AUTOMATIC INDEXED CALCULATION OF WACHSPRESS' RATIONAL FINITE ELEMENT FUNCTIONS

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Abstract—A numerical method of calculating all of the polynomial coefficients for Wachspress' rational finite element functions is presented. It requires only linear algebraic computations of matrix algebra and avoids the complicated constructions of geometric algebra used in the development of the rational elements.

1. INTRODUCTION

The rapid increase in the use of irregularly shaped finite elements has emphasized the inadequacy of simple polynomials to serve as basis functions to fit, interpolate, differentiate and integrate a function specified at various vertices, points on sides, and points interior to the elements. It is now becoming clear that higher-order approximations to functions over finite elements have the capacity to improve not only the appearance, but also the convergence properties of the approximations[1]. Recent numerical applications of higher-order approximations give specific examples of these improvements[2]. Recently, a new set of basis functions was presented by Wachspress[3]. The functions are ratios of polynomials which fit specified nodal values, and attain a prescribed degree of approximation over each element and its boundaries. Adjacent elements have the same polynomial variation along common edges, so that global continuity is assured. The continuous, analytic basis functions are amenable to interpolation, differentiation and integration. They are available for elements from triangles through N-sided polygons and their generalization, elements having sides which are algebraic curves. Any desire degree of approximation may be achieved in two-, three- and even higher-dimensional space.

Due to the manner in which these rational basis or wedge functions have been deduced from deep theorems of classical algebraic geometry, their construction tends to be heavily dependent on algebraic solutions to geometric problems. One may construct a unit wedge function for each node before one constructs the function that fits specified nodal values and has a prescribed degree of variation along the edges and over the element. This is essential for finite element application where nodal values are free parameters in a variational or Galarkin formulation. However, this is not essential for patchwork interpolation of data.

An alternate construction of both the basis wedges and composite fitting functions that depends only on the indexed computations of linear algebra is presented here. These matrix computations are implemented using a digital computer and require no geometric constructions whatsoever. Furthermore, the computation can be used to construct the general wedge basis functions of Wachspress as well as more general fitting functions over the element.

In Sec. 2, a few results from the theory of wedge construction will be described from the geometric point of view. It will be shown that a general interpolant over the element, represented as a linear combination of wedge functions times node values, has an alternate and useful linear-coefficient-polynomial-form that can be treated with the methods of conventional matrix algebra. In Sec. 3, a method for using linear algebra to solve for all of the unknown coefficients in the general function expression will be developed. This will be done in terms of specific polynomial fitting functions for the triangle, quadrilateral, etc. In Sec. 4 the theory for computation of the coefficients for up to degree-two fitting over polygons will be developed. In Sec. 5, comparisons with independent evaluations of several examples of basis functions will be presented and some of the computational difficulties encountered in the applications of the method introduced here will be presented. In Sec. 6, possible extensions of the method to elements with curved sides and to elements in three dimensions will be presented.

2. GEOMETRIC BACKGROUND

On reading the discussion of Wachspress[3], one observes that a rational function which is linear on all the edges and has zero value on all but one vertex of a convex polygon can be constructed. Given a polygon with N sides as seen in Fig. 1, one first proposes the degree-one wedge function for node i as the ratio of two polynomials:

$$W_i(x, y) = P_{N-2}(x, y)/Q_{N-3}(x, y), \tag{2.1}$$

where the subscripts $N-2$ and $N-3$ indicate the maximal degrees of the polynomials $P(x, y)$ and $Q(x, y)$. In particular, $Q_{N-3}(x, y)$ is a polynomial which vanishes on a curve of maximal degree-$(N-3)$. For example, for $N = 5$, one has

$$Q_{N-3}(x, y) = b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 y^2 + b_5 xy. \tag{2.2}$$

In fact, the coefficients b_i can be determined by the requirement that the curve $Q_{N-3}(x, y) = 0$ pass through all of the external intersection points (EIP's), of the polygon (see Fig. 1). The polynomial $P_{N-2}(x, y)$ is the product of the $N - 2$, order-one forms which vanish on the $N - 2$ polygon edges that do not contain point i. Note that the leading b_0 term will be 1 except for the case where an EIP is located at the origin of coordinates, in which case b_0 must be 0.

If one defines the degree-one form for a line 1, 2 as

$$L_{12}(x, y) = (L_{1,2}) = \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \tag{2.3}$$

then

$$P_{N-2}(x, y) = (L_{i+1,i+2}) \cdots (L_{N,1})(L_{1,2}) \cdots (L_{i-2,i-1}). \tag{2.4}$$

For the five-sided polygon (or 5-gon), $N - 2 = 3$, one has

$$\begin{aligned} P_{N-2}(x, y) = {} & a_1 + a_2 x + s_3 y + a_4 x^2 + a_5 y^2 + a_6 xy \\ & + a_7 x^3 + a_8 y^3 + a_9 x^2 y + a_{10} xy^2. \end{aligned}$$

It will also be of use to note the form, but not the details of the wedge for the ith node when the function is to be of degree two along each side as specified by the value of the function at an edge point as well as the corners. The function $Q_{N-3}(x, y)$ is the same for any degree of approximation. For the case of degree-two approximation over the N-sided polygon, the degree of P is $N - 1$. Thus for degree-two fitting one has
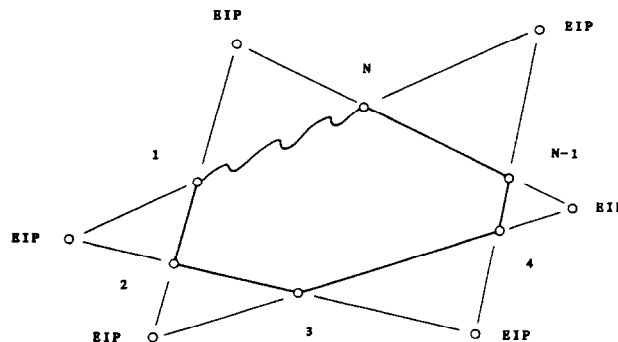
$$W_i(x, y) = P_{N-1}(x, y)/Q_{N-3}(x, y). \tag{2.5}$$

Fig. 1. A general N-sided polygon.

It is tempting to assume that $P_{N-1}(x, y)$ might be made up of products of quadric surfaces passing through the points on the nonadjacent sides. This is unfortunately not the case. However, the lack of such a simple extension causes no hardship. For degree-R approximation over the polygon one has

$$W_i(x, y) = P_{N+R-2}(x, y)/Q_{N-3}(x, y). \tag{2.6}$$

In order to be specific one can consider the degree-1 wedge for the triangle in Fig. 2. The basis function for point 1 is

$$
\begin{aligned}
W_1(x, y) &= \frac{P_1(x, y)Q_0(x_1, y_1)}{Q_0(x, y)P_1(x_1, y_1)} \\
&= \frac{L_{2,3}(x, y)}{L_{2,3}(x_1, y_1)} \\
&= \begin{vmatrix} x & y & 1 \\ x_2 & y_1 & 1 \\ x_3 & y_1 & 1 \end{vmatrix} \bigg/ \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_1 & 1 \\ x_3 & y_1 & 1 \end{vmatrix}.
\end{aligned}
\tag{2.7}
$$

The divisor has been chosen to normalize the wedge $W_1(x, y)$ to unity at the point (x_1, y_1).

For the convex quadrilateral (Fig. 3) the general degree-one wedge will have the form

$$
\begin{aligned}
W_1(x, y) &= \frac{P_2(x, y)}{Q_1(x, y)} \frac{Q_1(x_1, y_1)}{P_2(x_1, y_1)} \\
&= \frac{a_1 + a_2x + a_3y + a_4x^2 + a_5y^2 + a_6xy}{b_0 + b_1x + b_2y}.
\end{aligned}
\quad \text{(without normalization)} \tag{2.8}
$$

For the degree-one pentagon one has

$$W_i(x, y) = \frac{P_3(x, y)}{Q_2(x, y)} = \frac{a_1 + \cdots a_6xy + \cdots a_{10}xy^2}{b_0 + b_1x + b_2y + \cdots b_5xy}. \tag{2.9}$$

The number of terms in the general degree-N polynomial in x and y, such as $P_N(x, y)$ and $Q_N(x, y)$ is

$$\text{Dim } P_N(x, y) \text{ is} = (N + 1)(N + 2)/2!. \tag{2.10}$$

If the polynomial has three independent variables (x, y, z) then

$$\text{Dim } P_N(x, y, z) = (N + 1)(N + 2)(N + 3)/3!. \tag{2.11}$$

One will find it useful to generalize the concept of the matrix operator of powers to treat polynomial systems of degree-N with K degrees of freedom, or dimensions. This matrix will give all of the powers of each independent variable in the degree-N polynomial in K dimensions.
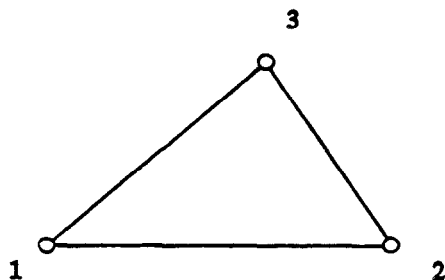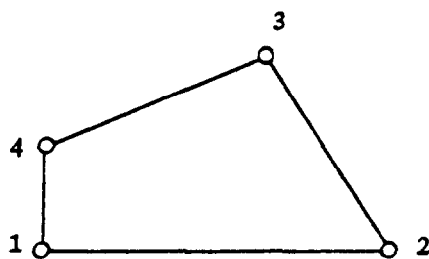

Fig. 2. A triangle with degree-one variation.

Fig. 3. A convex quadrilateral.

The rows will contain all monomials of degree m in K variables, where m runs from 1 to N. See Table 1 for examples of this matrix $[P, N, K]$. In addition one will need to define the matrices Q, R and S:

$$[Q, N, K] = [P, N, K] \tag{2.12}$$

with the leading row of zeros omitted,

$$[R, N, K + 1] = [P, N, K] \tag{2.13}$$

with an additional column of zeros added at the right,

$$[S, N, K + 1] = [Q, N, K] \tag{2.14}$$

with an additional column of ones added at the right.

Note that whenever any of the matrices P, Q, R or S occur before another matrix as in

$$[P, N, K][C] \tag{2.15}$$

it will be understood to mean

$$[P, N, K] \underset{k=1,K}{\pi} [C], \tag{2.16}$$

where the $\pi_{k=1,K}$ indicates that the N tuples of powers in each row of the $[P, N, K]$ matrix operate on the N tuples of the coordinates in the columns of the $[C]$ matrix to form the product of powers of coordinates in the terms in the polynomial $P_N(x, y, \cdots K)$.

One is now able to write Eq. (2.9) for the rational wedge for the 5-gon, using the $[P]$ and $[Q]$ matrices as

$$W(x\ y) = \frac{[a_1 \quad a_2 \quad \cdots \quad a_6][P, 3, 2]\begin{bmatrix} x \\ y \end{bmatrix}}{1 + [b_1 \quad b_2 \quad \cdots \quad b_5][Q, 2, 2]\begin{bmatrix} x \\ y \end{bmatrix}}. \tag{2.17}$$

Wachspress points out that a general function over the polygon can be constructed based on the wedge functions for each node times the values of the dependent variable z at each node. This will take the form

$$z(x, y) = \{\text{sum for } i \text{ over all nodes}\} \quad \text{of } z(x_i, y_i)W_i(x, y) \tag{2.18}$$

provided that each wedge is normalized to 1 at its node point. Then, the denominator Q is the same for all wedges and the general function $z(x, y)$ is

$$z(x, y) = P_{N+R-2}(x, y)/Q_{N-3}(x, y). \tag{2.19}$$

The coefficients of Q are determined by the requirement that the curve $Q(x, y)$ pass through the EIP's of the element.

3. DEGREE-ONE ALGEBRAIC THEORY

Rather than continue this geometric line of construction of the numerators, $P_{N+T-2}(x, y)$ and the denominators $Q_{N-3}(x, y)$, the tools of linear algebra will be applied to the problem of evaluating the unknowns a_1 a_2, . . . and b_1, b_2 . . . . If one has specified the value of the dependent variable $z(x, y)$ at the N corner points of a nonreentrant polygon with N sides as shown in Fig. 1, then using Equation (2.6) one has the N equations

$$z(x_i, y_i) = P_{N-2}(x_i, y_i)/Q_{N-3}(x_i, y_i) \quad \text{for } i = 1, 2, \ldots N. \tag{3.1}$$

In particular, for the quadrilateral, $N = 4$, one has

$$z_i(1 + b_1 x_i + b_2 y_i) = a_1 + a_2 x_i + \cdots a_6 x_i y_i. \quad \text{for } i = 1, 2, 3 \text{ and } 4. \tag{3.2}$$

Since there are eight unknowns a_1, ... a_6, b_1, b_2 and only four points, one selects one more point on each side and uses the requirement of linearity on the edges to interpolate a z value at each of four-side points for four more versions of Equation (3.2), for a total of eight equations in eight unknowns.

One can observe that for the degree-one N-sided polygon the ratio $P_{N-2}(x, y)/Q_{N-3}(x, y)$ could be of degree $N-2$ along an edge if $Q_{N-3}(x, y)$ were the constant $Q_0(x, y)$ or unity. A degree $(N-1)$ polynomial along a side can be determined by $(n-1)$ values along the side. By linear fitting along each side of the vertex, one guarantees that the ratio $P_{N-2}(x, y)/Q_{N-3}(x. y)$ will be of degree-one and not $N-2$.

Returning to the degree-one quadrilateral, one can easily write the equation for the interpolation along the sides in matrix form as

$$
\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ y_1 & & & \cdots\cdots\cdots & & & & y_8 \\ z_1 & & & \cdots\cdots\cdots & & & & z_8 \end{bmatrix}
$$

$$
= \begin{bmatrix} x_1 & x_3 & x_5 & x_7 \\ y_1 & \cdots & & y_7 \\ z_1 & \cdots & & z_7 \end{bmatrix} \begin{bmatrix} 1 & t_2 & & & & & & s_8 \\ & s_2 & 1 & t_4 & & & \\ & & & s_4 & 1 & t_6 & \\ & & & & & s_6 & 1 & t_8 \end{bmatrix}, \tag{3.3}
$$

where s_i and t_i are the fractional location of points along the sides

$$
\begin{aligned}
s_2 &= (x_2 - x_1)/(x_3 - x_1) \text{ and } t_2 = 1 - s_2 \\
&\vdots \\
s_8 &= (x_8 - x_7)/(x_1 - x_7) \text{ and } t_8 - 1 - s_8.
\end{aligned} \tag{3.4}
$$

Then Equation (3.2) becomes

$$
[a_1 \quad a_2 \quad \cdots \quad a_6] \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \pi_{k=1,3} \begin{bmatrix} x_1 & x_2 & \cdots & x_8 \\ y_1 & y_1 & \cdots & y_8 \\ z_1 & z_2 & \cdots & z_8 \end{bmatrix}
$$

$$
= [1 \quad b_1 \quad b_2] \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \pi_{k=1,3} \begin{bmatrix} x_1 & x_2 & \cdots & x_8 \\ y_1 & y_1 & \cdots & y_8 \\ z_1 & z_2 & \cdots & z_8 \end{bmatrix}, \tag{3.5}
$$

Table 1. Matrix of powers of the polynomial $P_N(x, y \cdots K)$

	$K = 1$	$K = 2$	$K = 3$
$N = 1$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
	$+$	$+$	$+$
$N = 2$	$[2]$	$\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$
	$+$	$+$	$+$
$N = 3$	$\begin{matrix} [3] \\ (N + 1)/1! \\ \text{lines} \end{matrix}$	$\begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$	$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$
		$\begin{matrix} (N + 1)(N + 2)/2! \\ \text{lines} \end{matrix}$	$\begin{matrix} (N + 1)(N + 2)(N + 3)/3! \\ \text{lines} \end{matrix}$

where the π of $k = 1, 3$ indicates that the triplets of powers in the power matrix operate on the triplets of x, y, z in the coordinate matrix to form the product terms in the polynomials in Equation (2.9). Note that on the LHS of Equation (3.5) the powers of z_i are all zero and z_i values are not required, but they add a symmetry very useful to the next equation. Rearranging Equation (3.5) and renaming $b_1 = -a_7$ and $b_2 = -a_8$ one will have

$$[a_1 \quad a_2 \quad \cdots \quad a_8] \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \underset{k=1,K}{\pi} \begin{bmatrix} x_1 & x_2 & \cdots & x_8 \\ y_1 & y_1 & \cdots & y_8 \\ z_1 & z_2 & \cdots & z_8 \end{bmatrix}$$
$$= [z_1 \quad z_2 \quad \cdots \quad z_8] \tag{3.6}$$

Making use of the $[R]$ and $[S]$ matrices described in Table 1, one may reduce Eq. (3.6) to:

$$[a] \begin{bmatrix} [R, 2, 3] & [C] \\ [S, 1, 3] & \end{bmatrix} = [z]. \tag{3.7}$$

One now takes the generalized inverse[4], indicated by the matrix superscript I of the rectangular matrix formed by

$$\begin{bmatrix} [R, 2, 3] & [C] \\ [S, 1, 3] & \end{bmatrix}. \tag{3.8}$$

Then one post multiplies Eq. (3.7) by the inverse, to solve for the $[a]$ vector:

$$[a] = [z] \begin{bmatrix} [R, 2, 3] & [C] \\ [S, 1, 3] & \end{bmatrix}^I. \tag{3.9}$$

The generalized inverse is used here to allow for the use of more than the minimum number of interpolation points when desired.

Once the (x, y, z) coordinates of four given points plus four points linearly interpolated around a quadrilateral are available, Equation (3.9) will allow one to use matrix algebra to solve for the eight coefficients in Wachspress' rational finite element functions. This can be done for either the unit wedge functions or for the general function over the quadrilateral.

In the general N-sided polygon with degree-one approximation one has

$$[a_1 \quad a_2 \quad \cdots \quad a_L] \begin{bmatrix} [R, N - 2, 3] \\ [S, N - 3, 3] \end{bmatrix}^* \begin{bmatrix} x_1 & x_2 & \cdots & x_L \\ y_1 & y_1 & \cdots & y_L \\ z_1 & z_2 & \cdots & z_L \end{bmatrix} = [z_1 \quad z_2 \quad \cdots \quad z_L]$$

where

$$L = (N + 1)(N + 2)/2 + N(N + 1)/2 - 1. \tag{3.11}$$

Here L is the sum of the number of unknowns in $P(x, y)$ plus the unknowns in $Q(x, y)$ [see Eq. (2.10)]. The degree-one wedge for the N-sided polygon will be

$$W_i(x, y) = \frac{[a_1 \quad \cdots \quad a_J][R, N - 2, 2] \begin{bmatrix} x \\ y \end{bmatrix}}{[1 - a_{J+1} \quad \cdots \quad a_L][P, N - 3, 2] \begin{bmatrix} x \\ y \end{bmatrix}}, \tag{3.12}$$

where

$$J = (N - 1)N/2! \text{ is the number of unknowns in } P(x, y). \tag{3.13}$$

In particular, for the triangle one has

$$\begin{bmatrix} [R, 1, 3] \\ [S, 0, 3] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.14}$$

For the degree-one quadrilateral using Eq. (3.11) and Table 1 one has

$$\begin{bmatrix} [R, 2, 3] \\ [S, 1, 3] \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}^T. \tag{3.15}$$

Note that the matrix has been transposed for ease of presentation. For the degree-one pentagon one has

$$\begin{bmatrix} [R, 3, 3] \\ [S, 2, 3] \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 1 & 3 & 0 & 2 & 1 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 & 0 & 3 & 1 & 2 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T. \tag{3.16}$$

One notes that the first ten columns in the $[P, 3, 3]$ matrix are just the ten triplets of powers of the three variables (x, y, z) in $P_3(x, y)$, the z's are all zero. The last five columns are just the five powers of x, y, z in $z*Q_2(x, y)$. Note that the leading 1 in the polynomial $Q_2(x, y)$ appears on the RHS of Eq. (3.11). Furthermore, the 15 z values used are the five corner values of z plus two more values, linearly interpolated, on each side.

4. HIGHER DEGREE FITTING FUNCTIONS

Using Eq. (2.6), degree-two fitting over a triangle, such as seen in Fig. 4, is of the form

$$[a_1 \quad a_2 \quad \cdots \quad a_6][P, 2, 2] \begin{bmatrix} x_1 & x_2 & \cdots & x_6 \\ y_1 & y_1 & \cdots & y_6 \end{bmatrix} = [z_1 \quad z_2 \quad \cdots \quad z_6] \qquad (4.1)$$

and fitting is over three corner points plus three more side points, one per side, to specify the quadratic behavior along each edge. This gives the following set of linear equations for the six unknown coefficients:

$$[a_1 \quad a_2 \quad \cdots \quad a_6][P, 2, 2] \begin{bmatrix} x_1 & x_2 & \cdots & x_6 \\ y_1 & y_1 & \cdots & y_6 \end{bmatrix} = [z_1 \quad z_2 \quad \cdots \quad z_6] \qquad (4.2)$$

or rearranging for compatibility with later cases:

$$[a_1 \quad a_2 \quad \cdots \quad a_6][R, 2, 3] \begin{bmatrix} x_1 & x_2 & \cdots & x_6 \\ y_1 & y_1 & \cdots & y_6 \\ z_1 & x_2 & \cdots & z_6 \end{bmatrix} = [z_1 \quad z_2 \quad \cdots \quad z_6] \qquad (4.3)$$

where

$$[R, 2, 3] = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T. \qquad (4.4)$$

Solving for the $[a]$ vector gives

$$[a_1 \quad a_2 \quad \cdots \quad a_6] = [z_1 \quad \cdots \quad z_6]$$
$$\times \left[[R, 2, 3] \begin{bmatrix} x_1 & \cdots & x_6 \\ y_1 & \cdots & y_6 \\ z_1 & \cdots & z_6 \end{bmatrix}^t \right] \qquad (4.5)$$

and reconstructing the wedge functions one has

$$W_i(x, y) = [a_1 \quad \cdots \quad a_6][P, 2, 2] \begin{bmatrix} x \\ y \end{bmatrix}. \qquad (4.6)$$

If one goes to the degree-three approximation over a triangle as seen in Fig. 5 one has

$$W_i(x, y) = P_3(x, y)/Q_0(x, y) = P_3(x, y) \qquad (4.7)$$
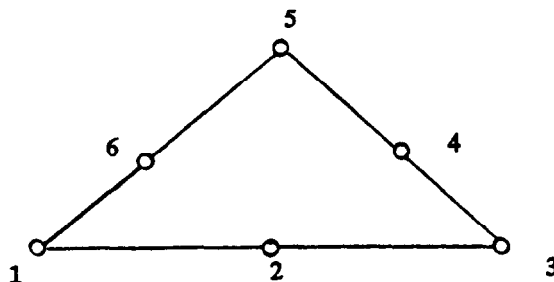

Fig. 4. Degree-two fitting over a triangle.

and one finds that the vector $[a]$ has $(3 + 1)(3 + 2)/2!$, or ten coefficients. From the previous arguements it can be seen that four points, including two corner points, are needed to govern a degree-three polynomial along each edge for a total of six edge and three corner points. The additional point must be off the triangle. With the ten points shown in Fig. 5 one can solve for the $[a_1 \cdots a_{10}]$ vector as

$$[z] = [z_1 \quad z_2 \quad \cdots \quad z_{10}] \left[[R, 3, 3] \begin{bmatrix} x_1 & \cdots & x_{10} \\ y_1 & \cdots & y_{10} \\ z_1 & \cdots & z_{10} \end{bmatrix} \right]' \tag{4.8}$$

and the reconstructed fitting function is

$$W_i(x, y) = [a][P, 3, 2] \begin{bmatrix} x \\ y \end{bmatrix}. \tag{4.9}$$

Turning to the quadrilateral with degree-two variation, as seen in Fig. 6, one has the general function of the form:

$$W_i(x, y) = P_3(x, y)/Q_1(x, y). \tag{4.10}$$

The first task is to interpolate one more, independent, quadratically consistent point along each side so that the 10 coefficients of $P_3(x, y)$ plus the two coefficients of $Q_1(x, y)$ can be specified by 12 node points around the quadrilateral. Simple degree-two fitting along a side, with coefficients q_1, q_3, q_4 to three points on a side, gives

$$[q_1 \quad q_3 \quad q_4] = [z_1 \quad z_3 \quad z_4][[P, 2, 1][0 \quad s3 \quad 1]]'. \tag{4.11}$$

The q_i's are numbered to reflect the points used to determine them. Note that s_3, as before, is the ratio of the distance from point 1 to 3 divided by the distance 1 to 4. One must interpolate one additional point along the line 1, 3, 4, with s_2 as the fractional distance from 1 to 2. Then, using the quadratic polynomial coefficients from Equation (4.4) one has

$$\begin{aligned}
[z_2] &= [q_1 \quad q_3 \quad q_4][P, 2, 1][s_2]. \\
&= [z_1 \quad z_3 \quad z_4][[P, 2, 1][0 \quad s_3 \quad 1]]'[P, 2, 1][s_2]. \\
&= [z_1 \quad z_3 \quad z_4] \begin{bmatrix} q_{1,2} \\ q_{3,2} \\ q_{4,2} \end{bmatrix}.
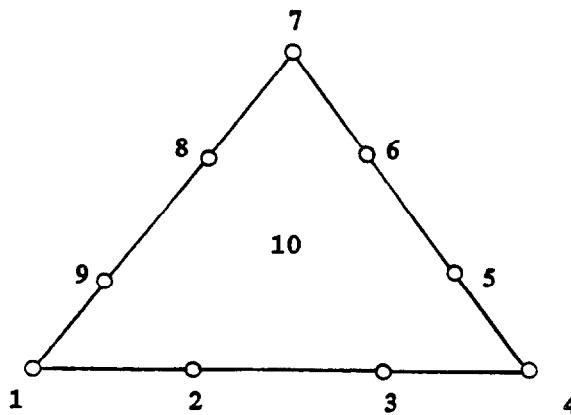\end{aligned} \tag{4.12}$$
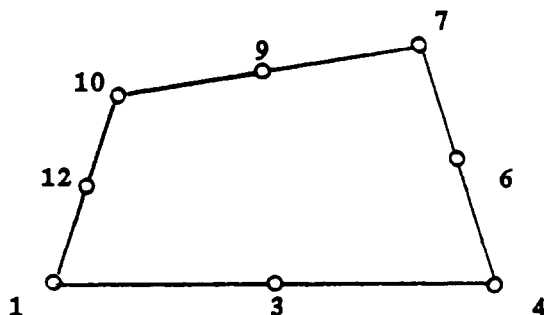

Fig. 5. The degree-three triangle.

Fig. 6. The degree-two quadrilateral.

where

$$
\begin{bmatrix} q_{1,2} \\ q_{3,2} \\ q_{4,2} \end{bmatrix} = [[P, 2, 1][0 \quad s3 \quad 1]]'[P, 2, 1][s_2].
$$

(4.13)

Thus one can interpolate the extra point as

$$
\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & \cdots & y_5 \\ z_1 & \cdots & z_4 \end{bmatrix} = \begin{bmatrix} x_1 & x_3 & x_4 \\ y_1 & y_1 & y_5 \\ z_1 & z_3 & z_4 \end{bmatrix} \begin{bmatrix} 1 & q_{1,2} & 0 & 0 \\ 0 & q_{3,2} & 1 & 0 \\ 0 & q_{4,2} & 0 & 1 \end{bmatrix}.
$$

(4.13)

This can easily be extended to a single degree-two matrix interpolation for all 12 points from the eight given points. The degree-two interpolated values at extra points 2, 5, 8 and 11, one on each of the sides, are determined in a similar fashion and give the quadratic interpolation of Equation (4.15):

$$
\begin{bmatrix} x_1 & \cdots & x_{12} \\ y_1 & \cdots & y_{12} \\ z_1 & \cdots & z_{12} \end{bmatrix} = \begin{bmatrix} x_1 & x_3 & x_4 & x_6 & x_7 & x_9 & x_{10} & x_{12} \\ x_1 & & & \cdots\cdots & & & & y_{12} \\ z_1 & & & \cdots\cdots & & & & z_{12} \end{bmatrix}
$$

$$
\times \begin{bmatrix} 1 & q_{1,2} & 0 & 0 & & & & & & 0 & q_{1,1} & 1 & 0 \\ 0 & q_{3,2} & 1 & 0 & & & & & & & & & \\ 0 & q_{4,2} & 0 & 1 & q_{4,5} & 0 & 0 & & & & & & \\ & & & 0 & q_{6,5} & 1 & 0 & & & & & & \\ & & & 0 & q_{7,5} & 0 & & q_{7,8} & 0 & 0 & & & \\ & & & & & & 0 & q_{9,8} & 1 & 0 & & & \\ & & & & & & 0 & q_{10,8} & 0 & 1 & q_{10,11} & & 0 \\ & & & & & & & & & 0 & q_{12,11} & & 1 \end{bmatrix}.
$$

(4.15)

One can rewrite Eq. (4.10) for the degree-two quadrilateral in the form:

$$
P_3(x, y) = z * Q_1(x, y)
$$

(4.16)

and thus in the form

$$
[a_1 \quad a_2 \quad \cdots \quad a_{12}] = [z_1 \quad z_2 \quad \cdots \quad z_{12}]
$$
$$
\times \left[\begin{bmatrix} [R, 3, 3] \\ [S, 1, 3] \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & x_{12} \\ y_1 & y_1 & \cdots & y_{12} \\ z_1 & z_2 & \cdots & z_{12} \end{bmatrix} \right]'.
$$

(4.17)

(0,1) 1

here a = 2/√3̄

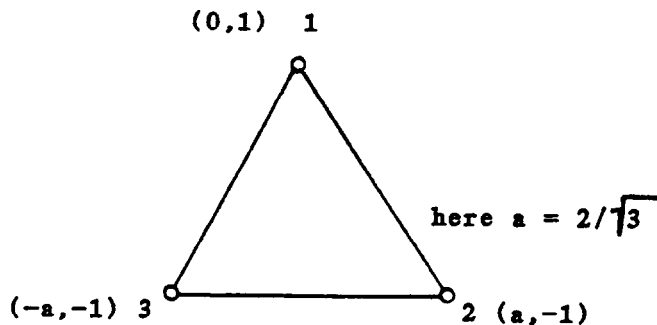**(-a,-1) 3**   **2 (a,-1)**

Fig. 7. A simple equilateral triangle.

As before, one can invert and solve for the twelve components of the $[a]$ vector and then construct the fitting function

$$W_i(x, y) = \frac{[a_1 \quad a_2 \quad \cdots \quad a_{10}][P, 3, 2]\begin{bmatrix} x \\ y \end{bmatrix}}{[1 - a_{11} - a_{12}][P, 1, 2]\begin{bmatrix} x \\ y \end{bmatrix}}. \tag{4.18}$$

The extension to order two functions over five- and six-sided polygons follows with no additional problems.

5. NUMERICAL APPLICATIONS AND COMPARISONS

In order to demonstrate the use of the method and validate the theory a series of numerical examples will be presented and comparisons made with the analytic-geometric constructions of the unit wedge functions for several of the elements presented in [3].

The first case considered is that of the centrally located equilateral triangle as shown in Fig. 7. For this case [with $a = 2/\text{SQRT}(3)$] one has for the wedge which is 1 at point 1:

$$[a_1 \quad a_2 \quad a_3] = [z_1 \quad z_2 \quad z_3]\left[[R, 1, 3]\begin{bmatrix} x_1 & x_2 & \cdots & x_8 \\ y_1 & y_1 & \cdots & y_8 \\ z_1 & z_2 & \cdots & z_8 \end{bmatrix}\right]^I$$

$$= [1 \quad 0 \quad 0]\left[\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & a & -a \\ a & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix}\right]^I$$

$$= [0.5 \quad 0 \quad 0.5] \tag{5.1}$$

and the wedge for point 1 is

$$W_1(x, y) = [a][P, 1, 2]\begin{bmatrix} x \\ y \end{bmatrix} = [a]\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}_{k=1,2}^{\pi} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= 0.5 + 0.5y. \tag{5.2}$$

This is the expected form shown in Fig. 8.

The next case is that of the degree-two wedge over a triangle. The same triangle, Fig. 7, is used with three additional points, one at the midpoint of each side. For the wedge that is
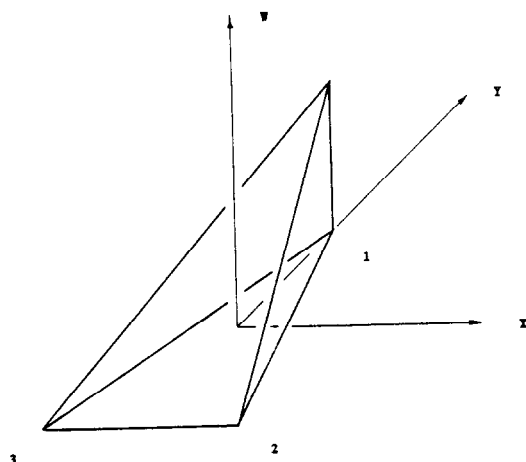
Fig. 8. The degree-one wedge.

unity at the first point one will have

$$[a_1 \quad a_2 \quad \cdots \quad a_6]$$

$$= [z_1 \quad z_2 \quad \cdots \quad z_8] \left[[R, 2, 3] \begin{bmatrix} x_1 & x_2 & \cdots & x_8 \\ y_1 & y_1 & \cdots & y_8 \\ z_1 & z_2 & \cdots & z_8 \end{bmatrix} \right]^I$$

$$= [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0] \left[\begin{matrix} [R, 2, 3] \\ [S, 1, 3] \end{matrix} \begin{bmatrix} 0 & a/2 & a & 0 & -a & -a/2 \\ 1 & 0 & -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right]$$

$$= [0 \quad 0 \quad 0.5 \quad 0 \quad 0.5 \quad 0]. \tag{5.3}$$

The general degree-two wedge for point 1 is, therefore,

$$W_1(x, y) = \frac{[0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0]}{2} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}_{k=1,2}^{\pi} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$= (y + y^2)/2. \tag{5.4}$$

There are two interesting points to be made concerning the six-point, degree-two triangle. First, one does not have to add any quadratically consistent extra points along the sides, because the denominator, $Q_0(x, y)$ is 1. Second, if one supplies a z vector with linear variation along each side (e.g. z goes from 1 to zero along side 1, 2 and along side 1, 3) then one finds

$$[a_1 \quad a_2 \quad \cdots \quad a_6] = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0] \left[[P, 2, 3] \right.$$

$$\times \left. \begin{bmatrix} 0 & a/2 & a & 0 & -a & -a/2 \\ 1 & 0 & -1 & -1 & -1 & 0 \end{bmatrix} \right]^I$$

$$= [1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]/2 \tag{5.5}$$

and one has, after constructing this quadratic fit to the linear wedge,

$$W_1(x, y) = \frac{[1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]}{2} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}_{k=1,2}^{\pi} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= (1 + y)/2. \tag{5.6}$$

as in Equation (5.3). One finds that the degree-one wedge is treated automatically as a subcase of the degree-two wedge in the case of a triangle and in fact for the general convex polygon.

The next case considered is that of a simple degree-one function over a convex quadrilateral. The example is that of [3, p. 40, Fig. 2.7] and is shown in Fig. 9. The $[a_1 \cdots a_8]$ vector for the $[z_1 \cdots z_8]$ vector which is nonzero at only the $(0, 0)$ corner node is

$$[a_1 \cdots a_8] = [z_1 \quad z_2 \quad \cdots \quad z_8] \begin{bmatrix} [R, 2, 2] \\ [S, 1, 2] \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & z_1 \\ y_1 & y_1 & \cdots & y_1 \\ z_1 & z_2 & \cdots & z_8 \end{bmatrix}'.$$

$$= [1 \quad 0.5 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.5] \begin{bmatrix} [R, 2, 3] \\ [S, 1, 3] \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 & 1.75 & 1.5 & 1 & 0.5 & 0.25 \\ 0 & 0 & 0 & 0.5 & 1 & 0.875 & 0.75 & 0.375 \\ 1 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}'. \qquad (5.7)$$

Solving Equation (5.7) with a simple computer code, Ref. 5 gives

$$W_1(x, y) = (1 - 0.1x - 0.85y - 0.2x^2 - 0.4y^2 + 0.7xy/(1 + 0.4x + 1.35y). \qquad (5.8)$$

If one starts with Wachspress's equation for $W_1(x, y)$ below Fig. 2.7, pg. 40, one finds

$$W_1(x, y) = (5 + 2x - 8y)(4 - 2x - y)/(20 + 8x - 17y)$$
$$= (20 - 2x - 27y - 4x2 - 8y2 + 14xy)/(20 + 8x - 17y). \qquad (5.9)$$

This is easily shown to be identical to Equation (5.8).

Next considered will be the degree-two wedges over the quadrilateral, in this case a trapezoid as shown in Fig. 6.4 of [3] and Fig. 10.

As noted in Sec. 3, one must add two additional quadratically interpolated points between the other three on each side. First, interpolating quadratically along the side 1, 8, 4 one finds

$$[q_1 \quad q_2 \quad q_3] = [1 \quad 0.5 \quad 0] | [P, 2, 1][0 \quad 0.5 \quad 1]|'$$
$$= [1 \quad -3 \quad 2]. \qquad (5.10)$$

Now calculating an interpolated z value at a point halfway between points 1 and 5 one finds

$$z_9 = [1 \quad -3 \quad 2][P, 2, 1][0.75] = 0.375. \qquad (5.11)$$
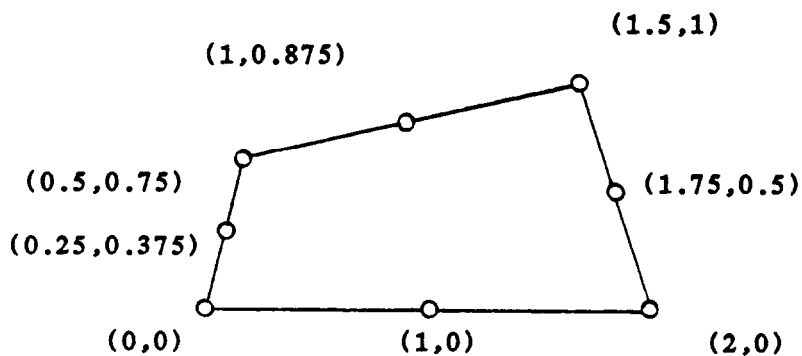
Note that the 4 side-node values are
derived by linear interpolation.

Fig. 9. A simple degree-one quadrilateral.

The z_{11} and the z_{12} values will both be zero and z_{10}, halfway between points 4 and 8 will be calculated from

$$z_{10} = [1 \quad -3 \quad 2][P, 2, 1][0.25].$$ (5.12)

In Fig. 10, the additional four-point locations are shown.

The wedge for degree-two approximation, that is 1 at point 1 and ϕ at the other 7 nodes will now be generated. The vector of fitting coefficients for this problem is

$$
\begin{aligned}
[a_1 \quad a_2 \quad &\cdots \quad a_{12}] \\
= [1 \quad 0 \quad &-0.125 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -0.125 \quad 0] \\
\times &\begin{bmatrix} [R, 3, 3] \\ [S, 1, 3] \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & -1 & 0.5 & 1 & 0.5 \\ -1 & 1 & 2 & -2 & 0 & 1.5 & 0 & -1.5 & -.5 & 1.25 & -1 & -1.75 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.375 & 0 & 0 & -.125 \end{bmatrix}' \\
= [-0.375 \quad &-0.125 \quad 0 \quad 0.375 \quad 0.16667 \quad 0.16667 \quad 0.125 \quad 0 \quad -0.16667 \quad -0.16667 \quad -0.3333 \quad 0].
\end{aligned}
$$ (5.13)

Thus one has

$$W_1(x, y) = (-0.375 - 0.125x + 0.375x^2 \cdots -0/16667xy^2)/(1 + 0.3333x).$$ (5.14)

Examination of Eq. 6.6 of [3] yields

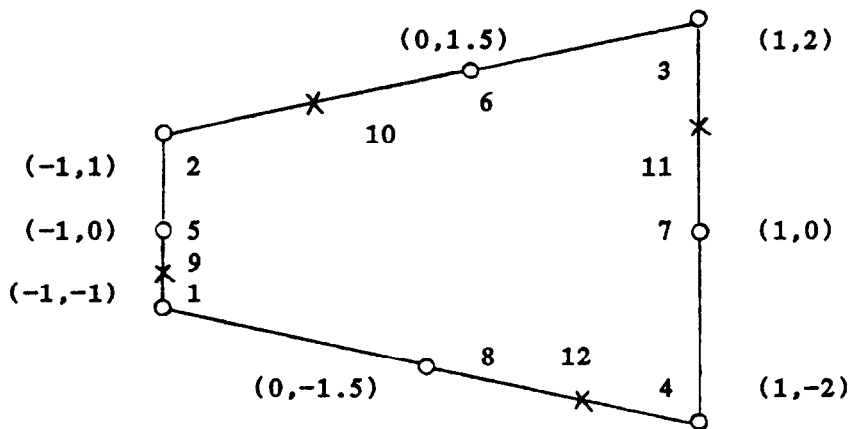$$W_1(x, y) = -3(1 - x)(x + 3 - 2y)(1 + x - 2y/3)/(8(3 + x)),$$ (5.15)

shows that $a_1 = -3 * 3 * 1/(8 * 3) = -0.375$. Similar agreement is obtained for the other coefficients.

For z nonzero ($= 1$) at side-node 6 one interpolates with

$$
\begin{aligned}
[q_2 \quad q_6 \quad q_3] &= [0 \quad 1 \quad 0][[P, 2, 1][0 \quad 0.5 \quad 1]]' \\
&= [0 \quad 4 \quad -4].
\end{aligned}
$$ (5.16)

Interpolated values are calculated for the $1/4$ and $3/4$ points using

$$
\begin{aligned}
[z_9 \quad z_{12}] &= [0 \quad 4 \quad -4][P, 2, 1][0.25 \quad 0.75] \\
&= [0.75 \quad 0.75].
\end{aligned}
$$ (5.17)

o indicate nodes with given values
x indicate nodes with interpolated values

Fig. 10. A degree-two quadrilateral.

It would seem that the stage is now set to solve for the ten coefficients that determine $P_3(x, y)$ and the two additional coefficients in the denominator $Q_1(x, y)$ of the degree-two wedge:

$$W_6(x, y) = P_3(x, y)/Q_1(x, y) \tag{5.18}$$

over the quadrilateral of Fig. 10. However, if one tries to solve Equations (3.16) one finds that the general inverse of $[A]$ times $[A]$ is not equal to the identity matrix $[I]$, and one must conclude that the problem is ill conditioned. It is known from determination of the 12 coefficients for the unit corner-wedge, Eq. (5.13), that $a_{11} = -0.3333$ and $a_{12} = 0$.

A simple resolution of this dilemma is obtained by reducing $Q_1(x, y)$ to $(1 - a_{11}x)$ by the simple expedient of removing the eleventh line $(1 \quad 0 \quad 1)$ from the power matrix and solving for $[a]$ less a_{12}. A more complete resolution is obtained by the use of the previous corner-node solution to specify all of the unknown coefficients in $Q_1(x, y)$. Thus one can rephrase the problem as

$$[a_1 \quad a_2 \quad \cdots \quad a_{10}][P, 3, 2] \begin{bmatrix} x_1 & \cdots & x_{12} \\ y_1 & \cdots & y_{12} \end{bmatrix}$$
$$= [1 - a_{11} - a_{12}][P, 1, 3] \begin{bmatrix} x_1 & \cdots & x_{12} \\ y_1 & \cdots & x_{12} \\ z_1 & \cdots & z_{12} \end{bmatrix}. \tag{5.19}$$

One then only solves for a_1 through a_{10}.

The basic flaw in the degree-one quadrilateral unit wedge for a lateral node is that nonzero information is given only along a single line in (x, y) space and this is not sufficient to determine the denominator $Q_1(x, y)$ which is the general equation of a line in (x, y) space. Taking the approach of excluding a_{12} one has

$$[a_1 \quad a_2 \quad \cdots \quad a_{11}] = [0.5 \quad 0.16667 \quad -0.3333 \quad -0.5 \quad 0 \quad 0$$
$$-0.16667 \quad 0 \quad 0.3333 \quad 0 \quad -0.3333]. \tag{5.20}$$

These coefficients are in exact agreement with the terms of $W_6(x, y)$ of [3, p. 182, Eq. 6.6]. It should be mentioned in passing that the method works equally well for the more general quadrilateral and is in no way restricted to the trapezoid.

Wachspress points out that the theoretical key to resolving this dilemma is to require that $Q(x, y)$ pass through all of the EIP's for which the interpolation data do not yield double values for the interpolant.

Two additional degree-one functions, over the plane pentagon and hexagon well be presented as a conclusion to the plane figures. The degree-one function for a pentagon is given by

$$W(x, y) = P_3(x, y)/Q_2(x, y). \tag{5.21}$$

Here the points are given on four corner plus two linearly interpolated points on each side. The four linearly related points along each side assure that the function $P_3(x, y)/Q_2(x, y)$ will in fact reduce to a degree-one function along each side.

Recalling the problem encountered with the quadrilateral when z values were nonzero along one line it is not surprising to find that the pentagon, which has a degree-two denominator, $Q_2(x, y)$, cannot be determined just by nonzero z values specified along only two lines. One may impose side conditions on $Q(x, y)$ which yield Q(all EIP's) $= 0$ as suggested by Wachspress. Alternatively one may specify z_i values that are nonzero and linear on three sides around the pentagon. A toothed set of z's works very well (since this results in double values at the five EIP's) to determine all of the a_i's including the a_{11} to a_{15} which govern the denominator, $Q_2(x, y)$. For the general five-sided polygon one can evaluate the $Q_2(x, y)$ in this manner, and then evaluate the $P_3(x, y)$ coefficients for the particular set of five z values desired. For the

equilateral pentagon it was sufficient to restrict the denominator to have only four of the five possible terms in $Q_2(x, y)$, excluding any term except the x or y squared terms. This is because the $Q_2(x, y)$ is uniquely determined by the circle containing the exterior intersection points. Thus one will have

$$Q_2(x, y) = 1 - (x^2 + y^2)/(r^2), \tag{5.22}$$

where r is the radius of the circle through the external intersection points.

The hexagon is treated in exactly the same fashion with no further difficulties and seems to warrant no further comment.

The degeneracy of the linear algebraic calculation of the fitting coefficients or the $[a]$ vector that contains the coefficients of $P_{N+T-2}(x, y)$ and $Q_{N-3}(x, y)$ in the expression

$$W_i(x, y) = P_{N+T-2}(x, y)/Q_{N-3}(x, y) \tag{5.23}$$

can be better understood if one considers a specific case, such as the degree-one quadrilateral with z values specified at the four vertices. The general equation for the z value at any point (x, y) is

$$P_2(x, y) = zQ_1(x, y) \tag{5.24}$$

then

$$a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy = z(1 - a_7 x - a_8 y). \tag{5.25}$$

If z happens to have some simple functional dependence on x and or y, then one can solve the system only for certain combinations of coefficients. For example, if $z = x$, then one can only find $(a_4 + a_7)$ and $(a_6 + a_8)$. In this case there is no multiplicity so one can choose $a_7 = a_8 = 0$. There are only six independent coefficients.

One way out of this dilemma is to use a set of z values that are unlikely to belong to a simple function of (x, y). For example, one can use z values that are alternately $+1$ and -1 at points around the polygon. This set may then be used to determine a full set of eight values for the $[a]$ vector. For the extremely unlikely case of this $+/-1$ choice leading to a single value at an EIP, and thus to a nonunique $Q(x, y)$ one could likely use randomly selected values between $+1$ and -1 in place of the simple $+/-1$ values at the nodes.

Then one takes the two coefficients, a_7 and a_8, which govern the denominator $Q_2(x, y)$ which is in turn governed by the $N - 2$ external intersection points which remain fixed and independent of z values used. After a_7 and a_8 are found using a set of $+1$ or -1 z values to calculate the $[a]$ for a general set of z values one need only determine the six values, $[a_1 \quad a_2 \quad \cdots \quad a_6]$. This can be done using

$$[a_1 \quad a_2 \quad \cdots \quad a_6] \left[[P, 2, 2] \begin{bmatrix} x_1 & x_2 & \cdots & x_8 \\ y_1 & y_1 & \cdots & y_8 \end{bmatrix} \right] = \tag{5.27}$$

$$[1 - a_7 - a_8][P, 1, 2] \begin{bmatrix} x_1 & x_2 & \cdots & x_8 \\ y_1 & y_1 & \cdots & y_8 \end{bmatrix}.$$

This can be solved by post multiplying by the general inverse of

$$\left[[P, 2, 2] \begin{bmatrix} x_1 & x_2 & \cdots & x_8 \\ y_1 & y_1 & \cdots & y_8 \end{bmatrix} \right].$$

The reconstruction of $z(x, y)$ follows as before. The same procedure is followed for the pentagon, hexagon etc.

6. EXTENSIONS OF THE WORK

Two classes of problems remain that want for numerical investigation. The first is the polygon with one, or more, curved conic sides, called a polycon. Preliminary numerical investigations indicate that the advantage of allowing a curved side on the element introduces (resolvable) theoretical and computational difficulties and such elements are recommended only where a rigorous representation of a curved boundary is required.

The second really challenging problem is that of the construction of a three-dimensional element that is bounded by eight corners, roughly forming a rectangular parallelepiped, with edges that are straight lines and with faces that will be, in particular, quadric surfaces (hyperboloids of one sheet). This "warped" parallelepiped occurs when one attempts to subdivide a three-dimensional space that has nonrectangular boundaries using points and straight lines rather than the more cumbersome planes. The surfaces may also be further warped by the relaxation of the location of grid points as seen in [7] to give reasonably sized and shaped elements while recognizing and avoiding the difficulty of maintaining plane surfaces during relaxation. From one point of view one can project any of these warped surfaces onto any convenient plane, roughly parallel to the surface. The surface is now the simple function

$$z = P_2(x, y)/Q_1(x, y) \tag{6.1}$$

in an (x, y) coordinate system in the plane of projection. From an alternate view point, Wachspress suggests that these surfaces may be closely related to the Steiner surfaces discussed by McLeod[8].

REFERENCES

1. C. Lecot, Spatial Convergence Properties of Some Characteristic Methods for the SN Equations in (x, y)-Geometry. *Proceedings of the ANS Topical Meeting on Advances in Reactor Computations, Salt Lake City, Utah,* March 28–31, 1983.
2. R. Patternoster, A Linear Characteristic-Nodal Transport Method for the Two-Dimensional (x, y)-Geometry Multigroup Discrete Ordinates Equations Over An Arbitrary Traigle Mesh. Ph.D. Dissertation, Department of Nuclear Engineering Sciences, University of Florida, Gainesville, 1983.
3. E. Wachspress, *A Rational Finite Element Basis.* Academic Press, New York (1975).
4. R. E. Cline, Elements of the Theory of the Generalized Inverses of Matrices, edc/umap, 55 Chapel St., Newton, Mass., 02160, 1979.
5. G. R. Dalton, SUPERPOLY A Generalized Polynomial Fitting Program. Dept. of Nuc. Eng. Sci., U. of Florida, Gainesville, Fl., 32611, 1983.
6. F. Scheid, *Theory and Problems of Numerical Analysis, Shawm's Outline Series.* McGraw Hill, New York (1968).
7. M. A. Yerry and M. S. Shephard, A Modified Quadtree Approach To Finite Element Mesh Generation, Computer Graphics. IEEE, Los Alamitos, Ca., pg. 39–46, January/February 1983.
8. R. J. Y. McLeod, The Steiner surface revisited. *Proc. R. Soc. London* **A369**, 157–174 (1979).