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On the Converse of a Well-Known Fact about Krull Domains

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I. INTRODUCTION

Although Krull domains have been an interesting subject in commutative ring theory, not many characterizations of Krull domains are known compared to those of Prüfer domains. However, it is known that every proper divisorial ideal of a Krull domain R is a v-product of prime ideals of R; i.e., every proper divisorial ideal I of R can be expressed in the form $I=(P_1 \cdots P_n)_v$ for some prime ideals $P_1, ..., P_n$ of R. In 1968, S. Tramel [22] proved that the converse of the above fact is true. (In fact, he proved a stronger result: If every proper principal ideal of a domain R is a v-product of prime ideals, then R is a Krull domain.) This was also proved by Nishimura [18] under the additional condition that the expression $I=(P_1 \cdots P_n)_v$ is unique. As Levitz showed in [17], by Tramel's result, we can easily solve Aubert's problem [6], obtaining in fact a stronger result. If every proper principal ideal of a domain R is a t-product of prime ideals of R, then R is a Krull domain.

In this paper we give a new characterization of a Krull domain: If every nonzero prime ideal of a domain R contains a t-invertible prime ideal, then R is a Krull domain. (Note that this is the corresponding part for Krull domains to the well-known fact about unique factorization domains (UFD): If every nonzero prime ideal of a domain R contains a nonzero principal prime ideal, then R is a UFD.) Then using this result, we can easily prove that the following statements are equivalent. (Recall that R is a π-domain if every principal ideal of R can be written as a product of prime ideals.)

(1) R is a UFD (resp., π-domain, Krull domain).

(2) Every t-ideal is principal (resp., invertible, t-invertible).

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(3) Every prime t-ideal is principal (resp., invertible, t-invertible).

(4) Every minimal prime ideal of a nonzero principal ideal is principal (resp., invertible, t-invertible).

(5) Every nonzero prime ideal contains a nonzero principal (resp., invertible, t-invertible) prime ideal.

(6) Every proper t-ideal is a t-product of principal (resp. invertible, t-invertible) prime ideals.

(7) Every proper principal ideal is a t-product of principal (resp., invertible, t-invertible) prime ideals.

Note that in order to get the corresponding statements for π-domains (resp., Krull domains), we just replace the term 'ideals' by 't-ideals' and the descriptive adjective 'principal' by 'invertible' (resp., 't-invertible') in the corresponding statements for UFDs.

It is obvious that every minimal prime ideal of a UFD (resp., π-domain, Krull domain) is principal (resp., invertible, t-invertible). However, it will be shown that the converse is not true. But if we add the condition that R satisfies Krull's principal ideal theorem (Every minimal prime of a nonzero principal ideal is a minimal prime ideal), then we can show that R is a Krull domain. In addition to these results, we give several new characterizations of π-domains. In particular, it will be shown that R is a π-domain if and only if R is a finite conductor Mori domain such that $((a)\cap(b))((c)\cap(d))=(ac)\cap(ad)\cap(bc)\cap(bd)$ for all elements a, b, c, d of R. Throughout this paper R will be an integral domain with quotient field K. The term 'finitely generated' will be abbreviated to 'f.g.' For the undefined terms and notation, the reader is referred to [14], [9], and [16].

II. Preliminary Results

The reader is reminded that throughout this paper R will be an integral domain with quotient field K. An R-submodule A of K is called a fractional ideal of R if $dA \subseteq R$ for some nonzero element d of R. $\mathscr{F}(R)$ will denote the set of nonzero fractional ideals of R. For $A \in \mathscr{F}(R)$, the v-operation is defined by $A_v = (A^{-1})^{-1}$ where $A^{-1} = \{x \in K \mid xA \subseteq R\}$. The t-operation is defined by $A_t = \bigcup\{(A_0)_v \mid A_0 \in \mathscr{F}(R), A_0 \subseteq A$, and A_0 is finitely generated$\}$. $A \in \mathscr{F}(R)$ is called a divisorial ideal or a v-ideal (resp., t-ideal) if $A_v = A$ (resp., $A_t = A$). A domain R is called a Mori domain if it satisfies the ascending chain condition (ACC) on divisorial ideals of R. A maximal divisorial ideal (maximal t-ideal) of a domain R is a proper divisorial ideal (proper t-ideal) of R which is maximal among proper divisorial ideals (proper t-ideals) of R. We denote the set of maximal divisorial ideals (maximal t-ideals) of R by $D_m(R)$ (t-Max(R)). By Zorn's Lemma, it is easy to see that $t\text{-Max}(R) \neq \phi$, while $D_m(R)$ can be an empty set. But if R is a Mori domain, then $D_m(R) \neq \phi$. Specifically, in a Mori domain, every proper v-ideal is contained in a maximal v-ideal.

We collect for ease of reference the following well-known results.

THEOREM 2.1. *Let R be an integral domain.*

(1) *The following are equivalent.*

(a) *R is a Mori domain.*

(b) *For every nonzero ideal I of R, there exists a* f.g. *ideal $I_0 \subseteq I$ such that $I_v = (I_0)_v$.*

(c) *If $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ is a descending sequence of fractional v-ideals such that $\bigcap_{n=1}^{\infty} F_n \neq 0$, then $\bigcap_{n=1}^{\infty} F_n = F_m$ for some m.*

(2) *Let P be a divisorial prime ideal of a Mori domain R. Then $P = (a:b)$ for some $a, b \in R$, where $(a:b) = \{x \in R \mid xb \in (a)\}$.*

(3) *Let I be an integral v-ideal of a Mori domain R. Then*

(a) *If I is a maximal divisorial ideal of R, then I is a prime ideal.*

(b) *There are only finitely many maximal v-ideals containing I.*

(4) *Let S be a multiplicatively closed subset of a Mori domain R. Let I be a nonzero* f.g. *ideal of R. Then $(I_S)_v = (I_v)_S$.*

(5) *If S is a multiplicatively closed subset of a Mori domain R, then R_S is a Mori domain.*

Proof. (1) Use I, Théorème 1 from [20]. (2) Use II, Théorème 1 from [21]. (3) These statements follow from Proposition 2.1 and Proposition 2.2, respectively, of [24]. (4) This result is well known and easily follows from (1) and Lemma 4 of [23]. (5) This is [19, Corollary 3].

A nonzero ideal I of R is said to be t-invertible if $(II^{-1})_t = R$. We prove the analogous result to [9, Theorem 7.6] for t-invertible prime ideals, which will be used in Section IV. We denote the multiplicatively closed subset $\{f \in R[X] \mid f \neq 0 \text{ and } (A_f)_v = R\}$ of $R[X]$ by N_v.

THEOREM 2.2. *Let P be a t-invertible prime ideal of an integral domain R such that $P_t \neq R$. Then*

(1) *P is a finite type divisorial ideal.*

(2) *$\{(P^n)_v\}_{n=1}^{\infty}$ is the set of all P-primary ideals of R.*

(3) *Suppose that $\bigcap_{n=1}^{\infty} P^n \neq 0$. Then $(\bigcap_{n=1}^{\infty} P^n)_v$ is a prime ideal of R. Moreover $(\bigcap_{n=1}^{\infty} P^n)_v = \bigcap_{n=1}^{\infty} (P^n)_v$.*

(4) *If A is a t-invertible ideal of R properly containing P, then $A_t = R$.*

Proof. (1) Let P be a t-invertible prime ideal of R. For an ideal I of R, I^e will denote the ideal $I[X]_{Nv}$. So $R^e = R[X]_{Nv}$. By [14, Corollary 2.5], P^e is invertible. And P^e is a prime ideal of R^e since $P_t \neq R$, i.e., $P[X] \cap N_v = \phi$. Since P^e is invertible, $P^e = (P^e)_v = (P_v)^e$ by [14, Proposition 2.2]. Hence $P = P^e \cap R = (P_v)^e \cap R = P_v$ by [14, Proposition 2.8]. The first of the above equalities follows from the fact that P^e is a prime ideal of R^e. Thus P is a divisorial ideal, and P is of finite type since P is t-invertible [13, Chapter I, Section 4, Theorem 8]. Thus (1) is proved.

(2) Let Q be a P-primary ideal of R. Then $Q[X]_{Nv}$ is a $P[X]_{Nv}$-primary ideal of $R[X]_{Nv}$. By [9, Theorem 7.6], $Q^e = (P^e)^k$ for some $k \in \mathbb{N}$, since P^e is invertible. Since $(P^e)^k$ is invertible, $(P^e)^k$ is a v-ideal. Hence $(P^e)^k = ((P^e)^k)_v = ((P^k)^e)_v = ((P^k)_v)^e$, where the last equality follows from [14, Proposition 2.2]. Now $Q = Q^e \cap R$ since $Q[X]$ is a $P[X]$-primary ideal such that $P[X] \cap N_v = \phi$. Hence $Q = Q^e \cap R = (P^e)^k \cap R = ((P^k)_v)^e \cap R = (P^k)_v$, where the last equality follows from [14, Proposition 2.8]. Thus $Q = (P^k)_v$ for some $k \in \mathbb{N}$. We complete the proof of (2) by showing that each $(P^k)_v$ is a P-primary ideal. By [9, Theorem 7.6], $(P^k)^e$ is a P^e-primary ideal. Hence $(P^k)_v = (P^k)^e \cap R$ is a $P^e \cap R$-primary ideal. Hence $(P^k)_v$ is a P-primary ideal.

(3) Assume that $\bigcap_{k=1}^{\infty} P^k \neq 0$. Then $\bigcap_k (P^e)^k \neq 0$. Hence $\bigcap_k (P^e)^k$ is a v-ideal since it is an intersection of v-ideals. It is easy to see that $\bigcap_k (P^e)^k = ((\bigcap_k P^k)_v)^e$, i.e., $\bigcap_k (P^k[X]_{Nv}) = (\bigcap_k P^k)_v [X]_{Nv}$. And $\bigcap_k (P^e)^k$ is a prime ideal of R^e by [9, Theorem 7.6]. Hence $(\bigcap_k P^k)_v = ((\bigcap_k P^k)_v)^e \cap R = \bigcap_k (P^e)^k \cap R$ is a prime ideal of R. Moreover $\bigcap_k (P^k)^e = \bigcap_k ((P^k)_v)^e$. So $((\bigcap P^k)_v)^e = \bigcap_k ((P^k)_v)^e$. Hence, contradicting back to R, we get $(\bigcap_k P^k)_v = \bigcap_k (P^k)_v$ by [14, Propositon 2.8].

(4) Let A be a t-invertible ideal of R properly containing P. Then $A^e \supsetneq P^e$ are invertible ideals of R^e. By [9, Theorem 7.6], either $A^e = P^e$ or $A^e = R^e$. We will show that $A^e \neq P^e$ so that $A^e = R^e$. Suppose $A^e = P^e$. Then $A \subseteq A^e \cap R = P^e \cap R = P$, which contradicts $P \nsubseteq A$. Therefore $A^e = R^e$, i.e., $A_t = R$.

III. Krull Domains

A domain R is called a Krull domain if R has a nonempty collection of prime ideals $\{P_\alpha\}$ such that $R = \bigcap R_{P_\alpha}$, each R_{P_α} is a principal ideal domain (PID), and every nonzero element of R is contained in only finitely many P_αs.

It is well known that a domain R is completely integrally closed if and only if every nonzero ideal is v-invertible, i.e., $(AA^{-1})_v = R$ for any nonzero ideal A of R ([9, 34.3]). Recall that a domain R is a Prüfer v-multiplication domain (PVMD) if every nonzero f.g. ideal is t-invertible, i.e., $(AA^{-1})_t = R$

for every nonzero f.g. ideal A of R. A domain R is an essential domain if $R=\bigcap_{P\in\Lambda} R_P$ where $\Lambda\subseteq \mathrm{Spec}(R)$ and each R_p, $P\in\Lambda$, is a valuation domain. A domain R is called a v-domain if $[A_v :_K A_v]=R$ for every nonzero f.g. ideal A of R, where K is the quotient field of R.

LEMMA 3.1. *An essential domain is a v-domain.*

Proof. Let $R=\bigcap_{P\in\Lambda} R_P$, where $\Lambda\subseteq \mathrm{Spec}(R)$ and each R_P, $P\in\Lambda$, is a valuation domain. Let A be a nonzero finitely generated ideal of R. Let $x\in [A_v :_K A_v]$. Then $xA_v\subseteq A_v$. Now for each $P\in\Lambda$, $x(A_v)_P\subseteq (A_v)_P$. Hence $x((A_v)_P)_v\subseteq ((A_v)_P)_v$. By [14, Lemma 3.4(2)], $(A_P)_v=((A_v)_P)_v$. Hence $((A_v)_P)_v$ is a principal ideal since R_P is a valuation domain. So $x\in R_P$. So $x\in\bigcap_{P\in\Lambda} R_P=R$. Therefore $[A_v :_K A_v]=R$ and hence R is a v-domain.

In the class of Mori domains, some subclasses of domains are identical with the class of Krull domains.

THEOREM 3.2. *The following are equivalent for an integral domain R.*

(1) *R is a Krull domain.*

(2) *R is a completely integrally closed Mori domain.*

(3) *R is a* PVMD *and R is a Mori domain.*

(4) *R is an essential Mori domain.*

(5) *R is a v-domain and R is a Mori domain.*

Proof. (1) ⇒ (2). This is [7, Theorem 3.6].

(2) ⇒ (3). Let A be a f.g. nonzero ideal of R. Then $(AA^{-1})_v=R$ since R is completely integrally closed. Hence $(AA^{-1})_t=R$ since R is a Mori domain. Hence R is a PVMD.

(3) ⇒ (4). Every PVMD is an essential domain by [14, Theorem 3.3] or by [11, Proposition 4 and Theorem 5].

(4) ⇒ (5). This implication follows from Lemma 3.1.

(5) ⇒ (2). Let A be a nonzero ideal of R. Then by Theorem 2.1(1), $A_v=(A_0)_v$ for some f.g. ideal $A_0\neq 0$ of R since R is a Mori domain. Now $[A_v :_K A_v]=[(A_0)_v :_K (A_0)_v]=R$ since R is a v-domain. Hence by [9, Theorem 34.3], R is completely integrally closed.

(2) ⇒ (1). By [14, Proposition 2.8(3)] or by [11, Proposition 4], $R=\bigcap_{M\in\Lambda} R_M$, where $\Lambda=t\text{-Max}(R)$. Since R is a Mori domain, $t\text{-Max}(R)=D_m(R)$. And the intersection is locally finite by Theorem 2.1(3). Each R_M is a Mori domain by Theorem 2.1(5). By [11, Theorem 5], R_M is a valuation domain since R is a PVMD by the implication (2) ⇒ (3). Hence each R_M, $M\in D_m(R)$, is a Mori valuation domain, i.e., a PID. Hence R is a Krull domain.

Remark. The implication (2)$\Rightarrow$(1) in the previous theorem can be found in [7, Corollary 3.13]. But we proved this noting that a completely integrally closed Mori domain is a Mori PVMD.

To prove the main result of this section, we need a couple of lemmata.

LEMMA 3.3. *Let R be an integral domain. If every nonzero prime ideal of R contains a t-invertible prime ideal, then there exists a nonempty collection Λ of minimal prime ideals of R such that $R = \bigcap_{P \in \Lambda} R_P$ and each R_P, $P \in \Lambda$, is a* PID.

Proof. Let M be a maximal t-ideal of R. Let $Q = P_M$ be a nonzero prime ideal of R_M, where P is a nonzero prime ideal of R. Then by assumption, P contains a t-invertible prime ideal P'. But P'_M is a principal (prime) ideal by [14, Corollary 2.7]. Thus $Q = P_M$ contains the nonzero principal prime ideal P'_M. Hence every nonzero prime ideal of R_M contains a principal prime ideal, hence R_M is a UFD. Let $\Lambda_M = \{P \in \mathrm{Spec}(R) \mid P \in X^{(1)}(R) \text{ and } P \subseteq M\}$, where $X^{(1)}(R)$ is the set of minimal prime ideals of R. Then $X^{(1)}(R_M) = \{P_M \mid P \in \Lambda_M\}$. Hence $\Lambda_M \neq \phi$. Now R_M is a Krull domain since it is a UFD. Hence as we showed in the proof of Theorem 3.2,

$$R_M = \bigcap_{Q \in X^{(1)}(R_M)} (R_M)_Q = \bigcap_{P \in \Lambda_M} (R_M)_{P_M} = \bigcap_{P \in \Lambda_M} R_P.$$

By [14, Proposition 2.8(3)], $R = \bigcap_{M \in \Gamma} R_M$, where $\Gamma = t\text{-Max}(R)$. Hence $R = \bigcap_{P \in \bigcup \Lambda_M} R_P$, and $\bigcup_M \Lambda_M \subseteq X^{(1)}(R)$.

Let $I, I_1, ..., I_n$ be nonzero ideals of R. If $I = (I_1 \cdots I_n)_t$, then I is called the t-product of $I_1, ..., I_n$.

LEMMA 3.4. *If every nonzero prime ideal of a domain R contains a t-invertible prime ideal, then every proper principal ideal of R is a t-product of prime ideals of R.*

Proof. Let S be the set of all nonzero non-units of $D = R[X]_{N_v}$ ideals generated by which are products of principal prime ideals of D. We will show that S contains every nonzero non-unit of R. Suppose not and let a be a nonzero non-unit of R such that $a \notin S$. Since $S \cup \{\text{units of } D\}$ is a saturated set of D which does not contain a, $aD \cap S = \phi$. So there exists a prime ideal Q of D such that $Q \supseteq aD$ and $Q \cap S = \phi$. By assumption, $Q \cap R$ contains a t-invertible prime ideal, say P. By [14, Corollary 2.5 and Theorem 2.14], PD is principal, say $PD = fD$ for some $f \in D$. Now $f \in PD \subseteq Q \subseteq P \backslash S$, which contradicts $f \in S$. Hence S contains all nonzero non-units of R. So if a is a nonzero non-unit of R, then $aD = Q_1 \cdots Q_n$ for some principal prime ideals $Q_1, ..., Q_n$ of D. By assumption, for each $1 \leqslant k \leqslant n$, $Q_k \cap R$ contains a t-invertible prime ideal of R, say P_k. Then $P_k D$ is a principal prime ideal by the previous argument. So $P_k D \subseteq Q_k$ implies $P_k D = Q_k$. So $aD = Q_1 \cdots Q_n = P_1 D \cdots P_n D = (P_1 \cdots P_n)D$. By [14, Corollary 2.3(3)], $aD = (aD)_t = ((P_1 \cdots P_n)D)_t = (P_1 \cdots P_n)_t D$. Now $aR = (P_1 \cdots P_n)_t$ by [14, Proposition 2.8(1)]. Thus every proper principal ideal of R is a t-product of prime ideals of R.

THEOREM 3.5. *If every nonzero prime ideal of a domain R contains a t-invertible prime ideal, then R is a Krull domain.*

Proof. By Lemma 3.3, $R = \bigcap_{P \in \Lambda} R_P$ for a collection Λ of minimal prime ideals of R and each R_P, $P \in \Lambda$, is a PID. It remains to show that every nonzero nonunit element a of R is contained in only finitely many Ps in Λ. By Lemma 3.4, $(a) = (P_1 \cdots P_n)_t$ for some prime ideals $P_1, ..., P_n$ of R. Let P be a minimal prime of (a). Then $P \supseteq (P_1 \cdots P_n)_t \supseteq P_1 \cdots P_n$. So P contains some P_i and hence by the minimality of P, $P = P_i$. Hence there are only finitely many minimal primes of (a). Since Λ is a collection of minimal prime ideals of R, the conclusion easily follows.

Now we characterize Krull domains in terms of t-invertibility. In the next theorem, the implications $(1) \Rightarrow (2) \Rightarrow \cdots \Rightarrow (7)$ are almost trivial, and the only seemingly nontrivial implication $(7) \Rightarrow (1)$ will follow from the previous theorem.

THEOREM 3.6. *The following are equivalent for a domain R.*

(1) *R is a Krull domain.*

(2) *Every t-ideal is t-invertible.*

(3) *Every nonzero ideal is t-invertible.*

(4) *Every nonzero prime ideal is t-invertible.*

(5) *Every prime t-ideal is t-invertible.*

(6) *Every minimal prime ideal of a nonzero principal ideal is t-invertible.*

(7) *Every nonzero prime ideal contains a t-invertible prime ideal.*

Proof. $(1) \Rightarrow (2)$. By Theorem 3.2, R is a completely integrally closed Mori domain. Hence $(AA^{-1})_v = R$ for every nonzero ideal A of R. Therefore $(AA^{-1})_t = R$ since R is a Mori domain.

$(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$. This is clear.

$(5) \Rightarrow (6)$. This follows from [12, Proposition 1.1].

$(6) \Rightarrow (7)$. Let P be a nonzero prime ideal of R. Let $a \in P \backslash O$. There exists a prime ideal $P' \subseteq P$ minimal over (a). Hence P contains a t-invertible prime ideal.

(7) ⇒ (1). This follows from Theorem 3.5.

It is well known that every Krull domain satisfies Krull's principal ideal theorem and that every minimal prime ideal is t-invertible. In Theorem 3.6, we proved the converse. Thus we isolate this result as Theorem 3.7, which will enable us to prove many famous facts about Krull domains and π-domains as well as several new results.

THEOREM 3.7. *The following are equivalent for a domain R.*

(1) *R is a Krull domain.*

(2) *R satisfies Krull's principal ideal theorem and every minimal prime ideal is t-invertible.*

(3) *Every minimal prime ideal of a nonzero principal ideal is t-invertible.*

Proof. (1) ⇒ (2). This is well known.

(2) ⇒ (3). This is clear.

(3) ⇒ (1). This follows from Theorem 3.6.

The next lemma for the single variable case is known [8, Theorem 2.2]. In this paper, we do not need the multi-variable case. However, for future application, we prove it for the multi-variable case. We will use the methods introduced in [14].

LEMMA 3.8. *Let $\{X_\alpha\}$ be a set of indeterminates over a domain R and let $N_v = \{f \in R[\{X_\alpha\}] \mid (A_f)_v = R\}$. If R is a Krull domain, then $R[\{X_\alpha\}]_{Nv}$ is a principal ideal domain.*

Proof. Suppose R is a Krull domain. By Theorem 3.2, R is a PVMD. So every nonzero ideal J of $R[\{X_\alpha\}]_{Nv}$ is of the form $J = IR[\{X_\alpha\}]_{Nv}$ for some nonzero ideal I of R by [14, Theorem 3.1]. Since R is completely integrally closed by Theorem 3.2, $(II^{-1})_v = R$. In a Mori domain, the v-operation is the same as the t-operation. So $(II^{-1})_t = R$, which implies that I is t-invertible. Hence by [14, Corollary 2.5 and Theorem 2.14], $J = IR[\{X_\alpha\}]_{Nv}$ is principal and therefore $R[\{X_\alpha\}]_{Nv}$ is a PID.

Now we characterize Krull domains in terms of t-products of prime ideals.

THEOREM 3.9. *The following are equivalent for a domain R.*

(1) *R is a Krull domain.*

(2) *Every proper principal ideal is a t-product of (t-invertible) prime ideals.*

(3) *Every proper t-ideal is a t-product of (t-invertible) prime ideals.*

(4) *Every proper t-invertible t-ideal is a t-product of (t-invertible) prime ideals.*

Proof. (1) ⇒ (3). Let R be a Krull domain. By Lemma 3.8, $R[X]_{Nv}$ is a PID. Let I be a proper t-ideal of R. Then $IR[X]_{Nv} = Q_1 \cdots Q_n$, where $Q_1, ..., Q_n$ are principal prime ideals of $R[X]_{Nv}$ since $IR[X]_{Nv}$ is a proper ideal of the PID $R[X]_{Nv}$. Since R is a PVMD, every ideal of $R[X]_{Nv}$ is extended from R by [14, Theorem 3.1]. Let $Q_i = P_i R[X]_{Nv}$, where P_i is an ideal of R. Now $IR[X]_{Nv} = Q_1 \cdots Q_n = P_1 R[X]_{Nv} \cdots P_n R[X]_{Nv} = (P_1 \cdots P_n)[X]_{Nv}$. Hence by [14, Lemma 3.13], $I = I_t = (P_1 \cdots P_n)_t$.

(3) ⇒ (4) ⇒ (2). This is obvious.

(2) ⇒ (1). Let P be a nonzero prime ideal of R. Choose $a \in P \setminus \{0\}$. By assumption $(a) = (P_1 \cdots P_n)_t$ for some prime ideals $P_1, ..., P_n$. Then $P_1 \cdots P_n \subseteq P$ implies that some $P_k \subseteq P$. Now P_k is t-invertible since it is a t-factor of the t-invertible ideal (a). Thus every nonzero prime ideal of R contains a t-invertible prime ideal. Hence by Theorem 3.5, R is a Krull domain.

Remark. The implication (2) ⇒ (1) in the previous Theorem 6.8 was proved by several authors; for example, see [17], [18], and [22]. But our proof is based on the new result Theorem 3.5.

A domain R is said to be t-locally UFD if R_M is a UFD for each $M \in t\text{-Max}(R)$.

THEOREM 3.10. *The following are equivalent for a domain R.*

(1) *R is a Krull domain.*

(2) *R is a t-locally* UFD *and every minimal prime ideal is a finite type t-ideal.*

Proof. (1) ⇒ (2). This is clear.

(2) ⇒ (1). Let a be a nonzero nonunit of R. Let P be a prime ideal minimal over (a). Then by [12, Proposition 1.1], P is a t-ideal. Hence P is contained in a maximal t-ideal M. Then P_M is a minimal prime of $(a)_M$ in the UFD R_M. Hence P_M is a principal (prime) ideal. For a maximal t-ideal M of R such that $P \nsubseteq M$, $P_M = R_M$. Hence in any case P_M is a principal ideal for every maximal t-ideal M. Hence by [14, Corollary 2.7], P is t-invertible since P is a finite type t-ideal. Then the conclusion follows from Theorem 3.6(6).

IV. π-DOMAINS

An integral domain R is defined to be a π-domain if each proper principal ideal is a finite product of prime ideals. In this section, we extend the results in Section III to π-domains.

THEOREM 4.1. *The following are equivalent for a domain R.*

(1) *R is a π-domain.*

(2) *Every nonzero prime ideal contains an invertible prime ideal.*

(3) *Every proper t-ideal is a finite product of (invertible) prime ideals.*

(4) *Every proper invertible ideal is a finite product of prime ideals.*

Proof. (1) $\Rightarrow$ (2). Let $P \neq 0$ be a prime ideal of R. Choose $a \in P \backslash 0$. Since R is a π-domain, $(a) = P_1 \cdots P_n$ where the P_is are prime ideals of R. Now $P_1 \cdots P_n \subseteq P$, hence some $P_i \subseteq P$. And it is clear that P_i is invertible since it is a factor of the invertible ideal (a).

(2) $\Rightarrow$ (3). Suppose every nonzero prime ideal contains an invertible prime ideal. Then by Theorem 3.6, R is a Krull domain, and hence if I is a proper t-ideal of R, then $I = (P_1 \cdots P_n)_t$ for some t-invertible prime ideals $P_1, ..., P_n$ by Theorem 3.9. By the given assumption, each P_i contains some invertible prime ideal P_i'. We may assume that $(P_i)_t \neq R$ for any i by discarding the P_is with $(P_i)_t = R$ from $I = (P_1 \cdots P_n)_t$, since $I \neq R$. Then by Theorem 2.2(4), $P_i = P_i'$. Thus P_i is invertible for every $i = 1, ..., n$. Hence the ideal $P_1 \cdots P_n$ is invertible. Therefore $I = (P_1 \cdots P_n)_t = P_1 \cdots P_n$ and every P_i is invertible.

(3) $\Rightarrow$ (4) $\Rightarrow$ (1). This is clear.

THEOREM 4.2. *The following are equivalent for a domain R.*

(1) *R is a π-domain.*

(2) *Every minimal prime of a nonzero principal ideal is invertible.*

Proof. (1) $\Rightarrow$ (2). Let P be minimal over $(a) \neq 0$. Then by [12, Proposition 1.1], P is a t-ideal. Hence by Theorem 4.1(3), P is invertible.

(2) $\Rightarrow$ (1). Let P be a nonzero prime ideal of R. Let $a \in P \backslash 0$. Then there exists a prime ideal $P' \subseteq P$ which is minimal over (a). This implication now follows from Theorem 4.1.

We prove the π-domain version of Theorem 3.7.

COROLLARY 4.3. *The following are equivalent for a domain R.*

(1) *R is a π-domain.*

(2) *R satisfies Krull's principal ideal theorem and the minimal prime ideals are invertible.*

Proof. (1) ⇒ (2). Let R be a π-domain. Let P be a minimal prime ideal. By Theorem 4.2, P is invertible. Hence by Theorem 3.7(3), R is a Krull domain and therefore R satisfies Krull's principal ideal theorem by Theorem 3.7(2).

(2) ⇒ (1). Let P be a minimal prime ideal of a nonzero principal ideal. By (2) P is invertible. Hence R is a π-domain by Theorem 4.2.

THEOREM 4.4. *The following are equivalent for a domain R.*

(1) *R is a π-domain.*

(2) *Every t-ideal is invertible.*

(3) *R is a Mori domain and every divisorial ideal is invertible.*

(4) *R is a Mori domain and every divisorial prime ideal is invertible.*

(5) *Every prime t-ideal is invertible.*

Proof. (1) ⇒ (2). This follows from Theorem 4.1(3).

(2) ⇒ (3). It suffices to show that R is a Mori domain. Let $\{I_\alpha\}_{\alpha \in \Lambda}$ be an ascending chain of integral t-ideals of R. Then $I = \bigcup_{\alpha \in \Lambda} I_\alpha$ is a t-ideal of R. Since I is invertible, it is f.g. Hence the chain stops, and so R satisfies ACC on t-ideals. Hence R satisfies ACC on v-ideals and therefore R is a Mori domain.

The implications (3) ⇒ (4) ⇒ (5) are clear.

(5) ⇒ (1). Let P be a prime ideal minimal over a nonzero principal ideal. Then P is a t-ideal by [12, Proposition 1.1] and hence P is invertible. Therefore R is a π-domain by Theorem 4.2.

V. MORE π-DOMAINS

A domain is a generalized GCD domain (G-GCD domain) if the intersection of any two invertible ideals is invertible.

For G-GCD domains, the reader is referred to [2]. A domain R is called a pseudo-Dedekind domain (resp., pseudo-principal domain) if every v-ideal of R is invertible (resp., principal). In [3], it is shown that R is a pseudo-Dedekind domain $\Leftrightarrow (AB)^{-1} = A^{-1}B^{-1}$ for all $A, B \in \mathscr{F}(R) \Leftrightarrow R$ is completely integrally closed and the v-ideals of R are closed under the usual ideal product. A domain R is called a *-domain if $(\bigcap_{i=1}^{n}(a_i))(\bigcap_{j=1}^{m}(b_j)) = \bigcap_{i,j}(a_ib_j)$ for all finite subsets $\{a_i\}$, $\{b_j\}$ of R. We will show that in the class of Mori domains, many subclasses of domains are identical with the class of π-domains.

THEOREM 5.1. *The following statements are equivalent for a Mori domain R.*

(1) *R is a $*$-domain.*

(2) *R is a pseudo-Dedekind domain.*

(3) *R is a* G-GCD *domain.*

(4) *R is a locally* GCD *domain.*

(5) *R is a locally* UFD *domain.*

(6) *R is a locally pseudo-Dedekind domain.*

(7) *R is a locally pseudo-principal domain.*

(8) *Every divisorial ideal is locally principal.*

(9) *R is a π-domain.*

Proof. (1)$\Rightarrow$(2). Suppose that R is a $*$-domain and let A, B be divisorial ideals of R. Since R is a Mori domain, $A = I_v$ and $B = J_v$ for some f.g. ideals I, J of R. Now $(AB)^{-1} = (I_v J_v)^{-1} = (IJ)^{-1} = I^{-1}J^{-1}$ since R is a $*$-domain. Thus $(AB)^{-1} = I^{-1}J^{-1} = (I_v)^{-1}(J_v)^{-1} = A^{-1}B^{-1}$. So R is a pseudo-Dedekind domain by [3, Corollary 2.5].

(2)$\Rightarrow$(3). This follows from the fact that the intersection of two divisorial ideals is a divisorial ideal.

(3)$\Rightarrow$(4). This follows from [2, Corollary 1].

(4)$\Rightarrow$(5). Suppose that R is a locally GCD domain. Since R is a Mori domain, R_M is a Mori domain for each maximal ideal M by Theorem 2.1(5). Thus R_M is a Mori GCD domain and therefore R_M is a UFD since any Mori domain satisfies the ascending chain condition on principal ideals.

The implications (5)$\Rightarrow$(6)$\Rightarrow$(7) are clear.

(7)$\Rightarrow$(8). Suppose that R is a locally pseudo-principal domain and let M be a maximal ideal of R. Let I be a divisorial ideal. Since R is a Mori domain, I_M is a divisorial ideal of R_M by Theorem 2.1(4), and hence I_M is principal by assumption. Therefore I is locally principal.

(8)$\Rightarrow$(9). Suppose that every divisorial ideal of R is locally principal. Let I be a divisorial ideal of R. Then I is of finite type since R is a Mori domain. So I is invertible by [1, Theorem 2.1], and hence every divisorial ideal of R is invertible, which implies that R is a π-domain by Theorem 4.4.

The implication (9)$\Rightarrow$(2) follows from the implication (1)$\Rightarrow$(3) of Theorem 4.4.

(2)$\Rightarrow$(1). This is clear.

In [4], it was proved that if R is a Noetherian domain satisfying $((a) \cap (b))^n = (a)^n \cap (b)^n$ for all a and b of R and some $n > 1$ depending on a and b, then R is a Krull domain. An appropriate change of the proof of [4, Theorem 3.2] will give us the following theorem.

An integral domain is called a finite conductor (FC) domain if the intersection of two principal ideals is finitely generated.

THEOREM 5.2. *Let R be a* FC *Mori domain satisfying* $((a)\cap(b))^n = (a)^n \cap (b)^n$ *for all* $a, b \in R$ *and some* $n > 1$ *depending on* a *and* b; *then* R *is a Krull domain.*

Proof. Let M be a maximal divisorial ideal of R. It is easy to see that R_M is a FC Mori domain satisfying $((a)\cap(b))^n = (a)^n \cap (b)^n$ for all $a, b \in R_M$ and some $n > 1$ depending on a and b. If we show that R_M is a PID, then it will follow that $R = \bigcap_{M \in D_m(R)} R_M$ is a Krull domain since the intersection is locally finite by Theorem 2.1(3). Thus we assume that (R, M) is a quasi-local domain whose maximal ideal M is a divisorial ideal. By Theorem 2.1(2), $M = (a : b)$ for some $a, b \in R$. Since $M \neq R$, $b \neq 0$. So $(a : b) = b^{-1}((a) \cap (b))$. There exists $n > 1$ such that $((a)\cap(b))^n = (a)^n \cap (b)^n$, and then $(a : b)^n = (a^n : b^n)$ by the previous observation. Since R is a FC domain, $M = (a : b)$ is f.g. So by Nakayama's Lemma, $M^n \circ M^{n-1}$. Choose $r \in M^{n-1} \backslash M^n$ so that $r \in M^{n-1} \subseteq (a^{n-1} : b^{n-1})$ but $r \notin (a^n : b^n) = M^n$. Then $rb^{n-1} = sa^{n-1}$ for some $s \in R$, so $rb^n = sa^{n-1}b$. But $r \notin (a^n : b^n)$, so $sa^{n-1}b \notin (a^n)$ and hence $sb \notin (a)$. Thus $s \notin (a : b) = M$, so that s is a unit. Hence $M^{n-1} \subseteq (a)^{n-1} : (b)^{n-1} = (s^{-1}rb^{n-1}) : (b)^{n-1} = (rb^{n-1}) : (b^{n-1}) = (r)$. Thus $M^{n-1} \subseteq (r)$ and hence $M^{n-1} = (r)$ since $r \in M^{n-1}$. Now M is invertible, so M is principal since R is quasi-local. Moreover $M^k \neq M^{k+1}$ for any $k \geqslant 1$ since M is invertible. By Theorem 2.1(1), $\bigcap_{k=1}^{\infty} M^k = 0$. Hence we conclude that R is a PID.

COROLLARY 5.3. *For a* FC *Mori domain* R, *the following are equivalent.*

(1) *R is a Dedekind domain.*

(2) $(A \cap B)^n = A^n \cap B^n$ *for all ideals* A *and* B *of* R *and all* $n > 1$.

(3) $(A \cap B)^n = A^n \cap B^n$ *for all ideals* A *and* B *of* R *and some* $n > 1$.

Proof. It is clear that (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1). By Theorem 5.2, R is a Krull domain. So R is integrally closed. By [10], R is a Prüfer domain. Hence R is a Dedekind doamin since it is a Mori Prüfer domain.

COROLLARY 5.4. *The following are equivalent.*

(1) *R is a π-domain.*

(2) *R is a* FC *Mori domain such that* $((a)\cap(b))((c)\cap(d)) = (ac) \cap (ad) \cap (bc) \cap (bd)$ *for all* a, b, c, d *of* R.

Proof. (1) $\Rightarrow$ (2). Every π-domain is a FC Mori domain by (1) $\Rightarrow$ (3) of Theorem 4.4. The conclusion now follows from (9) $\Rightarrow$ (1) of Theorem 5.1.

(2) ⇒ (1). By [4, Theorem 3.8] it suffices to show that R is a Krull domain. Since R is a Mori domain, it suffices to prove that R_M is a DVR for each maximal divisorial ideal M of R since the representation $R = \bigcap_{M \in D_m(R)} R_M$ is locally finite by Theorem 2.1(3). If R satisfies the conditions in (2), then so does R_M. So we may assume that (R, M) is a quasi-local domain whose maximal ideal M is divisorial. Now $M = (b:a)$ for some $a, b \in R$ by Theorem 2.1(2). As in the proof [4, Corollary 3.9], we can show that $M^2 = (b^2:a^2)$. Then by the proof of Theorem 5.2, R is a Krull domain.

VI. A Counterexample

In Sections III and IV, we showed that a domain R is a UFD (resp., π-domain, Krull domain) if (and only if) R satisfies Krull's principal ideal theorem and every minimal prime of a proper principal ideal is principal (resp., invertible, t-invertible). In this section we will show that the above result does not hold without the condition that R satisfies Krull's principal ideal theorem. Thus let us consider the following statement: If every minimal prime of a proper principal ideal of a domain R is principal (resp., inverible, t-invertible), then R is a UFD (resp., π-domain, Krull domain). Since there exists a non-Krull domain which does not have a minimal prime ideal (for example, see Exercise 8 on p. 221 in [9]), we will consider the nontrivial case when R has a minimal prime ideal and every minimal prime ideal of R is principal, invertible, or t-invertible.

THEOREM 6.1. *Let (V, M) be a valuation domain which does not have a minimal prime ideal (for example, see Exercise 8 on p. 221 in [9]). Let $R = V[X]_T$ where $T = V[X] \setminus (M[X] \cup (X))$. Then*

(1) *R is a Bézout domain.*

(2) *R is not a Krull domain.*

(3) *R has a minimal prime ideal and every minimal prime ideal of R is principal.*

Proof. Let $M_1 = M[X]_T$ and $M_2 = (X)_T$. Clearly $\text{Max}(R) = \{M_1, M_2\}$. So $R = R_{M_1} \cap R_{M_2} = V(X) \cap R[X]_{(X)}$. Here $V(X) = V[X]_{M[X]}$. Obviously, $R[X]_{(X)} = K[X]_{R[X] \setminus (X)}$ is a valuation domain. By [5, Lemma 2 and Theorem 4], $V(X)$ is a valuation domain. Hence by [16, Theorem 107], R is a Bézout domain.

(2) To show that R is not a Krull domain, we claim that R is not a Mori domain. For otherwise, R is a UFD since it is a GCD domain with ACC on principal ideals. Now $V(X) = R_{M_1}$ is a UFD. Hence $V(X)$ is a

UFD valuation domain, i.e., a DVR. So V is a DVR, which contradicts that V does not have a minimal prime ideal.

(3) Since xR_{M_2} is a minimal prime ideal of $R_{M_2} = V[X]_{(X)}$, XR is a minimal prime ideal of R, whence the first conclusion follows. Let $P_1 = P_T$ be a minimal prime ideal of R where P is a minimal prime ideal of $V[X]$. Now $P \subseteq M[X]$ or $P \subseteq (X)$. Suppose $P \subseteq M[X]$. Then $P = P_0[X]$ for some prime ideal P_0 of V by [14, Corollary 3.16]. Since P is a minimal prime ideal of R, P_0 is a minimal prime ideal of V. This contradicts that V does not have a minimal prime ideal. So $P \subseteq (X)$. Since (X) is a minimal prime idel of $V[X]$, $P = (X)$. Hence $P_1 = P_T$ is principal.

Remark. An example which is similar to the one above was used in [15] to construct a counterexample to Sheldon's conjecture. There we used the multi-variable case while we use a single variable in this paper.

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