 2,

\[
X := \left\{ h \in L^1_{\text{loc}}(J) \left| \int_{a}^{s} h(t) \, dt = \lim_{r \downarrow a} \int_{r}^{s} h(t) \, dt \text{ is finite for some } s \in J \right\} \right. \]

Assuming that \(L^1_{\text{loc}}(J)\) and \(X\) are ordered a.e. pointwise, and that \(Y\) is ordered pointwise, we shall show that the IVP (3.1) has extremal solutions in \(Y\) if the functions \(p, \phi, f, c\) and \(d\) satisfy the following hypotheses:

\begin{itemize}
  \item[(\phi)] \(\phi\) is an increasing homeomorphism from an open interval \(I\) of \(\mathbb{R}\) onto \(\mathbb{R}\).
  \item[(p\phi)] \(p\) is a.e. positive-valued, and \(\phi^{-1}(\frac{p}{p(t)}) \in X\) for all \(K \in \mathbb{R}\).
  \item[(f0)] \(f(\cdot, u, v, w)\) is Lebesgue measurable and \(X \ni h_- \leq f(\cdot, u, v, w) \leq h_+ \in X\) for all \(u, v, w \in L^1_{\text{loc}}(J)\).
  \item[(f1)] There exists a \(\lambda \geq 0\) such that \(f(\cdot, u_1, v_1, w_1) + \lambda w_1 \leq f(\cdot, u_2, v_2, w_2) + \lambda w_2\) whenever \(u_i, v_i, w_i \in L^1_{\text{loc}}(J), i = 1, 2, u_1 \leq u_2, v_1 \leq v_2\) and \(w_1 \leq w_2\).
\end{itemize}
(c0) $c_\pm \in \mathbb{R}$, and $c_- \leq c(u_1, v_1, w_1) \leq c(u_2, v_2, w_2) \leq c_+$ whenever $u_i, v_i, w_i \in L^1_{loc}(J)$, $i = 1, 2, u_1 \leq u_2, v_1 \leq v_2$ and $w_1 \leq w_2$.

(d0) $d_\pm \in \mathbb{R}$, and $d_- \leq d(u_1, v_1, w_1) \leq d(u_2, v_2, w_2) \leq d_+$ whenever $u_i, v_i, w_i \in L^1_{loc}(J)$, $i = 1, 2, u_1 \leq u_2, v_1 \leq v_2$ and $w_1 \leq w_2$.

Our first task is to convert the IVP (3.1) to a system of equations which do not contain derivatives.

Lemma 3.1. Assume that the hypotheses (φ) and (pφ) hold, and that $f(\cdot, u, v, w) \in X$ for all $u, v, w \in L^1_{loc}(J)$. Then $u$ is a solution of the IVP (3.1) in $Y$ if and only if $(u, v, w) = (u, u', w)$ for a.e. $t \in J$ in $L^1_{loc}(J) \times L^1_{loc}(J) \times L^1_{loc}(J)$.

Proof. Assume that $u$ is a solution of (3.1) in $Y$, and denote

$$w(t) = Lu(t) = \frac{d}{dt} p(t) \phi(u' (t))), \quad v(t) = u' (t), \quad t \in J. \quad (3.5)$$

The differential equation and the second initial condition of (3.1), the definition (3.2) of $Y$ and notations (3.5) ensure that first and third equations of (3.4) hold, and that

$$\int_r^s w(t) dt = \int_r^s \frac{d}{dt} p(t) \phi(v(t))) dt = p(s) \phi(v(s)) - p(r) \phi(v(r)), \quad a < r \leq s < b.$$

This result and the first initial condition of (3.1) imply that the second equation of (3.4) holds.

Conversely, let $(u, v, w)$ be a solution of the system (3.4) in $L^1_{loc}(J) \times L^1_{loc}(J) \times L^1_{loc}(J)$. The first equation of (3.4) implies that $v = u'$, that $u$ is locally absolutely continuous, and that the second initial condition of (3.1) holds. Since $v = u'$, it follows from the second equation of (3.4) that

$$p(t) \phi(u' (t)) = c(u, u', w) + \int_{a+}^t w(s) ds, \quad t \in J. \quad (3.6)$$

This equation implies that $p \cdot (\phi \circ u')$ is locally absolutely continuous and thus $u \in Y$. By differentiation we obtain from (3.6) that

$$w(t) = \frac{d}{dt} p(t) \phi(u' (t)) = Lu(t) \quad \text{for a.e. } t \in J. \quad (3.7)$$
This result and (3.6) imply that the first initial condition of (3.1) holds. The validity of the differential equation of (3.1) is a consequence of the third equation of (3.4), Eq. (3.7), and the fact that \( v = u' \).

Now we are ready to prove our main existence result for the IVP (3.1).

**Theorem 3.1.** Assume that the hypotheses \((\phi), (p\phi), (f0), (f1), (c0) and (d0)\) hold. Then the IVP (3.1) has least and greatest solutions in \( Y \), and they are increasing with respect to \( f, c \) and \( d \).

**Proof.** Assume that \( P = L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J) \) is ordered componentwise. The relations

\[
 x_\pm(t) := \left( d_\pm + \int_{a^+}^t \phi^{-1} \left( \frac{1}{p(s)} \left[ c_\pm + \int_{a^+}^s h_\pm(\tau) d\tau \right] \right) ds, \right.
\]

\[\left. \phi^{-1} \left( \frac{1}{p(t)} \left[ c_\pm + \int_{a^+}^t h_\pm(s) ds \right] \right), h_\pm(t) \right) \quad (3.8)\]

define functions \( x_\pm \in P \). If \((u, v, w) \in [x_-, x_+]\), then \( w \in [h_-, h_+] \), whence \( w \in X \).

Let \( W \) be a well-ordered chain in \( \text{ran} \, G \). The sets \( W_1 = \{u \mid (u, v, w) \in W\}, W_2 = \{v \mid (u, v, w) \in W\} \) and \( W_3 = \{w \mid (u, v, w) \in W\} \) are well-ordered and order-bounded chains in \( L^1_{\text{loc}}(J) \). It then follows from Lemma 2.3 that the suprema of \( W_1, W_2 \) and \( W_3 \) exist in \( L^1_{\text{loc}}(J) \). Obviously, \((\text{sup} W_1, \text{sup} W_2, \text{sup} W_3)\) is a supremum of \( W \) in \( P \). Similarly one can show that each inversely well-ordered chain of ran \( G \) has an infimum in \( P \).

The above proof shows that the operator \( G = (G_1, G_2, G_3) \) defined by (3.9) satisfies the hypotheses of Lemma 2.2, whence \( G \) has a least fixed point \( x_* = (u_*, v_*, w_*) \) and a greatest fixed point \( x^* = (u^*, v^*, w^*) \). It follows from (3.9) that \((u_*, v_*, w_*) \) and \((u^*, v^*, w^*) \) are solutions of the system (3.4). According to Lemma 3.1, \( u_* \) and \( u^* \) belong to \( Y \) and are solutions of the IVP (3.1).

To prove that \( u_* \) and \( u^* \) are least and greatest of all solutions of (3.1) in \( Y \), let \( u \in Y \) be a solution of (3.1). In view of Lemma 3.1, \((u, v, w) = (u, u', Lu)\) is a solution of the system (3.4). Applying the hypotheses \((f0), (c0) \) and \((d0)\) it is easy to show that \( x = (u, v, w) \in [x_-, x_+] \), where \( x_\pm \) are defined by (3.8). Thus \( x = (u, v, w) \) is a fixed point of \( G = (G_1, G_2, G_3) \) defined by (3.9). Because \( x_* = (u_*, v_*, w_*) \) and \( x^* = (u^*, v^*, w^*) \) are least and greatest fixed points of \( G \), then \((u_*, v_*, w_*) \leq (u, v, w) \leq (u^*, v^*, w^*) \). In particular, \( u_* \leq u \leq u^* \), whence \( u_* \) and \( u^* \) are least and greatest of all solutions of the IVP (3.1).
The last assertion is an easy consequence of the last conclusion of Lemma 2.2 and the definition (3.9) of $G = (G_1, G_2, G_3)$. □

If we replace the hypothesis (f0) by the following hypothesis:

\[(f2) \quad f(\cdot, u, v, w) \in X \quad \text{for all} \quad u, v, w \in L^1_{\text{loc}}(J), \quad \text{and there exist} \quad x_\pm = (u_\pm, v_\pm, w_\pm) \in P = L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J) \suchthat x_- \leq x_+ \text{ and } Gx_\pm \leq x_\pm, \text{where} \quad G = (G_1, G_2, G_3) \text{ is defined by (3.9)}, \]

we get the following result.

**Proposition 3.1.** Assume that the hypotheses \((\phi), (p\phi), (f1), (f2), (c0)\) and \((d0)\) hold. Then the IVP (3.1) has a least and a greatest solution in \(\{u \in Y \mid u_- \leq u \leq u_+\}\).

As a special case, we obtain an existence result for the IVP ,

\[
\frac{d}{dt}(p(t)\phi(u'(t))) = g(t, u(t), u'(t), \frac{d}{dt}(p(t)\phi(u'(t)))) \quad \text{for a.e.} \quad t \in J,
\]

\[
\lim_{t \to a^+} p(t)\phi(u'(t)) = c, \quad \lim_{t \to a^+} u(t) = d.
\]

**Corollary 3.1.** Let the hypotheses \((\phi)\) and \((p\phi)\) hold, and let \(g : J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) satisfy the following hypotheses:

\[(g0) \quad g(\cdot, u(\cdot), v(\cdot), w(\cdot)) \text{ is Lebesgue measurable and} \quad h_- \leq g(\cdot, u(\cdot), v(\cdot), w(\cdot)) \leq h_+ \text{ for all} \quad u, v, w \in L^1_{\text{loc}}(J) \text{ and for some} \quad h_\pm \in X.\]

\[(g1) \quad \text{There exists a} \quad \lambda \geq 0 \suchthat \quad g(t, x_1, x_2, x_3) + \lambda x_3 \leq g(t, y_1, y_2, y_3) + \lambda y_3 \text{ for a.e.} \quad t \in J \text{ and whenever} \quad x_1 \leq y_i \text{ in} \quad \mathbb{R}, \quad i = 1, 2, 3.\]

Then the IVP (3.10) has for each choice of \(c, d \in \mathbb{R}\) least and greatest solutions in \(Y\). Moreover, these solutions are increasing with respect to \(g, c\) and \(d\).

**Proof.** If \(c, d \in \mathbb{R}\), the IVP (3.10) is reduced to (3.1) when we define

\[
\begin{aligned}
&f(t, u, v, w) = g(t, u(t), v(t), w(t)), \quad t \in J, \quad u, v, w \in L^1_{\text{loc}}(J), \\
&c(u, v, w) \equiv c, \quad d(u, v, w) \equiv d, \quad u, v, w \in L^1_{\text{loc}}(J).
\end{aligned}
\]

The hypotheses \((g0)\) and \((g1)\) imply that \(f\) satisfies the hypotheses \((f0)\) and \((f1)\). The hypotheses \((c0)\) and \((d0)\) are also valid, whence (3.1), with \(f, c\) and \(d\) defined above, and hence also (3.10), has by Theorem 3.1 least and greatest solutions. The last assertion follows from the last assertion of Theorem 3.1. □

**Remarks 3.1.** If \(\lim_{t \to a^+} p(t) = 0\), the differential operator \(\frac{d}{dt}(p(t)\phi(u'(t)))\) in (3.1) is a singular phi-Laplacian operator. A special case of it is the \(p\)-Laplacian operator with

\[
\phi(x) = |x|^{p-2}x, \quad x \in (-\infty, \infty) \text{ and} \quad 1 < p < 2.
\quad (3.11)
\]

Another example of \(\phi\) is

\[
\phi(x) = \frac{x}{\sqrt{1 - x^2}}, \quad x \in (-1, 1).
\]
In this case the operator \( G \) given by (3.9) can be rewritten as

\[
\begin{align*}
G_1(u, v, w)(t) &= d(u, v, w) + \int_{a}^{b} v(s) \, ds, \quad t \in J, \\
G_2(u, v, w)(t) &= \frac{c(u, v, w) + \int_{a}^{b} w(s) \, ds}{\sqrt{p^2(t) + c(u, v, w) + \int_{a}^{b} w(s) \, ds}}, \quad t \in J, \\
G_3(u, v, w)(t) &= \frac{f(t, u, v, w) + \lambda w(t)}{1 + \lambda}, \quad t \in J.
\end{align*}
\] (3.12)

This formula shows that \(-1 \leq G_2(u, v, w)(t) \leq 1\) whenever \(G_2(u, v, w)\) is defined. Thus we have the following result.

**Corollary 3.2.** If \(-\infty < a < b \leq \infty\), the IVP

\[
\begin{align*}
Lu := \frac{d}{dt} \left( \frac{p(t)u''(t)}{\sqrt{1 - u'(t)^2}} \right) &= f(t, u, u', Lu) \quad \text{a.e. in } J = (a, b), \\
\lim_{t \to a+} \frac{p(t)u''(t)}{\sqrt{1 - u'(t)^2}} &= c(u, u', Lu), \quad u(a) = d(u, u', Lu),
\end{align*}
\] (3.13)

has least and greatest solutions if \(p \in L^1_{\text{loc}}(J)\) is positive-valued, if the hypotheses (f0), (f1) and (d0) hold, and if \(c(u_1, v_1, w_1) \leq c(u_2, v_2, w_2)\) in \(\mathbb{R}\) whenever \(u_1 \leq u_2, v_1 \leq v_2\) and \(w_1 \leq w_2\) in \(L^1_{\text{loc}}(J)\).

**Example 3.1.** Choose \(J = (0, \infty)\) and consider the IVP,

\[
\begin{align*}
Lu(t) := \frac{d}{dt} \left( \frac{p(t)u''(t)}{\sqrt{1 - u'(t)^2}} \right) &= h(t) + \frac{[K \tan h(q(t)) \int_{1}^{2} (u(s) + v(s) + w(s)) \, ds)}{K} \quad \text{a.e. in } J, \\
\lim_{t \to 0+} \frac{p(t)u''(t)}{\sqrt{1 - u'(t)^2}} &= c \cdot u'(1), \quad u(0) = \frac{[K \tan^{-1}(u(1)+u'(1))]}{2K},
\end{align*}
\] (3.14)

where \(p \in L^1_{\text{loc}}(J)\), \(p(t) > 0\) for \(t \in J, q \in L^1_{\text{loc}}(J)\), \(h \in L^1_{\text{loc}}(J)\), \(c \in \mathbb{R}, K, k > 0\) and \([z]\) denotes the greatest integer \(\leq z\). The problem (3.14) is of the form (3.13) with

\[
\begin{align*}
f(t, u, v, w) &= h(t) + \frac{[K \tan h(q(t)) \int_{1}^{2} (u(s) + v(s) + w(s)) \, ds)}{K}, \quad t \in J, \\
c(u, v, w) &= c \cdot v(1), \quad d(u, v, w) = \frac{[K \tan^{-1}(u(1)+v(1))]}{2K}.
\end{align*}
\] (3.15)

It is easy to see that the hypotheses of Corollary 3.2 hold, whence the IVP (3.14) has least and greatest solutions.

**Example 3.2.** The singular IVP

\[
\begin{align*}
Lu(t) := \frac{d}{dt} \left( \frac{tu''(t)}{\sqrt{1 - u'(t)^2}} \right) &= \frac{t}{10} - \frac{1}{10} + \frac{[100 \tan h(\frac{1}{10} u(t) + u'(t) + Lu(t)) \, dt]}{100} \quad \text{a.e. in } (0, \infty), \\
\lim_{t \to 0+} \frac{tu''(t)}{\sqrt{1 - u'(t)^2}} &= 0, \quad u(0) = \frac{[100 \tan^{-1}(u(1)+u'(1))]}{200}
\end{align*}
\] (3.16)
is a special case of (3.14) when \( p(t) = t, q(t) \equiv \frac{1}{2}, h(t) = \frac{\sqrt{t}}{10}, K = 100, k = 1000 \) and \( c = 0 \). Thus (3.16) has extremal solutions. The functions \( x_\pm \) defined by (3.8) with \( d_\pm = \pm 1, c_\pm = 0 \) and \( h_\pm (t) = \frac{\sqrt{t}}{10} \pm 1 \) can be calculated, and one obtains

\[
\begin{align*}
    x_-(t) &= \left( -1 - \frac{8}{2} \sqrt{21} + \frac{1}{2} \sqrt{25t^2 - 880t + 14414}, \frac{5r - 8}{\sqrt{25r^2 - 880r + 14414}}, \frac{r}{8} - \frac{11}{50} \right), \\
    x_+(t) &= \left( 1 - \frac{8}{2} \sqrt{181} + \frac{1}{2} \sqrt{25t^2 - 720t + 11584}, \frac{9 - 72}{\sqrt{25r^2 - 720r + 11584}}, \frac{r}{8} + \frac{9}{10} \right).
\end{align*}
\]

In this case the chains needed in the proof of Corollary 3.2 are reduced to ordinary iteration sequences \( (G^nx_\pm) \), where \( G = (G_1, G_2, G_3) \) is defined by (3.12). Calculating the iterations \( G^n x_- \), it turns out that \( G^n x_- = G^{10n} x_- \). Thus \( u_0 = G_1^n x_- \) is a least solution of (3.16). Similarly, one can show that \( u^* = G_1^n x_+ \) is a greatest solution of (3.16). The exact expressions of these solutions are

\[
\begin{align*}
    u_0(t) &= -\frac{698}{2009} - \frac{8}{5} \sqrt{109} + \frac{1}{2} \sqrt{25t^2 - 240t + 6976}, \\
    u^*(t) &= \frac{177}{1000} - \frac{4}{5} \sqrt{401} + \frac{1}{2} \sqrt{25t^2 + 40t + 6416}.
\end{align*}
\]

### 4. Existence results for second-order implicit boundary value problems

This section is devoted to the study of the implicit phi-Laplacian boundary value problem (BVP),

\[
\begin{align*}
    Lu(t) &:= -\frac{d}{dt} \left( p(t)\phi(u'(t)) \right) = f(t, u, u', Lu) \quad \text{for a.e. } t \in J := (a, b), \\
    \lim_{t \to a^+} p(t)\phi(u'(t)) &= \phi(u(a)), \quad \lim_{t \to b^-} u(t) = d(u, u', Lu),
\end{align*}
\]

where \(-\infty < a < b < \infty, p \in L^1_{\text{loc}}(J), \phi: I \to \mathbb{R}, f: J \times L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J) \to \mathbb{R}\) and \( c, d: L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J) \to \mathbb{R}\).

Denote

\[
Z := \left\{ h \in X \left| \int_a^b h(t) dt = \lim_{r \to b^-} \int_a^r h(t) dt \text{ is finite for some } r \in J \right. \right\},
\]

where \( X \) is defined by (3.3). Assuming that \( L^1_{\text{loc}}(J) \) and \( Z \) are ordered a.e. pointwise, we shall show that the BVP (4.1) has extremal solutions in the pointwise ordered set \( Y \) defined in (3.2) if the functions \( p, \phi, f, c \) and \( d \) satisfy the following hypotheses:

1. \((\phi)\) \( \phi \) is an increasing homeomorphism from an open interval \( I \) of \( \mathbb{R} \) onto \( \mathbb{R} \).
2. \((\phi p)\) \( p \) is a.e. positive-valued, and \( \int_a^b |p^{-1}(\frac{K}{p(t)})| ds | < \infty \) for all \( t \in J \) and \( K \in \mathbb{R} \).
3. \((f_0)\) \( f(\cdot, u, v, w) \) is Lebesgue measurable and \( Z \ni h_- \leq f(\cdot, u, v, w) \leq h_+ \in Z \) for all \( u, v, w \in L^1_{\text{loc}}(J) \).
4. \((f_1)\) There exists a \( \lambda \geq 0 \) such that \( f(\cdot, u_1, v_1, w_1) + \lambda w_1 \leq f(\cdot, u_2, v_2, w_2) + \lambda w_2 \) whenever \( u_i, v_i, w_i \in L^1_{\text{loc}}(J), i = 1, 2, u_1 \leq u_2, v_1 \geq v_2 \) and \( w_1 \leq w_2 \).
5. \((c_1)\) \( c_+ \in \mathbb{R} \) and \( c_- \leq c(u_2, v_2, w_2) \leq c(u_1, v_1, w_1) \leq c_+ \) whenever \( u_i, v_i, w_i \in L^1_{\text{loc}}(J), i = 1, 2, u_1 \leq u_2, v_1 \geq v_2 \) and \( w_1 \leq w_2 \).
\( d_+ \in \mathbb{R}, \) and \( d_- \leq d(u_1, v_1, w_1) \leq d(u_2, v_2, w_2) \leq d_+ \) whenever \( u_i, v_i, w_i \in L^1_{\text{loc}}(J), i = 1, 2, u_1 \leq u_2, v_1 \geq v_2 \) and \( w_1 \leq w_2. \)

The method is the same as in Section 3, that is, we shall first convert the BVP (4.1) to a system of three equations, and then apply the fixed point result of Lemma 2.3.

**Lemma 4.1.** Assume that the hypotheses \((\phi)\) and \((p\phi)\) hold, and that \(f(\cdot, u, v, w) \in Z\) for all \(u, v, w \in L^1_{\text{loc}}(J)\). Then \(u\) is a solution of the IVP (4.1) in \(Y\), defined by (3.2) if and only if \((u, u', Lu) = (u, v, w)\), where \((u, v, w)\) is a solution of the system

\[
\begin{align*}
  u(t) &= d(u, v, w) - \int_t^b v(s) \, ds, \quad t \in J, \\
  v(t) &= \phi^{-1}\left(\frac{1}{p(t)}\left[c(u, v, w) - \int_t^a w(s) \, ds\right]\right), \quad t \in J, \\
  w(t) &= f(t, u, v, w) \quad \text{for a.e.} \ t \in J,
\end{align*}
\]

in \(L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J)\).

**Proof.** Assume that \(u\) is a solution of (4.1) in \(Y\), and denote

\[
w(t) = Lu(t) = -\frac{d}{dt}\left[p(t)\phi(u'(t))\right], \quad v(t) = u'(t), \quad t \in J.
\]

The differential equation and the second initial condition of (4.1), the definition (3.2) of \(Y\) and notations (4.4) ensure that first and third equations of (4.3) hold, and that

\[
\int_a^r w(t) \, dt = -\int_a^r \frac{d}{dt}\left[p(t)\phi(v(t))\right] \, dt = p(r)\phi(v(r)) - p(s)\phi(v(s)),
\]

\(a < r \leq s < b\).

This result and the first initial condition of (4.1) imply that the second equation of (4.3) holds.

Conversely, let \((u, v, w)\) be a solution of the system (4.3) in \(L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J)\). The first equation of (4.3) implies that \(v = u'\), that \(u\) is locally absolutely continuous, and that the second initial condition of (4.1) holds. Since \(v = u'\), it follows from the second equation of (4.3) that

\[
p(t)\phi(u'(t)) = c(u, u', w) - \int_a^t w(s) \, ds, \quad t \in J.
\]

This equation implies that \(p \cdot (\phi \circ u')\) is locally absolutely continuous, and thus \(u \in Y\). It follows from (4.5) by differentiation that

\[
w(t) = -\frac{d}{dt}\left[p(t)\phi(u'(t))\right] = Lu(t) \quad \text{for a.e.} \ t \in J.
\]

This result and (4.5) imply that the first initial condition of (4.1) holds. The validity of the differential equation of (4.1) is a consequence of the third equation of (4.3), Eq. (4.6), and the fact that \(v = u'\). □
The main existence result for the BVP (4.1) reads as follows.

**Theorem 4.1.** Assume that the hypotheses \((\phi), (\phi p), (f_0), (f_1), (c_1)\) and \((d_1)\) hold. Then the BVP (4.1) has least and greatest solutions in \(Y\), and they are increasing with respect to \(f\) and \(d\) and decreasing with respect to \(c\).

**Proof.** Assume that \(P = L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J) \times L^1_{\text{loc}}(J)\) is ordered by

\[(u_1, v_1, w_1) \preceq (u_2, v_2, w_2) \quad \text{if and only if} \quad u_1 \leq u_2, \quad v_1 \geq v_2 \quad \text{and} \quad w_1 \leq w_2.
\]

The relations

\[
x_-(t) := \left( d_+ - \int_t^b \phi^{-1} \left( \frac{1}{p(t)} \left[ c_+ - \int_{a+}^t h_-(s) d\tau \right] \right) ds, \right.
\]

\[
\left. \phi^{-1} \left( \frac{1}{p(t)} \left[ c_+ - \int_{a+}^t h_-(s) d\tau \right] \right), \quad h_-(t) \right),
\]

\[
x_+(t) := \left( d_+ - \int_t^b \phi^{-1} \left( \frac{1}{p(t)} \left[ c_- - \int_{a+}^t h_+(s) d\tau \right] \right) ds, \right.
\]

\[
\left. \phi^{-1} \left( \frac{1}{p(t)} \left[ c_- - \int_{a+}^t h_+(s) d\tau \right] \right), \quad h_+(t) \right),
\]

define functions \(x_+ \in P, \) and \(x_- \in P, \) Moreover, it is easy to show, by applying the given hypotheses, that the relations

\[
G_1(u, v, w)(t) := d(u, v, w) - \int_t^b v(s) ds, \quad t \in J,
\]

\[
G_2(u, v, w)(t) := \phi^{-1} \left( \frac{1}{p(t)} \left[ c(u, v, w) - \int_t^b w(s) ds \right] \right), \quad t \in J,
\]

\[
G_3(u, v, w)(t) := \frac{f(t, u, v, w) + \lambda w(t)}{1 + \lambda}, \quad t \in J,
\]

define an increasing mapping \(G = (G_1, G_2, G_3): [x_-, x_+] \to [x_-, x_+].\)

Let \(W\) be a well-ordered chain in \(\text{ran} \, G, \) The sets \(W_1 = \{u \mid (u, v, w) \in W\}\) and \(W_2 = \{v \mid (u, v, w) \in W\}\) are well-ordered, \(W_2 = \{v \mid (u, v, w) \in W\}\) is inverse well-ordered, and all three are order-bounded in \(L^1_{\text{loc}}(J)\). It then follows from Lemma 2.3 that the supremums of \(W_1\) and \(W_3\) and an infimum of \(W_2\) exist in \(L^1_{\text{loc}}(J)\). Obviously, \((\sup W_1, \inf W_2, \sup W_3)\) is a supremum of \(W\) in \((P, \preceq)\). Similarly one can show that each inversely well-ordered chain of \(\text{ran} \, G\) has an infimum in \((P, \preceq)\).

The above proof shows that the operator \(G = (G_1, G_2, G_3)\) defined by (4.9) satisfies the hypotheses of Lemma 2.2, whence \(G\) has a least fixed point \(x_* = (u_*, v_*, w_*)\) and a greatest fixed point \(x^* = (u^*, v^*, w^*)\). It follows from (4.9) that \((u_*, v_*, w_*)\) and \((u^*, v^*, w^*)\) are solutions of the system (4.3). According to Lemma 4.1 \(u_*\) and \(u^*\) belong to \(Y\) and are solutions of the IVP (4.1).

To prove that \(u_*\) and \(u^*\) are least and greatest of all solutions of (4.1) in \(Y\), let \(u \in Y\) be a solution of (4.1). In view of Lemma 4.1, \((u, v, w) = (u, u', Lu)\) is a solution of the system (4.3). Applying the hypotheses \((f_1), (c_1)\) and \((d_1)\) it is easy to show that \(x = (u, v, w) \in [x_-, x_+]\), where \(x_\pm\) are defined by (4.8). Thus \(x = (u, v, w)\) is a fixed point of \(G = (G_1, G_2, G_3): [x_-, x_+] \to [x_-, x_+]\), defined by (4.9). Because \(x_* = (u_*, v_*, w_*)\) and \(x^* = (u^*, v^*, w^*)\) are least and greatest fixed points of \(G\), then \((u_*, v_*, w_*) \preceq (u, v, w) \preceq (u^*, v^*, w^*)\). In particular, \(u_* \leq u \leq u^*\), whence \(u_*\) and \(u^*\) are least and greatest of all solutions of the IVP (4.1).
The last assertion is an easy consequence of the last conclusion of Lemma 2.2 and the definition (4.9) of $G = (G_1, G_2, G_3)$.

If we replace the hypothesis $(f_0)$ by the following hypothesis:

$$(f_2) \quad f(\cdot, u, v, w) \in Z$$

for all $u, v, w \in P = L^1_{loc}(J) \times L^1_{loc}(J) \times L^1_{loc}(J)$, and there exist $x \pm = (u \pm, v \pm, w \pm) \in P$ such that $x_- \approx x_+, x_- \approx Gx_-$ and $Gx_+ \approx x_+$, where $G = (G_1, G_2, G_3)$ is defined by (4.9),

we get the following result.

**Proposition 4.1.** Assume that the hypotheses $(\phi)$, $(\phi p)$, $(f_2)$, $(c_1)$ and $(d_1)$ hold. Then the BVP (4.1) has a least and a greatest solution in \( \{ u \in Y | u_- \leq u \leq u_+ \} \).

As a special case, we obtain an existence result for the BVP,

$$(-d_t (p(t)\phi(u'(t)))) = g(t, u(t), v(t), w(t)), \quad \text{for a.e.} \quad t \in J,$$

(4.10)

**Corollary 4.1.** Let the hypotheses $(\phi)$ and $(\phi p)$ hold, and let $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the following hypotheses:

$$(g_0) \quad g(\cdot, u(\cdot), v(\cdot), w(\cdot)) \text{ is Lebesgue measurable and } h_- \leq g(\cdot, u(\cdot), v(\cdot), w(\cdot)) \leq h_+ \text{ for all } u, v, w \in L^1_{loc}(J) \text{ and for some } h_{\pm} \in \mathbb{Z}.$$

$$(g_1) \quad \text{There exists a } \lambda \geq 0 \text{ such that } g(t, x_1, y_1, z_1) + \lambda z_1 \leq g(t, x_2, y_2, z_2) + \lambda z_2 \text{ for a.e. } t \in J \text{ and whenever } x_1 \leq x_2, y_1 \geq y_2 \text{ and } z_1 \leq z_2 \text{ in } \mathbb{R}.$$

Then the BVP (4.10) has for each choice of $c, d \in \mathbb{R}$ least and greatest solutions in $Y$. Moreover, these solutions are increasing with $g$ and $d$ and decreasing with respect to $c$.

**Proof.** If $c, d \in \mathbb{R}$, the BVP (4.10) is reduced to (4.1) when we define

$$f(t, u, v, w) = g(t, u(t), v(t), w(t)), \quad \text{for a.e. } t \in J, \quad u, v, w \in L^1_{loc}(J),$$

$c(u, v, w) \equiv c, \quad d(u, v, w) \equiv d, \quad u, v, w \in L^1_{loc}(J)$.

(4.11)

The hypotheses $(g_0)$ and $(g_1)$ imply that $f$ satisfies the hypotheses $(f_0)$ and $(f_1)$. The hypotheses $(c_1)$ and $(d_1)$ is also valid, whence (4.1), with $f$, $c$ and $d$ defined by (4.11), and hence also (4.10), has by Theorem 4.1 least and greatest solutions. The last assertion follows from the last assertion of Theorem 4.1.

In the case when $\phi$ is defined by

$$\phi(x) = \frac{x}{\sqrt{1 - x^2}}, \quad x \in (-1, 1),$$
the operator $G$ given by (4.9) can be rewritten as
\[
\begin{align*}
G_1(u, v, w)(t) &= d(u, v, w) - f^b_1 v(s) \, ds, \quad t \in J, \\
G_2(u, v, w)(t) &= \frac{c(u,v,w)-f^b_1 w(s) \, ds}{\sqrt{p^2(t)+(c(u,v,w)-f^b_1 w(s))d^2}}, \quad t \in J, \\
G_3(u, v, w)(t) &= \frac{f(t,u,v,w)+\lambda w(t)}{1+\lambda}, \quad t \in J.
\end{align*}
\] (4.12)

This formula shows that $-G$ the operator $G$.

Remark 4.1. If $c$ has least and greatest solutions if $p \in L^1_{\text{loc}}(J)$ is positive-valued, if the hypotheses $(f_0)$, $(f_1)$ and $(d_1)$ hold, and if $c(u_2, v_2, w_2) \leq c(u_1, v_1, w_1)$ in $R$ whenever $u_1 \leq u_2$ and $v_1 \geq v_2$ and $w_1 \leq w_2$ in $L^1_{\text{loc}}(J)$.

Corollary 4.2. If $-\infty \leq a < b < \infty$, the IVP
\[
\begin{align*}
Lu(t) &:= -\frac{d}{dt} \left( \frac{p(t)u'(t)}{1-u'(t)^2} \right) = f(t, u, u', Lu) \quad \text{a.e. in } J = (a, b), \\
\lim_{t \to a+} p(t)u'(t) &\to c(u', u', Lu), \quad (b) = d(u, u', Lu),
\end{align*}
\] (4.13)

has least and greatest solutions if $p \in L^1_{\text{loc}}(J)$ is positive-valued, if the hypotheses $(f_0)$, $(f_1)$ and $(d_1)$ hold, and if $c(u_2, v_2, w_2) \leq c(u_1, v_1, w_1)$ in $R$ whenever $u_1 \leq u_2$ and $v_1 \geq v_2$ and $w_1 \leq w_2$ in $L^1_{\text{loc}}(J)$.

Example 4.1. Determine least and greatest solutions of the BVP,
\[
\begin{align*}
-\frac{d}{dt} \left( \frac{u'(t)}{1-u'(t)^2} \right) &= \frac{2r-1}{8} + \frac{[100\tan^{-1}(f^b_1 u(t)-u'(t)+Lu(s))]_{0}}{200} \quad \text{a.e. in } (0, 3), \\
\lim_{t \to 0+} \frac{tu'(t)}{1-u'(t)^2} &= 0, \quad (3) = \frac{[100\tanh(u(1)-u'(1)+Lu(1))]_{0}}{1000}.
\end{align*}
\] (4.14)

Solution. (4.14) is a special case of (4.13) when $p(t) = t$. It is also easy to see that the hypotheses of Corollary 4.2 are satisfied. Thus (4.14) has extremal solutions. The functions $x_{\pm}$ defined by (4.8) can be calculated, and one obtains
\[
\begin{align*}
x_-(t) &= \left( 9 - \sqrt{t^2 - 18t + 145} \right), \\
x_+(t) &= \left( 1 + 2\sqrt{4t - \sqrt{t^2 + 14t + 113}} \right).
\end{align*}
\]

In this case the chains needed in the proof of Corollary 4.2 are reduced to ordinary iteration sequences $(G^n x_{\pm})$, where $G$ is defined by (4.9). Calculating the iterations $G^n x_{\pm}$, it turns out that $G^5 x_- = G^5 x_-$. Thus $u_{\pm} = G^5 x_{\pm}$ is a least solution of (4.14). Similarly, one can show that $u^* = G^5 x_{\pm}$ is a greatest solution of (4.14). The exact expressions of these solutions are
\[
\begin{align*}
u_{\pm}(t) &= -\frac{57}{1005} + \frac{1}{\sqrt{2}} \sqrt{41369} - \frac{1}{4} \sqrt{625t^2 - 5600t + 52544}, \\
u^*(t) &= \frac{247}{1005} + \frac{1}{\sqrt{2}} \sqrt{68561} - \frac{1}{4} \sqrt{625t^2 + 4700t + 48836}.
\end{align*}
\]
Remarks 4.3. Problems of the form (2.1), (3.1) and (4.1) include many kinds of special types. For instance, they can be

- implicit because both differential equations and initial/boundary conditions depend on the differential operator $Lu$;
- singular, because a case $\lim_{t \to a} p(t) = 0$ is allowed;
- functional, because the functions $c$, $d$ and $f$ may depend functionally on $u$, $Lu$ and/or $u'$;
- discontinuous, since the dependencies of $c$, $d$ and $f$ on $u$, $u'$ and $Lu$ can be discontinuous;
- problems on unbounded intervals, because cases $a = -\infty$ and/or $b = \infty$ are included;
- $p$-Laplacian when $\phi$ is defined by (3.11).

Explicit problems, i.e., cases where neither differential equations nor initial/boundary conditions depend on the differential operator $Lu$, are considered in [8]. As for uniqueness results for phi-Laplacian initial and boundary value problems see, e.g., [4,7]. In [1–3,9,10] existence results are introduced for these problems.

References

[10] D. O’Regan, Existence theory for $(\phi(y'))' = af(t, y, y')$, $0 \leq t \leq 1$, Comm. Appl. Anal. 1 (1997) 35–52.