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On Scale Embeddings of Graphs into Hypercubes

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We investigate graphs that are isometrically embeddable into the metric space l_1.

1. Introduction

All graphs considered in this paper are finite, undirected, without loops or multiple edges. For graphs Γ and Δ a mapping $\varphi: V(\Gamma) \to V(\Delta)$ is called *a scale embedding* if there is an integer λ such that for any $x, y \in V(\Gamma)$ one has

$$d_\Delta(\varphi(x), \varphi(y)) = \lambda d_\Gamma(x, y),$$

where d_Δ and d_Γ denote the usual path distances in Δ and Γ, respectively. If we want to specify the value of λ exactly, we speak about embeddings with scale λ, or λ-embeddings; 1-embeddings are also called isometric embeddings. In what follows we investigate scale embeddings of graphs into hypercubes; i.e. Δ is supposed to be an n-cube for some dimension n, throughout. The paper was inspired by the following question.

Question. Is it true that a graph which is embeddable into a hypercube with an odd scale is also 1-embeddable into a hypercube?

In what follows we give the affirmative answer to this question, but also develop a kind of theory which covers both the cases of odd and even scale. The main result in this theory is Theorem 2 below, but first we formulate the answer to the above question.

Theorem 1. *If Γ is embeddable into an n-cube with an odd scale λ, then Γ is 1-embeddable into a k-cube, where $k = [n/\lambda]$.*

Theorem 2. *Whenever Γ is a graph, scale embeddable into a hypercube, there is another graph $\hat{\Gamma}$ and a 1-embedding φ of Γ into $\hat{\Gamma}$, such that:*
(1) $\hat{\Gamma} \simeq \hat{\Gamma}_1 \times \cdots \times \hat{\Gamma}_r$, where each $\hat{\Gamma}_i$ is isomorphic either to a complete graph, or to a cocktail party graph, or to a halved cube;
(2) for any scale embedding ψ of Γ into a hypercube, there is a scale embedding $\hat{\psi}$ of $\hat{\Gamma}$ into the same hypercube, such that $\psi = \hat{\psi}\varphi$.

Remark. In the assumption part of Theorem 2, Γ has at least one scale embedding for *at least one* scale. In the part (2) of the conclusion, ψ is any scale embedding of Γ for *any* scale.

The relation between Theorems 1 and 2 is as follows. A hypercube is itself a direct product of a number of complete graphs, each having two vertices. Only such factors can appear in Theorem 2 when the scale is odd.

In [1] it was proved that a graph can be isometrically embedded into the metric space l_1 (such graphs are called l_1-graphs) iff it is scale embeddable into a hypercube. So one can say that in this paper we are studying l_1-graphs. Since this language is probably more customary among specialists, we will use it for formulate the consequences of Theorem 2.

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COROLLARY 1. *A graph is an l_1-graph iff it is an isometric subgraph of a direct product of cocktail party graphs and halved cubes.*

COROLLARY 2. *The property of being l_1-graph is recognizable in polynomial time.*

An l_1-graph is called l_1-rigid if up to a natural equivalence it has only one isometric embedding into a hypercube (see [3]). An application of Theorem 2 gives the following criterion of l_1-rigidity.

COROLLARY 3. *An l_1-graph Γ is l_1-rigid iff $\hat{\Gamma}$ is l_1-rigid.*

Since $\hat{\Gamma}$ is a direct product graph, it is l_1-rigid iff all its factors are l_1-rigid. It is equivalent to the condition that no factor is isomorphic to a complete graph with more than 3 vertices, or to a cocktail party graph with more then 6 vertices (see [3]). In particular, we have the following:

COROLLARY 4. *Every l_1-rigid graph is an isometric subgraph of a hypercube, or a halved cube.* □

For an arbitrary l_1-graph Γ, the minimal scale λ, such that Γ is λ-embeddable into a hypercube, can also be bounded.

COROLLARY 5. *If Γ is an l_1-graph with $v \geqslant 4$ vertices then there exists $\lambda < v - 1$, such that Γ is λ-embeddable into a hypercube.*

All the proofs given in this paper are elementary and rely only on very basic definitions. It must be mentioned that there is a different method (see [2]) based on the consideration of the L-polytope of the lattice defined by the embedding of the graph into a euclidean space. This approach is applicable to a wider class of distance spaces. In the latest version of [2] there is a result (Lemma 3.1) showing that the L-polytope graph of a graph has properties similar to those of $\hat{\Gamma}$. Also, our Corollary 1 appears in [2] (as Theorem 3.6).

Another paper to be mentioned here is [5]. Since in Theorem 2 we embed Γ in the direct product $\hat{\Gamma}$ as an isometric subgraph, the irredundant (for definitions, see [5]) part of $\hat{\Gamma}$ must be an isometric subgraph of the canonical direct product Γ^* constructed in [5]. Applying Theorem 2 we obtain that that isometric embedding is actually an isomorphism (and Γ^* coincides with our intermediary direct product $\bar{\Gamma}$). We formulate this as follows:

COROLLARY 6. *Let $\alpha: \Gamma \to \Gamma^*$ be the canonical embedding constructed in [5]. Then for every scale embedding ψ of Γ into a hypercube Δ there is a scale embedding ψ^* of Γ^* into Δ such that $\psi = \psi^* \alpha$.*

In the whole paper we are actually proving only Theorem 2, while Theorem 1 is obtained as a by-product quite early in the proof. The proof of Theorem 2 proceeds in two steps, which is just a trick to ease the understanding of what is happening. First, in Section 3, we consider the case in which Γ is already embedded in a hypercube, and construct $\hat{\Gamma}$ in there. In Section 4 we follow the construction from Section 3 and demonstrate that it does not actually depend on the concrete embedding, so establishing Theorem 2. All relevant definitions and preliminary results are given in Section 2.

2. Definitions and Preliminary Results

The definition of a scale embedding was given in the introduction. Let us now recall the definitions of the particular graphs mentioned there. The *cocktail party graph* $K_{k\times 2}$ is a complete multipartite graph with k parts, each of cardinality 2. It is directly by definition that for each vertex in such a graph there is exactly one another vertex not adjacent to the first one. In what follows these two vertices are called *opposite* to each other.

The *hypercube* associated with dimension n is also known as the Hamming graph $H(n, 2)$ of words of length n in the alphabet of two letters. Another definition, given below, is equivalent, but better fits the arguments in this paper. Let Δ be a set of n elements. Then the vertices of the hypercube are all the subsets of Δ. Two subsets are adjacent whenever their symmetric difference has cardinality 1. It easily follows that the distance in the hypercube between any two subsets equals the cardinality of their symmetric difference. The hypercube is bipartite, and the *halved cube* is just the graph defined on one of the parts, where the adjacency relation is given by being distance 2 apart in the hypercube. It is a little bit more convenient to choose the part containing the subsets of even cardinality, so we do that. If the dimension n of the hypercube is 2 or 3, then the corresponding halved cube is a complete graph with 2 and 4 vertices. If $n = 4$ then it is the cocktail party $K_{4\times 2}$. *In what follows we use the name 'halved cube' only for the case $n \geqslant 5$.* By definition the halved cube is 2-embedded into the corresponding hypercube.

To simplify the notation the same letter Δ will be used to denote both the basic set of n elements and the whole hypercube. Accordingly, we will use upper case letters to denote vertices of a hypercube and write $X \subseteq \Delta$ to abbreviate the statement that X is such a vertex. All other graphs are considered sometimes abstractly, simply as graphs, but sometimes as being embedded into a hypercube. This may concern even a particular graph, and we will use the same convention to distinguish the two possible contexts. If a graph is embedded in a hypercube, but only its own properties are concerned, then the lower case letters serve to denote the vertices. On the other hand, if we want to emphasize the fact that the vertices are actually subsets, then the respective upper case letters are used instead.

Now all the graphs are defined and this is the right moment for several observations concerning scale embeddings. In Lemmas 2.1–2.5 it is assumed that $x \mapsto X$ is a scale embedding of a graph Γ into a hypercube Δ. Then by definition

$$|X\Delta Y| = \lambda d(x, y),$$

where x and y are any two vertices of Γ, λ is the scale and d is the path distance in Γ.

Lemma 2.1. *For any $S \subseteq \Delta$, the mapping $x \mapsto X\Delta S$ also is a scale embedding of Γ, having the same scale λ.* □

Two embeddings that can be obtained from one another, as in Lemma 2.1, are called *equivalent*. We will consider embeddings up to this equivalence and, in particular, it is always assumed that *$\varnothing$ is in the image of embedding*. Let v throughout denote the vertex such that $V = \varnothing$.

Lemma 2.2. *Let λ be the scale of the embedding $x \mapsto X$. Then*:
(1) *For any $x \in V(\Gamma)$ one has that $|X| = \lambda d(x, v)$ is an integer multiple of λ.*
(2) *For any $x, y \in V(\Gamma)$ one has that $|X \cap Y| = \lambda/2[d(x, v) + d(y, v) - d(x, y)]$ is an integer multiple of $\lambda/2$.* □

The following definition gives us something very similar to what is called a root in [2] and [4]. Let x, y be adjacent in Γ. Then the *atom* defined by the edge $\{x, y\}$ is by definition the set $X \Delta Y$. Clearly, an atom consists of exactly λ elements, and also different edges may define the same atom. The following lemma shows how an atom may appear.

LEMMA 2.3. *For adjacent $x, y \in V(\Gamma)$ either $X \subset Y$, or $Y \subset X$, or $|X \backslash Y| = |Y \backslash X| = \lambda/2$.*

PROOF. This follows from Lemma 2.2, since both $|X \backslash Y|$ and $|Y \backslash X|$ are integer multiples of $\lambda/2$, while $X \Delta Y = (X \backslash Y) \cup (Y \backslash X)$ has cardinality λ. □

An atom is called *proper* if it is defined by an edge $\{x, y\}$ such that $X \subset Y$ or $Y \subset X$. In such a case we will always assume that $X \subset Y$, i.e. the corresponding proper atom can be defined also as $Y \backslash X$. Some more observations follow.

LEMMA 2.4. *If $v = x_0, x_1, \ldots, x_s = x$ is a shortest path from v to x, then X is the disjoint union of the proper atoms $A_i = X_i \backslash X_{i-1}$, $i = 1, \ldots, s$.*

PROOF. The proof follows by definition, since $V = \emptyset$. □

LEMMA 2.5. *If A is a proper atom and B is either a vertex, or another proper atom, then $|A \cap B| = 0$, $\lambda/2$ or λ.*

PROOF. By definition $A = Y \backslash X$, for two adjacent vertices $x, y \in V(\Gamma)$ such that $d(x, v) + 1 = d(y, v)$. If now B is the image of a vertex b then $|A \cap B| = |Y \cap B| - |X \cup B|$ is an integer multiple of $\lambda/2$ by Lemma 2.2. The case in which B is a proper atom is quite similar. □

At this point we already can prove our Theorem 1.

PROOF OF THEOREM 1. If λ is odd then all atoms are proper and, by Lemma 2.5, no two different atoms have a non-trivial intersection. This, together with Lemma 2.4, means that for any $x \in V(\Gamma)$ the subset X can be uniquely represented as a (disjoint) union of atoms. Clearly, it gives us a 1-embedding of Γ into the hypercube defined by the set of atoms, the cardinality of which is at most $[n/\lambda]$, n the cardinality of the set Δ. □

If $\Gamma_1, \ldots, \Gamma_s$ are graphs then the *direct product* $\Gamma_1 \times \cdots \times \Gamma_s$ is the graph having $V(\Gamma_1) \times \cdots \times V(\Gamma_s)$ as its vertex set, two vertices $(x_1, \ldots, x_s)$ and $(y_1, \ldots, y_s)$ adjacent iff there is an index i such that $x_j = y_j$, for $j \neq i$, and x_i is adjacent to y_i.

LEMMA 2.6. *Let $\Gamma_1, \ldots, \Gamma_s$ be graphs embedded with the same scale into a hypercube, each into an independent part of its basic set. Then the mapping*

$$(x_1, \ldots, x_s) \mapsto X_1 \cup \cdots \cup X_s$$

gives a natural scale embedding of $\Gamma_1 \times \cdots \times \Gamma_s$ into the hypercube. □

The following lemma is a converse of Lemma 2.6. Observe that the factor Γ_i can be naturally identified with the subgraph in the direct product, induced by the vertices $(x_1, \ldots, x_s)$, where $x_j = y_j$, for $j \neq i$, and x_i is arbitrary. Here $(y_1, \ldots, y_s)$ is any fixed vertex. In what follows we take as such the vertex v.

LEMMA 2.7. *Let $\Gamma_1 \times \cdots \times \Gamma_s$ be scale embedded into a hypercube. Suppose that the factors Γ_i's are chosen, passing through v with $V = \varnothing$. Then each Γ_i is embedded into an independent subset of the basic set.*

PROOF. It suffices to consider the case $s = 2$. Let x be any vertex of Γ_1 and y be any vertex of Γ_2. Since v is the common vertex of Γ_1 and Γ_2, we have $d(x, y) = d(x, v) + d(y, v)$. Now, by Lemma 2.2, one has $|X \cap Y| = \lambda/2[d(x, v) + d(y, v) - d(x, y)] = 0$. □

Now several lemmas follow about scale embeddings of the particular graphs mentioned above.

LEMMA 2.8. *If a graph Γ, isomorphic to a cocktail party graph, is embedded with a scale λ into a hypercube Δ then the vertices of Γ cover all together exactly 2λ elements of the set Δ. Moreover, if any graph Γ has a scale embedding with this property, then it is isomorphic to a subgraph of a cocktail party graph.*

PROOF. Recall that we assume that there is a vertex $v \in V(\Gamma)$ with $V = \varnothing$. Let x, y be a pair of opposite vertices of Γ, such that $x, y \neq v$. Clearly, $|X \cup Y| = 2\lambda$. If z is any other vertex then $|Z \cap X| = |Z \cup Y| = \lambda/2$, since $d(z, x) = d(z, y) = 1$. Hence $Z \subseteq X \cup Y$.

The reverse statement is clear, since one can add to Γ all the complements of vertices in that 2λ-element set. □

The following lemma claims that all scale embeddings of a halved cube can be produced from its natural 2-embedding. Recall that the halved cube graph was defined as a certain 2-embedded subgraph of a hypercube. In particular, each vertex x of the halved cube has its original cardinality, which we will refer to as card(x). In the following lemma we also suppose that card(v) = 0. Clearly, we may suppose this without loss of generality.

LEMMA 2.9. *If Γ is a halved cube and $x \mapsto X$ is a λ-embedding of Γ into a hypercube Δ, then there is a family of pairwise disjoint subsets of Δ, each or cardinality $\lambda/2$, such that X is a union of exactly* card(x) *of these subsets for any $x \in V(\Gamma)$. If the vertices of Γ cover the whole of Δ then such a family is unique.*

PROOF. Recall that by definition vertices of Γ are all even subsets of a certain set $\{1, \ldots, k\}$, $k \geqslant 5$. Two subsets are adjacent whenever their symmetric difference has cardinality 2. Since by assumption $v = \varnothing \in V(\Gamma)$ is mapped to $V = \varnothing \in V(\Delta)$, it follows that $|X| = \lambda.\text{card}(x)/2$ for any $x \in V(\Gamma)$.

For $i = 2, \ldots, k$ let $x_i = \{i-1, i\} \in V(\Gamma)$. Clearly, all these vertices are adjacent to v, and each x_i is adjacent to x_{i-1} and x_{i+1}, and non-adjacent to all other x_j's. For $i = 3, \ldots, k$ let $P_i = X_i \backslash X_{i-1}$. Let $P_2 = X_3 \cap X_2$ and $P_1 = X_2 \backslash X_3$. Clearly, P_i's are pairwise disjoint subsets of cardinality $\lambda/2$.

Now let $x = \{s, t\}$ be any other 2-element subset. If x is adjacent to x_i and non-adjacent to x_{i+1} then, clearly, $X \supset P_{i-1}$. Likewise, if x is adjacent to x_i and non-adjacent to x_{i-1}, then $X \supset P_i$. Using these two tricks and the condition $k \geqslant 5$, it is easy to prove in each case that $X = P_s \cup P_t$. Finally, if $x = \{s, t, r, \ldots, f\}$ is any vertex of Γ then, for $y = \{s, t\}$ and $z = \{r, \ldots, f\}$, it follows from pairwise distances that Y and Z are disjoint, and $X = Y \cup Z$. Hence each X is a union of an even number of P_i's. The uniqueness, which is claimed in the lemma, follows from the observation that every P_i must belong to any such family. □

REMARK. It was pointed out by one of the referees that this lemma, claiming in fact the l_1-rigidity of the halved cube, has already appeared in [3].

The final statement of the section follows:

LEMMA 2.10. *Let Γ be λ-embedded into a hypercube Δ, and Ω be a subset of Δ. Let Γ' be the subgraph of Γ induced by all the vertices x with $X \subseteq \Omega$. Then:*
(1) if Γ is isomorphic to a complete graph or a cocktail party graph, then so is Γ';
(2) if Γ is isomorphic to a halved cube, then Γ' is either a halved cube, or a complete graph K_s, $s = 1, 2, 4$, or a cocktail party graph $K_{4\times 2}$.
In either case, Γ' is scale embedded into Δ with the same scale λ.

PROOF. The proof easily follows from Lemmas 2.8 and 2.9. □

REMARK. The strange formulation of claim (2) of Lemma 2.10 is due to our decision to consider the halved cubes in small dimensions as 'wrong' ones.

3. THE ATOM GRAPH

In this section we consider only subgraphs embedded with scale λ in a fixed hypercube Δ. Such a graph Γ can be identified with its image; namely, with a subset of the vertex set of Δ (equivalently, a set of subsets of the basic set Δ). On such a family the adjacency (as in the original Γ) is to be defined by

$$X \sim Y \quad \Leftrightarrow \quad |X \Delta Y| = \lambda.$$

Moreover, the distance defined by this adjacency must possess the condition of embedding:

$$d(X, Y) = |X \Delta Y|/\lambda,$$

for any two subsets from the family. In this section we regard the subgraphs, λ-embedded in Δ, as such families, and for two Γ, Γ' of them with $\Gamma \subseteq \Gamma'$ we say that Γ' is an *extension* of Γ. Clearly, in such a situation we have that Γ is naturally 1-embedded into Γ'.

The main result of this section is the following:

PROPOSITION 3.1. *Any graph Γ, scale embedded into Δ, possesses a unique extension $\hat{\Gamma}$ minimal with respect to the following conditions:*
(1) $\hat{\Gamma} = \hat{\Gamma}_1 \times \cdots \times \hat{\Gamma}_s$;
(2) $\hat{\Gamma}_i$, $i = 1, \ldots, s$, is isomorphic to a complete graph, cocktail party graph or halved cube.

In order to prove Proposition 3.1 we develop some further theory related to the notion of an atom. Throughout, we suppose that Γ is λ-embedded into Δ.

For a given Γ let us define the *atom graph* $\Lambda(\Gamma)$ as the graph defined on the set of proper atoms of Γ by the following: two proper atoms A and B are adjacent if $|A \cap B| = \lambda/2$. By Lemma 2.5, distinct proper atoms are either adjacent, or disjoint.

LEMMA 3.2. *If Γ' is an extension of Γ then $\Lambda(\Gamma)$ is a subgraph of $\Lambda(\Gamma')$. In particular, each connected component of $\Lambda(\Gamma)$ is contained in a connected component of $\Lambda(\Gamma')$.* □

Let Λ be a connected component of $\Lambda(\Gamma)$ and Ω be the union of all proper atoms—vertices of Λ. By definition of Ω each proper atom of Γ is either contained in, or is disjoint of Ω. We will need a more general statement.

LEMMA 3.3. *For each atom A of Γ (not necessarily proper), either $A \subseteq \Omega$, or $A \cap \Omega = \varnothing$.*

PROOF. By definition $A = X \Delta Y$ for some adjacent vertices X and Y. By the above remark we may restrict ourselves to the case in which A is not proper. By Lemma 2.3 one has $|X \backslash Y| = \lambda/2$. By Lemma 2.4 both X and Y can be represented as disjoint unions of proper atoms—$X = \bigcup_{i=1}^{s} B_i$ and $Y = \bigcup_{i=1}^{s} C_i$. Let $\mathscr{B} = \{B_1, \ldots, B_s\}$ and $\mathscr{C} = \{C_1, \ldots, C_s\}$. If $\alpha \in X \backslash Y$ then there is an atom $B \in \mathscr{B}$ such that $\alpha \in B$. It follows from Lemma 2.5 that $|B \cap Y|$ is an integer multiple of $\lambda/2$. Now $|X \backslash Y| = \lambda/2$ implies $X \backslash Y \subset B$. Symmetrically, there is $C \in \mathscr{C}$ such that $Y \backslash X \subset C$. Furthermore, since C_i's are disjoint, each B_j, which is not equal to some C_i, has non-trivial intersections with exactly two proper atoms C_i's, unless $B_j = B$, in which case B_j has non-trivial intersection with exactly one proper atom C_i. Clearly, a symmetric statement holds if we switch the roles of C's and B's.

This means that the subgraph in $\Lambda(\Gamma)$ generated by all B_i's and all C_i's is a disjoint union of isolated vertices, cycles and exactly one string. Clearly, the end vertices of that string S are B and C. By definition, all proper atoms along S belong to one connected component of $\Lambda(\Gamma)$. Now the statement of the lemma follows, since $A \subseteq B \cup C$. □

For $X \in V(\Gamma)$ let $\bar{X} = X \cap \Omega$.

LEMMA 3.4. *The set $\{\bar{X} \mid X \in V(\Gamma)\}$ forms a graph $\bar{\Gamma}$ embedded into Δ with the scale λ. Moreover, atoms (proper atoms) of $\bar{\Gamma}$ are just those atoms (proper atoms) of Γ which are contained in Ω. In particular, Λ coincides with $\Lambda(\bar{\Gamma})$.*

PROOF. Let us define the adjacency on $\{\bar{X} \mid X \in V(\Gamma)\}$ as in the beginning of the section. With this definition it suffices to prove that for any $X, Y \in V(\Gamma)$ there is an integer s such that $|\bar{X} \Delta \bar{Y}| = \lambda s$, and there is a path in $\bar{\Gamma}$ between $\bar{X}$ and $\bar{Y}$, having length s. As in Lemma 2.4, $X \Delta Y$ is the disjoint union of the atoms (not necessarily proper) defined by the edges on a shortest path $X = X_0, \ldots, X_k = Y$ from X to Y. If X_{i-1} and X_i are consecutive vertices on that path, then by Lemma 3.3 the atom $X_{i-1} \Delta X_i$ is either contained in, or disjoint of Ω. Let s be the number of those atoms which are contained in Ω. Then, clearly, $|\bar{X} \Delta \bar{Y}| = \lambda s$ and the path $\bar{X}_0, \ldots, \bar{X}_k$, with repetitions omitted, is the desired path between $\bar{X}$ and $\bar{Y}$, having length s.

Clearly, if an atom $A = X \Delta Y$ is contained in Ω then $A = \bar{X} \Delta \bar{Y}$, proving that (proper) atoms of Γ are also (proper) atoms of $\bar{\Gamma}$. Now let $A = \bar{X} \Delta \bar{Y}$ be any atom of $\bar{\Gamma}$. As above, $X \Delta Y$ is a disjoint union of atoms defined by the edges on a shortest path from X to Y. Since by Lemma 2.5 each atom is either contained in Ω, or disjoint of it, one has that A itself is an atom of Γ, proper if it is such in $\bar{\Gamma}$. □

The graph $\bar{\Gamma}$ can be considered as a projection of Γ, defined by the subset Ω (or component Λ). Let $\Lambda_1, \ldots, \Lambda_r$ be the full set of connected components of $\Lambda(\Gamma)$. Let Ω_i, $i = 1, \ldots, r$, be the part of the basic set of Δ, covered by atoms from $V(\Lambda_i)$, and let $\bar{\Gamma}_i$ be the above-defined projection of Γ on the subset Ω_i. By Lemma 3.4, $\Lambda_i = \Lambda(\bar{\Gamma}_i)$, $i = 1, \ldots, r$. By Lemma 2.6 the projections $\bar{\Gamma}_i$ give a certain extension $\bar{\Gamma}$ of Γ, which is simply the direct product of $\bar{\Gamma}_i$'s.

Now let $\hat{\Gamma}$ be a minimal extension of Γ possessing (1) and (2) of Proposition 3.1. The existence and uniqueness of $\hat{\Gamma}$ is to be proved. Here we reduce the general situation to the case in which $\Lambda(\Gamma)$ is connected. By Lemma 2.7 we may assume that the factors $\hat{\Gamma}_i$ are embedded into independent parts of Δ. First we observe the following:

LEMMA 3.5. *$\Lambda(\hat{\Gamma})$ is a disjoint union of $\Lambda(\hat{\Gamma}_i)$, $i = 1, \ldots, s$.* □

Clearly, all $\Lambda(\hat{\Gamma}_i)$ are connected. By Lemma 3.2 each Λ_i is a subgraph of some $\Lambda(\hat{\Gamma}_j)$.

LEMMA 3.6. *If Λ_i is a subgraph of $\Lambda(\hat{\Gamma}_j)$, then any vertex of $\hat{\Gamma}_j$ is contained in Ω_i.*

PROOF. The proof follows from Lemma 2.10, in view of the minimality of $\hat{\Gamma}$. □

It follows from Lemma 3.6 that Proposition 3.1 would be a consequence of the following:

PROPOSITION 3.7. *If $\Lambda = \Lambda(\Gamma)$ is connected then there exists exactly one minimal extension $\hat{\Gamma}$ of Γ isomorphic to a complete graph, a cocktail party graph or a halved cube.*

Before proving Proposition 3.7 we insert a lemma which helps to distinguish the degenerate and non-degenerate cases in the proof.

LEMMA 3.8. *Suppose that a connected graph Ξ is not a subgraph of a cocktail party graph. Suppose, furthermore, that every vertex $x \in V(\Xi)$ is represented by a vertex X of the hypercube Δ in such a way that $|X| = \lambda$ for every $x \in V(\Xi)$, and for any $x, y \in V(\Xi)$ one has $|X \cap Y| = \lambda/2$ or 0, depending on whether or not x and y are adjacent. Then there are $x, y \in V(\Xi)$ with $d(x, y) = 2$, and a $z \in V(\Xi)$ such that $d(x, z) \neq 1$ and $d(y, z) = 1$.*

PROOF. Suppose, to the contrary, that, for x and y at distance 2 from each other and $z \in V(\Xi)$, the statements $d(x, z) = 1$ and $d(y, z) = 1$ imply each other. Let z be any neighbour of x and y. Since X and Y are disjoint, one has $Z \subseteq X \cup Y$. Now any neighbour of z must be a neighbour of x or y; hence it must be a neighbour of both. By connectivity all vertices of Ξ are represented by subsets of the 2λ-element set $X \cup Y$. Now it is easy to see that $x \mapsto X$ is a scale embedding and hence, by Lemma 2.8, Ξ is a subgraph of a cocktail party graph, a contradiction. □

PROOF OF PROPOSITION 3.7. First we consider the degenerate case. If Λ is complete then by Lemma 2.4 any vertex of Γ is either $V = \varnothing$, or a vertex of Λ. Hence Γ is itself complete and $\hat{\Gamma} = \Gamma$. If Λ now contains two proper atoms A and B at distance 2 from each other (i.e. disjoint), but Λ is still a subgraph of a cocktail party graph, then every other proper atom C is adjacent to both A and B, and hence $C \subseteq A \cup B$. By Lemma 2.4 the graph Γ is situated within a 2λ-element subset $A \cup B$, and, clearly, $\hat{\Gamma}$, which is in this case a cocktail party graph, consists of all the vertices of Γ and all their complements in $A \cup B$.

Now suppose that Λ is not a subgraph of a cocktail party graph. By Lemma 3.8 there exist proper atoms A, C at distance 2 from each other, and a proper atom D, such that D is disjoint of A and intersects C non-trivially. Let B be a proper atom which is adjacent to both A and C.

We will call *halves* the subsets of Δ of the following two kinds—$X \cup Y$ and $X \backslash Y$, where X and Y are adjacent proper atoms. Clearly, each half has cardinality $\lambda/2$. In order to prove the lemma in the remaining case it suffices to show that different halves are disjoint. Then, by connectivity of Λ, each proper atom is uniquely represented as a pair of halves, and every vertex X is uniquely represented (by Lemma 2.4) as a disjoint union of $2d(X, V)$ halves. This brings us directly to an isometric embedding of Γ into the halved cube defined by the set of halves.

We will prove the disjointness of the halves by induction. By connectivity we can order the set $\{A_1, \ldots, A_d\}$ of proper atoms in such a way that each A_j, $j \neq 1$, is adjacent to at least one A_s with $s < j$. Then the subgraph of Λ, induced by the first j of its vertices, is connected. We may also assume that $A_1 = A$, $A_2 = B$, $A_3 = C$ and $A_4 = D$.

Let us start with $j = 4$. Set $H_1 = A \backslash B$, $H_2 = A \cap B$, $H_3 = B \cap C$ and $H_4 = C \backslash B$. Since A is not adjacent to C, all these four halves are disjoint. Now if $X = D \cap C$ coincides with neither H_3, nor H_4, then $0 < |X \cap H_3| < \lambda/2$. It means that $B \cap D$ (which must be of cardinality $\lambda/2$, if non-empty) non-trivially intersects both halves H_2 and H_3 of B. Since $H_2 \subseteq A$, one has that D is adjacent to A; a contradiction. We have proved that $X = H_3$ or H_4 and, hence, $H_5 = D \backslash C$ is disjoint of H_1, H_2, H_3 and H_4.

Now let us suppose that all the halves defined by the first $j - 1$ vertices of Λ are disjoint, $j > 4$. Let us denote the set of those halves by $\mathcal{H}$. Two halves $X, Y \in \mathcal{H}$ will be called *neighbours* if their union is one of the atoms A_s, $s < j$. This clearly provides a graph structure on $\mathcal{H}$ and, moreover, the graph $\mathcal{H}$ is connected since the subgraph in Λ induced by $\{A_1, \ldots, A_{j-1}\}$ is such.

By assumption A_j is adjacent to some A_s, with $s < j$. Let $A_s = X_1 \cup X_2$, $X_1, X_2 \in \mathcal{H}$. Suppose that $A_j \cap A_s$ is equal to neither X_1 nor X_2. Then, clearly, $|A_j \cap X_1| = \alpha$ and $|A_j \cap X_2| = \beta$ for some α and β, such that $\alpha, \beta > 0$ and $\alpha + \beta = \lambda/2$. More generally, if Y_1 and Y_2 are two halves, which are neighbours, and $|A \cap Y_1| = \alpha$ or β then $|A \cup Y_2| = \beta$ or α, respectively. Now the connectivity of the graph $\mathcal{H}$ implies that for any neighbours $Y_1, Y_2 \in \mathcal{H}$ one has $|A \cap Y_1| = \alpha$ and $|A_j \cap Y_2| = \beta$, or vice versa.

Now, it is easy to see that $|A \cap (H_1 \cup H_2 \cup H_3 \cup H_4 \cup H_5)| = 2\alpha + 3\beta$ or $3\alpha + 2\beta$, which is more than λ in either case. The contradiction proves that $A \cap A_s$ is either X_1, or X_2. Hence $A_j \backslash A_s$ is either a half from $\mathcal{H}$, or a new half which is, clearly, disjoint of all halves from $\mathcal{H}$. Induction implies that all halves are disjoint. This completes the proof of Proposition 3.7. □

As already mentioned, Proposition 3.1 is a consequence of Proposition 3.7 and Lemma 3.6.

4. The Abstract Construction

In this section we present a sequence of constructions, depending only on the internal features of Γ, which turn out to result in the same structures as the embedding-dependent constructions from Section 3. Therefore in this section we keep more or less parallel with Section 3, pointing out from time to time what our current construction means in terms of that section. It must be stressed that we now consider the embedding $x \mapsto X$ as an 'arbitrary' embedding and check on it that our constructions are correct. *Throughout this section we assume that Γ is scale embeddable into a hypercube*, i.e. it has such an arbitrary embedding.

Let us choose an initial vertex $v \in V(\Gamma)$. Replacing if necessary our arbitrary embedding by an equivalent one, we may assume without loss that $v \mapsto V = \varnothing$ holds.

Proper atoms, the atom graph. Let us define a function $x \cap y$ with $x, y \in \mathscr{V} = V(\Gamma)$ by

$$x \cap y = \tfrac{1}{2}[d(v, x) + d(v, y) - d(x, y)].$$

Let $\mathscr{E}$ be the set of edges $\{y, z\}$ of Γ, such that $d(z, v) = d(y, v) + 1$. Let us extend the function $\cap$ onto $\mathscr{V} \cup \mathscr{E}$ by

$$x \cap \{y, z\} = x \cap z - x \cap y,$$

$$\{y, z\} \cap \{y', z'\} = z' \cap \{y, z\} - y' \cap \{y, z\}.$$

Here x, y, y', z and z' are vertices of Γ, $d(v, z) = d(v, y) + 1$ and $d(v, z') = d(v, y') + 1$, so that $\{y, z\}, \{y', z'\} \in \mathscr{E}$.

LEMMA 4.1. (1) *If $e \in \mathscr{E}$ and $x \in \mathscr{V} \cup \mathscr{E}$ then $e \cap x \in \{0, 1/2, 1\}$.*
(2) *The relation on $\mathscr{E}$ defined by*

$$e \sim e' \quad \Leftrightarrow \quad e \cap e' = 1$$

is an equivalence relation.
(3) *For $e, e' \in \mathscr{E}$ and $x \in \mathscr{V}$ the values $e \cap e'$ and $e \cap x$ do not depend on the choice of e and e' within their equivalence classes.*

PROOF. Consider an arbitrary scale embedding and apply Lemmas 2.2 and 2.4. □

The set of equivalence classes on $\mathscr{E}$, defined in Lemma 4.1, will be denoted by $\mathscr{A}$. Clearly, elements of $\mathscr{A}$ are in a natural bijection with the proper atoms from Section 3. In view of Lemma 4.1 we will no longer distinguish edges within the equivalence classes on $\mathscr{A}$; in particular, we now consider $\cap$ as defined on $\mathscr{V} \cup \mathscr{A}$.

The set $\mathscr{A}$ carries a structure of a graph defined by: $a, b \in \mathscr{A}$ are adjacent iff $a \cap b = 1/2$. Let $\mathscr{L}$ denote this graph as a whole and let $\mathscr{L}_1, \ldots, \mathscr{L}_r$ be all its connected components. The following is straightforward.

LEMMA 4.2. *The natural bijection between $\mathscr{A}$ and the set of proper atoms gives an isomorphism of $\mathscr{L}$ and the atom graph Λ. In particular, r has the same meaning as in Section 3 and, up to renumbering, $\mathscr{L}_i$ is naturally isomorphic to Λ_i, $i = 1, \ldots, r$.* □

The projections of Γ. Now we are going to construct the graphs which correspond to the projections $\bar{\Gamma}_1, \ldots, \bar{\Gamma}_r$ of Γ.

Let us decide for each edge $\{x, y\}$ of Γ (not necessarily from $\mathscr{E}$) to which of r 'components' it belongs. This will be our analogue of the belonging of atoms to the subsets Ω_i, given in Lemma 3.3. Let $v = x_0, \ldots, x_s = x$ and $v = y_0, \ldots, y_t = y$ be any shortest paths from v to y, respectively. Then every edge on either of these paths belongs to $\mathscr{E}$ and hence defines an element from $\mathscr{A}$. Let $\mathscr{S}$ be the subgraph of $\mathscr{L}$ generated by all those elements of $\mathscr{A}$.

LEMMA 4.3. (1) *$\mathscr{S}$ is a disjoint union of a number of isolated vertices, a number of cycles, and of exactly one string;*
(2) *The component $\mathscr{L}_j$, to which the string belongs, depends only on the edge $\{x, y\}$ itself, but not on the chosen shortest paths.*

PROOF. See the proof of Lemma 3.3. □

Now each edge in Γ has a certain label on it, belonging to the set $\{1, \ldots, r\}$. For i in this set, we say that two vertices $x, y \in V(\Gamma)$ are i-equivalent whenever x and y are joined by a path, no edge on which has i as its label.

LEMMA 4.4. *Vertices $x, y \in V(\Gamma)$ are i-equivalent iff they map onto the same vertex of $\bar{\Gamma}_i$. In particular, the folding $\mathcal{G}_i$ of Γ over the i-equivalence relation is naturally isomorphic to $\bar{\Gamma}_i$.*

PROOF. If an edge $\{x, y\}$ has label $j \neq i$, then the atom $X\Delta Y$ is contained in Ω_j. In particular, the projections of x and y into $\bar{\Gamma}_i$ coincide. Now let vertices $x, y \in V(\Gamma)$ be given, such that their images in $\bar{\Gamma}_i$ coincide. Then we choose a shortest path $x = x_0, \ldots, x_t = y$ between x and y and see that atoms $X_{t-1}\Delta X_i$ are contained in $X\Delta Y$. Since the latter set intersects with Ω_i trivially, no edge on the path has label i. □

The set of natural morphisms from Γ to $\mathcal{G}_i$, $i = 1, \ldots, r$, defines an embedding of Γ into $\mathcal{G}_1 \times \cdots \times \mathcal{G}_r$, which is clearly the same as the embedding of Γ into $\bar{\Gamma} = \bar{\Gamma}_1 \times \cdots \times \bar{\Gamma}_r$.

Extending $\mathcal{G}_i$ to $\hat{\mathcal{G}}_i$. Now the last step is to embed each factor $\mathcal{G}_i$ into a certain graph $\hat{\mathcal{G}}_i$ isomorphic to either a complete graph, a cocktail party graph or a halved cube. As in Section 3, we have degenerate and non-degenerate cases.

If $\mathcal{G}_i$ is a complete graph then we take $\hat{\mathcal{G}}_i = \mathcal{G}_i$. If $\mathcal{G}_i$ is not a complete graph, but is still a subgraph of a cocktail party graph, then to obtain $\hat{\mathcal{G}}_i$ we add an opposite for each vertex of $\mathcal{G}_i$ which does not yet have it.

Now suppose that $\mathcal{G}_i$ is not a subgraph of a cocktail party graph. According to the proof of Proposition 3.7, $\mathcal{G}_i$ is isometrically embeddable into a halved cube. Hence our purpose is to reconstruct the set of what was called halves. For simplicity, let us represent halves by the integers from $\{1, \ldots, N\}$, where N would be the number of halves. Then reconstructing the halves means attaching to every vertex a of $\mathcal{L}_i$ a pair $h(a) = \{m, n\}$, with the meaning that a 'consists' of the mth and nth halves. Such an attachment must possess the following conditions:

(1) The set $\mathcal{H}_m$ of the vertices a of $\mathcal{L}_i$, for which $m \in h(a)$, generates a complete subgraph of $\mathcal{L}_i$, and every edge of $\mathcal{L}_i$ belongs to such a subgraph.

(2) Every vertex belongs to exactly two $\mathcal{H}_k$'s; two sets $\mathcal{H}_k$ and $\mathcal{H}_j$ intersect each other in at most one vertex.

The first statement simply means that two atoms, having a half in common, must be adjacent, and vice versa. The second statement means that we attach pairs, and no pair can repeat. Since $\mathcal{G}_i \simeq \bar{\Gamma}_i$ is 1-embeddable into a halved cube, there must be at least one such attachment.

LEMMA 4.5. *If $\mathcal{L}_i$ is not a subgraph of a cocktail party graph then, up to renumbering, there exists exactly one family $\{\mathcal{H}_m\}$ that possesses* (1) *and* (2).

PROOF. We repeat with certain variations the proof of Proposition 3.7. Let us order the set of vertices $\mathcal{A}_i = \{a_1, \ldots, a_f\}$ of $\mathcal{L}_i$ in such a way that each a_j is adjacent to at least one vertex a_k with $k < j$. By Lemma 3.8 we may also assume that $d(a_1, a_3) = 2$, $d(a_2, a_4) = 1$ and $d(a_1, a_4) \neq 1$. Set $h(a_1) = \{1, 2\}$, $h(a_2) = \{2, 3\}$, $h(a_3) = \{3, 4\}$ and set $h(a_4) = \{3, 5\}$ or $\{4, 5\}$ depending on whether a_4 is adjacent, or non-adjacent to a_2. Clearly, it is the only way to attach pairs to a_1, a_2, a_3 and a_4.

Suppose that we have already uniquely defined the pairs $h(a_k)$ for each $k<j$ and let N_{j-1} be the number of the halves defined so far. Let $\mathcal{H}_m(j-1)$ denite the set of all a_k with $k<j$, such that $m \in h(a_k)$. Now, what pair should be attached to a_j? Since a_j is adjacent to at least one a_k with $k<j$, the pair $h(a_j)$ cannot consist of two new halves. Hence, $h(a_j)=\{m, n\}$ or $\{m, N_{j-1}+1\}$, where $m, n \leq N_{j-1}$. Let $\mathcal{N}$ be the set of neighbours of a_j in $\{a_1, \ldots, a_{j-1}\}$. In the first case $\mathcal{N}=\mathcal{H}_m(j-1) \cup \mathcal{H}_n(j-1)$ and, moreover, $\mathcal{H}_m(j-1)$ and $\mathcal{H}_n(j-1)$ are disjoint. In the second case $\mathcal{N}=\mathcal{H}_m(j-1)$. It remains to prove that there is only one way to represent $\mathcal{N}$ as $\mathcal{H}_m(j-1)$, or as a disjoint union of $\mathcal{H}_m(j-1)$ and $\mathcal{H}_n(j-1)$, where m, n run through $\{1, \ldots, N_{j-1}\}$.

Let '+' denote disjoint union and suppose for some pairwise different k, l, m and n that one has either $\mathcal{H}_k(j-1)=\mathcal{H}_l(j-1)$, or $\mathcal{H}_k(j-1)=\mathcal{H}_l(j-1)+\mathcal{H}_m(j-1)$, or $\mathcal{H}_k(j-1)+\mathcal{H}_l(j-1)=\mathcal{H}_m(j-1)+\mathcal{H}_n(j-1)$. In either case it is easy to see that such an equality means that $\mathcal{N}$ forms a connected component of $\mathcal{L}_i$ with at most 4 vertices. Moreover, if $|\mathcal{N}|=4$ then the generated subgraph is a 4-cycle. This gives the desired contradiction, since the subgraph generated by $\{a_1, \ldots, a_{j-1}\}$ is connected, contains at least 4 vertices, and is not a 4-cycle. □

This lemma provides the means of recovering the set of halves and, moreover, representing any proper atom of $\mathcal{L}_i$ by a pair of halves. Now Lemma 2.4 provides an obvious way to define a mapping from the vertex set of $\mathcal{L}_i$ into the even part of the hypercube defined by the set of halves. Clearly, this even part is just our halved cube $\hat{\mathcal{G}}_i$.

Proof of Theorem 2. By the above arguments, for any scale embedding ψ of Γ, such that v is mapped onto $\varnothing$, the 'embedding-dependent' graph $\hat{\Gamma}$ is canonically isomorphic to the graph $\hat{\mathcal{G}}=\widehat{\mathcal{G}_1 \times \cdots \times \mathcal{G}_r}$, and this gives us the embedding $\hat{\psi}$. In the case $\psi(v)=V \neq \varnothing$ we define $\hat{\psi}$ as $(\widehat{\psi \Delta V})\Delta V$, where the shift $\psi \rightarrow \psi\Delta V$ and then back is as in Lemma 2.1. □

In the above tricky proof if remains unclear why the graph $\hat{\mathcal{G}}=\hat{\Gamma}$ does not depend on the choice of the initial vertex v, whereas the given construction evidently depends on it in many places. We have no option but to regard this as a property of graphs that are scale embeddable into hypercubes.

Proof of Corollary 1. Since a complete graph is a subgraph of, say, a cocktail party graph, we need only two types of factors if we do not assume the direct product graph to be the minimal possible. □

Proof of Corollary 2. The above construction of the graph $\hat{\mathcal{G}}$ can clearly be performed in a polynomial time. The verification that the natural mapping from Γ into $\hat{\mathcal{G}}$ is indeed a 1-embedding also requires a polynomial time. □

In order to give a proof of Corollary 3, we must first discuss the notion of *l_1-rigidity*. Our way of giving it will differ from that of [3], although of course it is completely equivalent. It was mentioned above that rigidity means having only one scale embedding up to some natural equivalence. It remains to define the equivalence. First of all, every embedding $x \mapsto X \subseteq \Delta$ is equivalent by definition to the *blown* embedding, which is obtained by substituting every element of Δ by a separate set of a fixed cardinality k. Clearly, the scale of the blown embedding is equal to $k\lambda$, where λ is the original scale. Now, given two scale embeddings $x \mapsto X_1 \subseteq \Delta_1$ and $x \mapsto X_2 \subseteq \Delta_2$ of the same graph Γ, we may assume, up to blowing, that the scales of the embeddings coincide. Also, up to adding redundant elements we may assume that Δ_1 and Δ_2 have the same cardinality. In such a case, we call the two embeddings equivalent if there is a bijection σ between Δ_1 and Δ_2, and a subset $S \subseteq \Delta_2$, such that $X_2 = \sigma(X_1)\Delta S$ for every $x \in V(\Gamma)$.

PROOF OF COROLLARY 3. Blowing of a graph Γ, embedded into a hypercube Δ, is equally a blowing of its extension $\hat{\Gamma}$ (cf. Section 3). Recall that the graph $\hat{\Gamma}$ was defined as the minimal extension of Γ in Δ, having properties (1) and (2) from Proposition 3.1. Let us say that $\hat{\Gamma}$ is the *closure* of Γ in Δ. Now suppose that $x \mapsto X_1 \subseteq \Delta_1$ and $x \mapsto X_2 \subseteq \Delta_2$ are two equivalent embeddings with the same scale and the same cardinality of Δ_1 and Δ_2, and suppose that σ and S establish the equivalence of these two embeddings. Let $\hat{\Gamma}_i$ denote the closure of (the image of) Γ in Δ_i, $i = 1, 2$. Then, by minimality of $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$, we have that σ and S also establish the equivalence of the two embeddings of the abstractly defined graph $\hat{\mathcal{G}}$, so that the rigidity of Γ implies the rigidity of $\hat{\mathcal{G}}$. The reverse implication follows from Theorem 2(2). □

PROOF OF COROLLARY 5. The empty set, the point set of an n-dimensional affine space over $GF(2)$ and all the hyperplanes of that affine space give an embedding of the cocktail party graph $K_{2^n \times 2}$ with scale 2^{n-1}. This means that every graph $K_{k\times 2}$ is embeddable with scale 2^{n-1}, where n is defined by $k \leqslant 2^n < 2k$.

Now consider an arbitrary l_1-graph Γ. By Theorem 2, the minimal scale for Γ is equal to the minimal scale for $\hat{\Gamma} = \hat{\Gamma}_1 \times \cdots \times \hat{\Gamma}_r$. Consider a particular $\hat{\Gamma}_i$. If it is isomorphic to a complete graph or a halved cube, then it is scale embeddable with any even scale. If $\hat{\Gamma}_i \simeq K_{k\times 2}$ then, clearly, $k < v$, where v is the number of the vertices of Γ. Now the above argument shows that every $\hat{\Gamma}_i$ (and hence $\hat{\Gamma}$ itself) is embeddable with the scale 2^{n-1}, where $v - 1 \leqslant 2^n < 2(v-1)$, provided that $2^{n-1} \geqslant 2$ (i.e. $v > 3$). □

PROOF OF COROLLARY 6. According to [5], we must check that all edges of $\bar{\Gamma}_i$ (same as $\mathcal{G}_i$) are equivalent under the equivalence relation $\hat{\theta}$, which is the transitive closure of the relation θ defined by

$$\{x, y\}\theta\{x', y'\} \quad \Leftrightarrow \quad d(x, x') + d(y, y') \neq d(x, y') + d(y, x').$$

If $\bar{\Gamma}_i$ is a subgraph of a cocktail party graph, then checking is straightforward. Otherwise, $\bar{\Gamma}_i$ is a subgraph of a halved cube and hence it is 2-embeddable in a hypercube. Now for a 2-embedding it is easy to check that $d(x, x') + d(y, y') \neq d(x, y') + d(y, x')$ iff the corresponding atoms intersect each other in exactly one element. Since the atom graph of $\bar{\Gamma}_i$ is connected, we have that all proper atoms are $\hat{\theta}$-equivalent. Now Lemma 3.3 implies that all other atoms of $\bar{\Gamma}_i$ lie in the same equivalence class. □

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