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Total excess and Tits metric for piecewise Riemannian 2-manifolds

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Abstract

A *piecewise Riemannian* 2-*manifold* is a combinatorial 2-manifold with a triangulation such that each 2-simplex is a geodesic triangle of some Riemannian 2-manifold. In this paper, we study the *total excess* $e(X)$ of a simply connected nonpositively curved piecewise Riemannian 2-manifold X in connection with the Tits metric on the boundary at infinity $X(\infty)$.

Keywords: Piecewise Riemannian manifolds; Total excess; Boundary at infinity; Tits metric; Geodesic

AMS classification: Primary 53C23; 57N05, Secondary 53C45; 53C70

1. Introduction

A *piecewise Riemannian* 2-*manifold* is a combinatorial 2-manifold with a triangulation such that each 2-simplex is a geodesic triangle in some Riemannian 2-manifold. In a previous paper [6], the authors studied the total excess of piecewise Riemannian 2-manifolds in connection with the existence of straight lines. In this paper, we study the relation between the total excess of simply connected nonpositively curved piecewise Riemannian 2-manifolds and the Tits metric on the boundary at infinity. A piecewise Riemannian 2-manifold is said to be *nonpositively curved* if the sectional curvature at any interior point of each 2-simplex is nonpositive with respect to the Riemannian metric and the *angle excess*, defined in Section 2, at each vertex is also nonpositive. A simply connected nonpositively curved piecewise Riemannian 2-manifold X is a Hadamard space in the sense of Ballmann [2] and the boundary at infinity $X(\infty)$ is well defined. We

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introduce a topology, called the *standard topology*, as an analogue of the sphere topology for Hadamard manifolds. Also, a metric Td, called the *Tits metric*, is introduced in a similar fashion to the one for Hadamard manifolds [2, Chapter II].

The second author in [9] proved that for any 2-dimensional Hadamard manifold X, the total curvature $C(X)$ of X satisfies $C(X) = 2(\pi - \mathrm{diam}_{\mathrm{Td}}(X(\infty)))$, where $\mathrm{diam}_{\mathrm{Td}}(X(\infty))$ denotes the diameter of the metric space $(X(\infty), \mathrm{Td})$. We prove that the same formula holds for the total excess of each simply connected nonpositively curved piecewise Riemannian 2-manifold without boundary.

We then study the topology on the boundary at infinity induced by the Tits metric which is finer than the standard topology in general. We prove that these two topologies coincide with each other if and only if the total excess is finite.

For a simply connected nonpositively curved piecewise Riemannian 2-manifold X without boundary, the collection of all connected component of $(X(\infty), \mathrm{Td})$ provides a decomposition of the unit circle $\mathbb{S}^1$ into points and subsets homeomorphic to open, closed or half-open intervals. Conversely, we show that *any* decomposition of $\mathbb{S}^1$ into points and subsets homeomorphic to intervals is realized as the boundary at infinity $(X(\infty), \mathrm{Td})$ for some simply connected nonpositively curved piecewise Riemannian 2-manifold X.

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2. Preliminaries

In this section, we introduce some definitions and related results. First we define the concept of a piecewise Riemannian 2-manifold.

For a metric space (X, d), a continuous map on an interval I into X is called a curve. A curve $\alpha : I \to X$ is called a *geodesic* if it is locally distance minimizing, i.e., for any point $t \in I$, there exists a neighborhood U of t such that $d(\alpha(s_1), \alpha(s_2)) = |s_1 - s_2|$ for any points $s_1, s_2 \in U$. In what follows, we assume that α is parameterized proportional to arc length. If the above equality holds for any points $s_1, s_2 \in I$, then we call α a *minimizing geodesic*. In particular, a minimizing geodesic defined on $[0, \infty)$ is called a *ray* and that defined on $(-\infty, \infty)$ a *straight line*. We occasionally identify a geodesic with its image. For a geodesic segment $\alpha : [a, b] \to X$ on a compact interval $[a, b]$, let

$$\mathring{\alpha} := \alpha|_{(a,b)} : (a, b) \to X,$$

and a point on $\alpha(a, b)$ is called an interior point of α. Also the points $\alpha(a)$ and $\alpha(b)$ are called the end points of α.

A metric space (X, d) is called a *geodesic space* if for any pair of points x, y on X, there exists a minimizing geodesic segment from x to y. A metric space (X, d) is said to be *proper* if any bounded subset has the compact closure. Any proper metric space is locally compact and separable.

Let X be a topological 2-manifold with a triangulation such that each 2-simplex is a geodesic triangle in some Riemannian 2-manifold. We introduce a natural metric as follows.

For any pair of points $x, y \in X$, let $c:[a,b] \to X$ be a piecewise smooth curve from x to y, that is, a curve with a sequence $a = t_0 < t_1 < \cdots < t_k = b$ such that $c|_{[t_{i-1},t_i]}$ is contained in a 2-simplex for each i and is a smooth curve with respect to the Riemannian metric on the simplex. The length of c is denoted by

$$l(c) := \sum_{i=1}^{k} l\big(c|_{[t_{i-1},t_i]}\big),$$

where $l(c|_{[t_{i-1},t_i]})$ is the length with respect to the Riemannian metric on the simplex. Now we define the metric d by

$$d(x,y) := \inf\big\{l(c) \mid c \text{ is a piecewise smooth curve from } x \text{ to } y\big\}.$$

It is easy to see that the metric space (X,d) is a proper geodesic space.

Definition 2.1. We call such a space (X,d) a *piecewise Riemannian* 2*-manifold*.

A piecewise Riemannian 2-manifold X is said to be *piecewise flat* if each 2-simplex is isometric to a 2-simplex in the Euclidean plane $\mathbb{R}^2$.

2.1. Total excess

Here we will review the concept of total excess on piecewise Riemannian 2-manifold. Although the total excess is defined for more general spaces, we confine ourselves to piecewise Riemannian 2-manifold, for simplicity. See [8,12] for more details.

For a proper geodesic space (X,d), a *geodesic triangle with vertices* p, q *and* r, denoted by $\Delta(p,q,r)$, is the union $\alpha_{pq} \cup \alpha_{qr} \cup \alpha_{rp}$, where α_{ab} is a minimizing geodesic segment from a to b. For a real number k, let $M(k)$ be the 2-dimensional space form of constant sectional curvature k. A geodesic triangle $\widetilde{\Delta}(p,q,r) := \Delta(\tilde{p},\tilde{q},\tilde{r}) = \alpha_{\tilde{p}\tilde{q}} \cup \alpha_{\tilde{q}\tilde{r}} \cup \alpha_{\tilde{r}\tilde{p}}$ in $M(k)$ is called a *comparison triangle* of $\Delta(p,q,r)$ if $l(\alpha_{ab}) = l(\alpha_{\tilde{a}\tilde{b}})$ for any $a, b \in \{p,q,r\}$. The angle subtended by $\alpha_{\tilde{p}\tilde{q}}$ and $\alpha_{\tilde{p}\tilde{r}}$ is denoted by $\widetilde{\angle}_k(qpr)$. In this paper, a closed disk domain bounded by a geodesic triangle is also called a geodesic triangle.

For a point p on a piecewise Riemannian 2-manifold X, let $\mathcal{R}_p$ be the set of all geodesics emanating from p. For $\alpha, \beta \in \mathcal{R}_p$ and a real number k, it is known that the limit

$$\overline{\angle}_p(\alpha,\beta) := \lim_{s,t\to 0} \widetilde{\angle}_k\big(\alpha(s)p\beta(t)\big)$$

exists and does not depend on the choice of k. It is called the *upper angle* at p subtended by α and β. For a geodesic space in general, the limit does not necessarily exist and the upper limit is defined as the superior limit of the above term. See Section 4 of [12].

In general, the upper angle $\overline{\angle}_p$ is known to be a pseudo-metric on $\mathcal{R}_p$ and induces an equivalence relation $\sim$ defined as follows: $\alpha \sim \beta$ if and only if $\overline{\angle}_p(\alpha,\beta) = 0$. The completion of the metric space $(\mathcal{R}_p/{\sim}, \overline{\angle})$ is denoted by $(\Sigma_p, \overline{\angle})$ and is called the *space of directions* at p. For a subset Y of X, let

$$\mathcal{R}_p^Y := \big\{\alpha \in \mathcal{R}_p \mid \alpha([0,\varepsilon]) \subset Y \text{ for some } \varepsilon > 0\big\}.$$

The *space of directions with respect to* Y, denoted by Σ_p^Y, is the completion of the metric space $(\mathcal{R}_p^Y/\sim, \overline{\angle})$.

If a point x is on $\mathring{X}$, the interior of X, the space Σ_x is homeomorphic to $\mathbb{S}^1$, the unit circle on the plane $\mathbb{R}^2$. Moreover, if x is not a vertex of the triangulation of X, then Σ_x is isometric to $\mathbb{S}^1$.

For a point $p \in \mathring{X}$, let $k(p) = 2\pi - L(\Sigma_p)$, where L is the one-dimensional Hausdorff measure on Σ_p. $k(p)$ is called the *angle excess* at the point p in this paper. The following is clear from the above:

$$\text{If } p \text{ is not a vertex, then } k(p) = 0. \tag{$*$}$$

Note that, when X is piecewise flat, $k(p)$ is called the curvature at p in [5]. However, we would like to avoid the use of the terminology "curvature" here to prevent a possible confusion with the Gaussian curvature at a point on the interior of a 2-simplex.

For a Riemannian manifold without boundary, each geodesic is locally extended in a unique way, but this does not hold for a piecewise Riemannian manifold. Suppose that a piecewise Riemannian 2-manifold X has a minimizing geodesic α with an end point p. If $k(p) > 0$, then it is easily seen that α cannot be extended, as a geodesic, beyond p. On the other hand, if $k(p) < 0$, there are infinitely many minimizing geodesic-extensions beyond p. In this sense, a point with nonzero angle excess is "singular" with respect to the extension of geodesics. We define the *positive singular set* $\mathrm{Sing}^+(X)$ and the *negative singular set* $\mathrm{Sing}^-(X)$ of X, respectively by

$$\mathrm{Sing}^{\pm}(X) := \{ p \in \mathring{X} \mid k(p) \gtrless 0 \},$$

and the *singular set* $\mathrm{Sing}(X)$ by $\mathrm{Sing}(X) := \mathrm{Sing}^+(X) \cup \mathrm{Sing}^-(X)$. By the property $(*)$ above, $\mathrm{Sing}(X)$ is a subset of the vertices of the triangulation of X. It is also clear that there is no positive singular point on the interior of any minimizing geodesic.

Now we define the total excess of X as follows. Let $C(\Delta)$ be the total curvature of the Riemannian 2-manifold Δ with boundary, and $e_{\mathrm{reg}}(X) := \sum_{\Delta:\text{2-simplex}} C(\Delta)$ provided the sum is absolutely convergent, $e_{\mathrm{sing}}(X) := \sum_{p \in \mathrm{Sing}(X)} k(p)$ if the sum converges absolutely. Then the *total excess* $e(X)$ of X is defined by

$$e(X) := e_{\mathrm{reg}}(X) + e_{\mathrm{sing}}(X),$$

when the sum of the right hand side makes sense.

We illustrate typical cases. If M is a Riemannian 2-manifold triangulated by geodesic triangles, then $e_{\mathrm{sing}}(M) = 0$ and $e(M) = C(M)$, the total curvature of M. If M is a piecewise flat 2-manifold, then $e_{\mathrm{reg}}(M) = 0$ and $e(M) = \sum_{p \in \mathrm{Sing}(M)} k(p)$, the total curvature of M in the sense of [5].

Remark. Each piecewise Riemannian 2-manifold is a good surface in the sense of [8], and the above definition coincides with the one given in [8].

The remark above allows us to apply the following analogue of the Gauss–Bonnet theorem in [8], which play the fundamental role in our argument.

A curve $c:[a,b]\to X$ is called a *broken geodesic* if there is a subdivision $a=x_0<\cdots<x_n=b$ such that $c|_{[x_{i-1},x_i]}$ is a geodesic segment. The point $c(x_i)$ $(i=0,\ldots,n)$ is called a *vertex* of the broken geodesic c.

Theorem 2.1 (The generalized Gauss–Bonnet theorem [8, Theorem 3.1]). *Let X be a piecewise Riemannian 2-manifold without boundary and Y a compact domain of X such that ∂Y consists of simple closed broken geodesics without self-intersection. Then*

$$e(Y)=2\pi\chi(Y)-\sum_{p\in\partial Y}\theta^Y(p),$$

where $\theta^Y(p)=\pi-L(\Sigma_p^Y)$.

Remark. For a Riemannian 2-manifold X and its compact domain Y, the nontrivial contribution to the sum of the above equality is made only at the vertices of the broken geodesics. However in our setting, a geodesic may pass through points of negative singularity and those singular points may contribute to that sum. Also notice that, if $p\in\partial Y\setminus \mathrm{Sing}(X)$ is not a vertex of the boundary ∂Y of Y, then $\theta^Y(p)=0$. Since there are only finitely many singular points on ∂Y, the second term of the right side of the above equality makes sense.

In what follows, for brevity, $\sum_{p\in S}f(p)$ is often denoted by $\sum_S f$ for a function $f:S\to\mathbb{R}$ defined on a set S. For example, $\sum_{p\in\partial Y}\theta^Y(p)$ is abbreviated to $\sum_{\partial Y}\theta^Y$.

2.2. Boundary at infinity

For a proper geodesic space X, an open set U of X is called a CAT_0 *domain* if, for each geodesic triangle $\Delta(p,q,r)$ in U and the corresponding comparison triangle $\widetilde{\Delta}(p,q,r)$ in $\mathbb{R}^2$, we have the following inequality

$$d(x,y)\leqslant d(\tilde{x},\tilde{y}),$$

for each pair of points x,y on the edges of $\Delta(p,q,r)$ and the corresponding points $\tilde{x},\tilde{y}$ on $\widetilde{\Delta}(p,q,r)$. If each point on X belongs to a CAT_0 domain, then we say that X has *nonpositive Alexandrov curvature*. After Ballmann [2], a simply connected complete geodesic space of nonpositive Alexandrov curvature is called a *Hadamard space*. It is known that for a Hadamard space X, X itself is a CAT_0 domain. Hence it is clear that any geodesic is a minimizing geodesic, and for each pair of two points of X, there exists the unique geodesic on X joining these points.

A piecewise Riemannian 2-manifold is said to be *nonpositively curved* if the sectional curvature at an interior point of each 2-simplex is nonpositive with respect to the Riemannian metric and further $k(p)\leqslant 0$ for each vertex p. In what follows, we are concerned with a noncompact simply connected nonpositively curved piecewise Riemannian 2-manifold without boundary. It is known that such a space is a Hadamard space (the CAT_0-condition above is verified directly for a small neighborhood of each vertex).

The following is a brief review of the concepts of the boundary at infinity $X(\infty)$ of a Hadamard space X, the standard topology and the Tits metric on $X(\infty)$. See [2] for more detail.

Let X be a Hadamard space and p a point on X. We denote the set of all geodesic rays on X and all geodesic rays emanating from p by $\mathcal{R}$ and $\mathcal{R}_p$, respectively. Two geodesic rays α and β are said to be *asymptotic* if there exists a constant K such that $d(\alpha(t), \beta(t)) < K$ for any $t \geqslant 0$. This is an equivalence relation and the *boundary at infinity* $X(\infty)$ of X is defined as the equivalence classes $\mathcal{R}/\!\sim$. For a geodesic ray σ, the equivalence class of σ is denoted by $\sigma(\infty)$. It is known that for any point $\xi \in X(\infty)$ and for any $p \in X$, there exists the unique geodesic ray $\sigma \in \mathcal{R}_p$ such that $\sigma(\infty) = \xi$, which is denoted by $\sigma_{p\xi}$.

Next we introduce a topology on the set $\overline{X} = X \cup X(\infty)$. Fix a point $p \in X$. The basis of open sets of $\overline{X}$ consists of all open sets of X together with the sets of the form:

$$U_p(\xi, R, \varepsilon) = \bigl\{z \in \overline{X} \bigm| z \in \overline{X} \setminus B(p, R),\ d\bigl(\sigma_{pz}(R), \sigma_{p\xi}(R)\bigr) < \varepsilon\bigr\},$$

where $\xi \in X(\infty)$ and $B(p, R) := \{x \in X \mid d(x, p) \leqslant R\}$. It is known that the above topology does not depend on the choice of p, and the space $\overline{X}$ with the above topology is a compactification of X. The relative topology on $X(\infty)$ is called the *standard topology* on $X(\infty)$, denoted by $(X(\infty), st)$ in the sequel.

For points $\xi, \eta \in X(\infty)$, we define the *angle* by

$$\angle(\xi, \eta) := \sup_{p \in X} \overline{\angle}_p(\sigma_{p\xi}, \sigma_{p\eta}),$$

where $\overline{\angle}_p$ is the upper angle. Then $(X(\infty), \angle)$ is a complete metric space, and the induced topology is finer than the standard topology.

The *Tits metric* Td on $X(\infty)$ is defined as the interior metric of $\angle$. Namely for $\xi, \eta \in X(\infty)$, if there is a continuous curve from ξ to η on $(X(\infty), \angle)$, then $\mathrm{Td}(\xi, \eta)$ is the infimum of the lengths of such curves and otherwise $\mathrm{Td}(\xi, \eta) = \infty$.

It is known that Td is a complete metric, and for any two points $\xi, \eta \in X(\infty)$ with $\mathrm{Td}(\xi, \eta) < \infty$, there exists a minimizing geodesic from ξ to η with respect to Td.

3. Tits metrics and the total excess of simply connected nonpositively curved piecewise Riemannian 2-manifolds

Throughout this section, X denotes a simply connected nonpositively curved piecewise Riemannian 2-manifold without boundary. We first prove that the boundary at infinity $(X(\infty), st)$ with the standard topology is homeomorphic to $\mathbb{S}^1$. This is trivially true for a Hadamard 2-manifold, because $X(\infty)$ is homeomorphic to the unit tangent sphere $S_p(X)$ at any fixed point p, via the map $\Psi_p : S_p(X) \to X(\infty)$ defined by $\Psi_p(v) = \gamma_v(\infty)$, where γ_v is the unique geodesic ray from p such that $\gamma_v'(0) = v$. However, on a Hadamard space, geodesics may branch off in various directions and the map Ψ_p above is not well defined. To avoid this difficulty, we represent $(X(\infty), st)$ as the projective limit of geodesic spheres as follows.

Fix a point $p \in X$ and let $S(p,r) := \{x \in X \mid d(p,x) = r\}$. For two positive numbers $0 < r < R$, a continuous map $\varphi_{rR} : S(p,R) \to S(p,r)$ is defined by

$$\varphi_{rR}(x) := S(p,r) \cap \sigma_{px} \quad \text{for } x \in S(p,R),$$

where σ_{px} is the unique geodesic segment from p to x. It is clear that the above map is well defined and continuous. Then we obtain a projective system $\mathbb{S} = \{S(p,r), \varphi_{rR}\}$ and from the definition of the projective limit, we have the following result which provides a useful tool to study the topology of $X(\infty)$.

Lemma 3.1. *For a Hadamard space X, $X(\infty)$ with the standard topology is homeomorphic to $\varprojlim \mathbb{S}$, the projective limit of $\mathbb{S}$.*

The following result provides information on the system above. A continuous map $f : X \to Y$ is called a *near-homeomorphism* if, for each $\varepsilon > 0$, there exists a homeomorphism $h : X \to Y$ such that $d(f(x), h(x)) < \varepsilon$ for any $x \in X$.

Lemma 3.2. *For any point $p \in X$ and $R > r > 0$, $S(p,R)$ is homeomorphic to $\mathbb{S}^1$ and each map $\varphi_{rR} : S(p,R) \to S(p,r)$ is a near-homeomorphism.*

Proof. First we note that $S(p,r)$ is homeomorphic to $\mathbb{S}^1$ for sufficiently small $r > 0$.

Since $\mathrm{Sing}(X)$ is countable and discrete, for any $R > r$, there is a sequence $r = r_1 < r_2 < \cdots < r_k = R$ such that

$$\mathrm{Sing}(X) \cap \bigl\{x \in X \mid r \leqslant d(p,x) \leqslant R\bigr\} \subset \bigcup_{i=1}^{k} S(p, r_i).$$

If there is no singular point on $S(p,r_i)$, then it is clear that $\varphi_{r_i r_{i+1}}$ is a homeomorphism.

If y is a singular point on $S(p,r_i)$, then geodesics from p through y branch off at y. Suppose that there are two distinct points $x_1, x_2 \in S(p, r_{i+1})$ such that $\varphi_{r_i r_{i+1}}(x_1) = \varphi_{r_i r_{i+1}}(x_2) = y \in S(p,r_i)$ as in Fig. 1. It is clear that one of two sectors bounded by σ_{yx_1} and σ_{yx_2}, denoted by S, satisfies $\varphi_{r_i r_{i+1}}(x) = y$ for each $x \in S \cap S(p, r_{i+1})$, and hence each fiber of $\varphi_{r_i r_{i+1}}$ over a singular point is homeomorphic to $[0,1]$. Note that the singular points on $S(p,r_i)$ is finite and $\varphi_{r_i r_{i+1}}$ is a homeomorphism over $S(p,r_i) \setminus \mathrm{Sing}(X)$. Hence $S(p,r_{i+1})$ is homeomorphic to $S(p,r_i)$ and also it is easy to see that, for each $\varepsilon > 0$ there exists a homeomorphism $h : S(p,r_{i+1}) \to S(p,r_i)$ such that $d(h(x), \varphi_{r_i r_{i+1}}(x)) < \varepsilon$ for each $x \in S(p, r_{i+1})$.

Therefore $S(p,R)$ is homeomorphic to $\mathbb{S}^1$ and $\varphi_{rR} : S(p,R) \to S(p,r)$ is a near-homeomorphism. $\square$

Applying Brown's approximation theorem [4] together with above lemmas, we have the following proposition.

Proposition 3.3. *For a simply connected nonpositively curved piecewise Riemannian 2-manifold X without boundary, $(X(\infty), st)$ is homeomorphic to $\mathbb{S}^1$.*

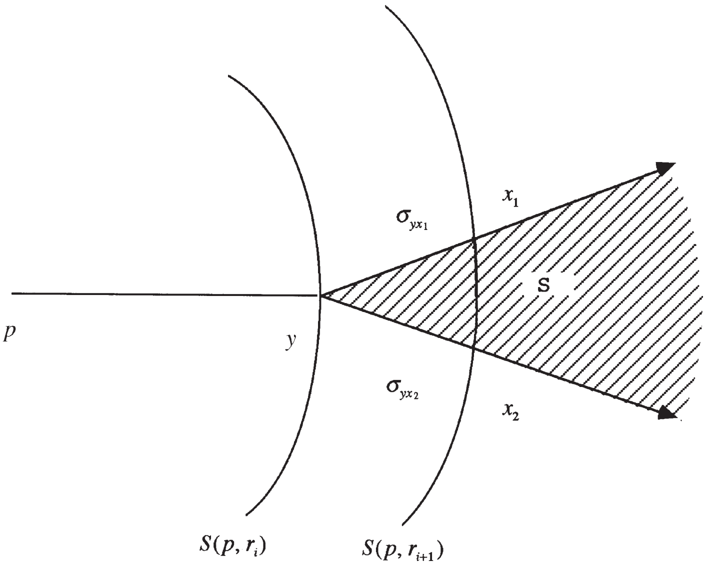

Fig. 1.

In general, the topology induced by the Tits metric Td on $X(\infty)$, called the Tits topology, is finer than the standard topology. The following is an answer to the question as to when these topologies coincide.

Proposition 3.4. $(X(\infty), \mathrm{Td})$ *is homeomorphic to* $(X(\infty), st) \approx \mathbb{S}^1$ *if and only if* $\mathrm{diam}_{\mathrm{Td}}\, X(\infty)$ *is finite.*

Proof. Assume that $\mathrm{diam}_{\mathrm{Td}}\, X(\infty)$ is finite, and we derive a contradiction by supposing that $id : (X(\infty), st) \to (X(\infty), \mathrm{Td})$ is not continuous at $z \in X(\infty)$. Take a sequence $\{z_i\}$ of points on $X(\infty)$ such that

$$\lim_{i\to\infty} \mathrm{Td}(z, z_i) = a := \sup_{y \in X(\infty)} \mathrm{Td}(z, y) \leqslant \mathrm{diam}_{\mathrm{Td}}\, X(\infty) < \infty.$$

Since $(X(\infty), \mathrm{Td})$ is a geodesic space, there exists a geodesic segments c_i on $X(\infty)$ from z to z_i for each i. Now we prove that $c_i \subset c_j$ or $c_i \supset c_j$ for any i, j. Suppose not. Then $c_i \cup c_j$ forms a neighborhood of z in $(X(\infty), st)$ for some i and j, and the compactness of $(c_i \cup c_j, \mathrm{Td})$ easily implies that $id : (c_i \cup c_j, st) \to (c_i \cup c_j, \mathrm{Td})$ is continuous, and in particular, is continuous at z, a contradiction. Therefore, as $i, j \to \infty$,

$$\mathrm{Td}(z_i, z_j) = \left|\mathrm{Td}(z, z_i) - \mathrm{Td}(z, z_j)\right| \to 0.$$

By the completeness of $(X(\infty), \mathrm{Td})$, the Cauchy sequence $\{z_i\}$ converges to a point z_∞. Note that $z_\infty \neq z$, since $\mathrm{Td}(z_\infty, z) = a > 0$. There are exactly two simple curves on $(X(\infty), st)$ from z to z_∞, only one of which is the geodesic segment on $(X(\infty), \mathrm{Td})$. We denote the geodesic segment by A, and take a point $w \in X(\infty) \setminus A$. Then a geodesic

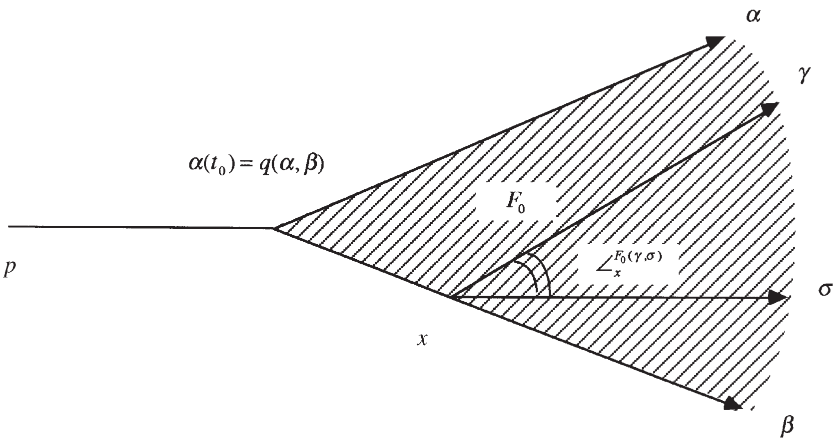

Fig. 2.

from z to w must contain A because of the discontinuity of $id:(X(\infty), st) \to (X(\infty), \mathrm{Td})$ at z. Hence we have that

$$\mathrm{Td}(w, z) > \mathrm{Td}(z_\infty, z) = \sup_{y \in X(\infty)} \mathrm{Td}(z, y),$$

a contradiction.

The reverse implication is obvious and this completes the proof. □

The following theorem is our main result of this section.

Theorem 3.5. *Let X be a simply connected nonpositively curved piecewise Riemannian 2-manifold without boundary. Then we have that*

$$e(X) = 2\big(\pi - \mathrm{diam}_{\mathrm{Td}}\, X(\infty)\big).$$

The proof is similar to the one for a Hadamard 2-manifold in [9] or [11]. First we prove the following lemma.

Let $\alpha, \beta : [0, \infty) \to X$ be two geodesic rays on X emanating from p. Suppose that $\mathrm{Td}(\alpha(\infty), \beta(\infty)) < \pi$, and let F be the domain defined by

$$F := \bigcup_{t \geqslant 0} \sigma_{\alpha(t)\beta(t)}.$$

Let $t_0 := \sup\{t \mid \alpha(t) = \beta(t)\}$, $q(\alpha, \beta) := \alpha(t_0) = \beta(t_0)$ and $F_0 := \bigcup_{t \geqslant t_0} \sigma_{\alpha(t)\beta(t)}$. We call F_0 *the surface component of* F and $q(\alpha, \beta)$ *the vertex of* F_0 in this paper. Clearly $q(\alpha, \beta)$ is a negative singular point if $t_0 > 0$. For geodesic rays γ and σ on F emanating from $x \in \partial F$, let $\angle_x^F(\gamma, \sigma)$ be the angle at x subtended by γ and σ with respect to F, which is defined as follows (cf. Fig. 2). Since Σ_x is homeomorphic to $\mathbb{S}^1$, Σ_x is divided into two closed intervals I, J whose end points are the equivalence classes of γ and σ. Then one of these intervals is contained in Σ_x^F. If $I \subset \Sigma_x^F$, then $\angle_x^F(\gamma, \sigma)$ is defined to be $L(I)$,

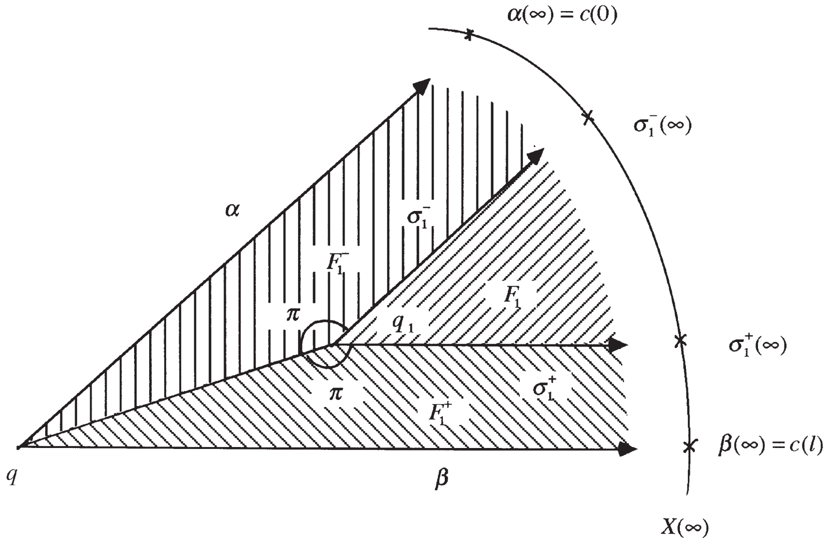

Fig. 3.

where L is the one-dimensional Hausdorff measure on Σ_x. Notice that if $L(I) \leqslant \pi$, then $\angle_x^F(\gamma, \sigma) = \overline{\angle}_x(\gamma, \sigma)$.

To simplify the notation, let $q := q(\alpha, \beta)$, $\alpha := \alpha|_{[t_0,\infty)}$ and $\beta := \beta|_{[t_0,\infty)}$.

Lemma 3.6. *If* $\mathrm{Td}(\alpha(\infty), \beta(\infty)) < \pi$, *then*

$$\mathrm{Td}\big(\alpha(\infty), \beta(\infty)\big) \geqslant \angle_q^{F_0}(\alpha, \beta) - \sum_{\mathring{F}_0} k,$$

where $\mathring{F}_0$ *is the interior of* F_0 *and* $k(x) = 2\pi - L(\Sigma_x)$. *In particular,* $\sum_{\mathring{F}_0} k$ *is finite.*

Proof. Let $l := \mathrm{Td}(\alpha(\infty), \beta(\infty))$ and take a minimizing geodesic $c : [0, l] \to (X(\infty), \mathrm{Td})$ from $\alpha(\infty)$ to $\beta(\infty)$. Since $l < \pi$, it is clear that

$$\bigcup_{0 \leqslant t \leqslant l} \sigma_{qc(t)} = F_0,$$

where σ_{qz} is the ray from q with $\sigma_{qz}(\infty) = z \in X(\infty)$.

Enumerate all the singular points on $\mathring{F}_0$ as $\{q_i \mid i = 1, 2, \ldots\}$ such that $d(q_i, q) \leqslant d(q_{i+1}, q)$ for each i. We can extend the geodesic segment σ_{qq_1} from q to q_1 to two geodesic rays σ_1^+ and σ_1^- such that

$$\overline{\angle}_{q_1}\big(\sigma_{q_1 q}, \tilde{\sigma}_1^{\pm}\big) = \pi, \quad \text{where } \tilde{\sigma}_1^{\pm} := \sigma_1^{\pm} \setminus \sigma_{qq_1}.$$

Let F_1 be the subdomain of F_0 bounded by $\tilde{\sigma}_1^{\pm}$. We may assume that $\alpha(\infty) = c(0) < \sigma_1^-(\infty) < \sigma_1^+(\infty) < c(l) = \beta(\infty)$ with respect to the natural order of $c([0, l])$ as in Fig. 3. Since c is a geodesic on $(X(\infty), \mathrm{Td})$, we have that

$$\begin{aligned}\mathrm{Td}\big(\alpha(\infty),\beta(\infty)\big) &= \mathrm{Td}\big(\alpha(\infty),\sigma_1^-(\infty)\big)+\mathrm{Td}\big(\sigma_1^-(\infty),\sigma_1^+(\infty)\big)\\ &\quad+\mathrm{Td}\big(\sigma_1^+(\infty),\beta(\infty)\big)\\ &\geqslant \angle_q^{F_0}\big(\alpha,\sigma_1^-(\infty)\big)+\big(L(\Sigma_{q_1})-2\pi\big)+\angle_q^{F_0}(\sigma_1^+,\beta)\\ &= \angle_q^{F_0}(\alpha,\beta)-k(q_1).\end{aligned}$$

Next let U_1^-, U_1 and U_1^+ be the interiors of the subdomains F_1^-, F_1 and F_1^+ of F_0 bounded by α and σ_1^-, bounded by σ_1^+ and σ_1^- and bounded by β and σ_1^+, respectively.

If $q_2 \in U_1^-$, then we have that

$$\mathrm{Td}\big(\alpha,\sigma_1^-(\infty)\big)\geqslant \angle_q^{F_1^-}\big(\alpha,\sigma_1^-(\infty)\big)-k(q_2)$$

by the same way as above. Using this in the above estimation, it follows that

$$\mathrm{Td}\big(\alpha(\infty),\beta(\infty)\big)\geqslant \angle_q^{F_0}(\alpha,\beta)-\sum_{i=1}^{2}k(q_i). \tag{$*$}$$

The case $q_2 \in U_1^+$ can be treated similarly. If $q_2 \in U_1$, then

$$\mathrm{Td}\big(\sigma_1^-(\infty),\sigma_1^+(\infty)\big)\geqslant \angle_{q_1}^{F_1}\big(\sigma_1^+(\infty),\sigma_1^-(\infty)\big)-k(q_2),$$

and we may repeat the above argument to obtain the inequality $(*)$ again.

Suppose that $q_2 \in \sigma_1^-$. Then we extend σ_{qq_2} to geodesic rays $\sigma_2^\pm$ such that

$$\angle_{q_2}^{F_2}\big(\sigma_{q_2q},\sigma_2^\pm\big)=\pi,$$

where F_2 is the subdomain of F_0 bounded by $\sigma_2^\pm$. We may assume that $\alpha(\infty)=c(0)<\sigma_2^-(\infty)<\sigma_2^+(\infty)<\sigma_1^+(\infty)<c(l)=\beta(\infty)$ with respect to the natural order of $c([0,l])$ as in Fig. 4. Then we have the inequality $(*)$ as follows:

$$\begin{aligned}\mathrm{Td}\big(\alpha(\infty),\beta(\infty)\big) &= \mathrm{Td}\big(\alpha(\infty),\sigma_2^-(\infty)\big)+\mathrm{Td}\big(\sigma_2^-(\infty),\sigma_2^+(\infty)\big)\\ &\quad+\mathrm{Td}\big(\sigma_2^+(\infty),\sigma_1^+(\infty)\big)+\mathrm{Td}\big(\sigma_1^+(\infty),\beta(\infty)\big)\\ &\geqslant \angle_q^{F_0}(\alpha,\sigma_2^-)+\angle_{q_2}^{F_2}(\sigma_2^-,\sigma_2^+)+\angle_{q_1}^{F_1}(\sigma_2^+,\sigma_1^+)+\angle_q^{F_0}(\sigma_2^+,\beta)\\ &= \angle_q^{F_0}(\alpha,\beta)+\angle_{q_1}^{F_1}(\sigma_1^-,\sigma_1^+)+\angle_{q_2}^{F_2}(\sigma_2^-,\sigma_2^+)\\ &= \angle_q^{F_0}(\alpha,\beta)-\big\{k(q_1)+k(q_2)\big\}.\end{aligned}$$

The case that $q_2 \in U_1^-$ can be proved similarly.

By repeating this argument, we see that, for each n,

$$\mathrm{Td}\big(\alpha(\infty),\beta(\infty)\big)\geqslant \angle_q^{F_0}(\alpha,\beta)-\sum_{i=1}^{n}k(q_i),$$

which clearly implies the desired inequality.

Now the above implies that

$$\angle_q^{F_0}(\alpha,\beta)-\mathrm{Td}\big(\alpha(\infty),\beta(\infty)\big)\leqslant\sum_{i=1}^{n}k(q_i)\leqslant 0\quad\text{for each } n,$$

and hence $\sum_{\mathring{F}_0} k$ is finite. □

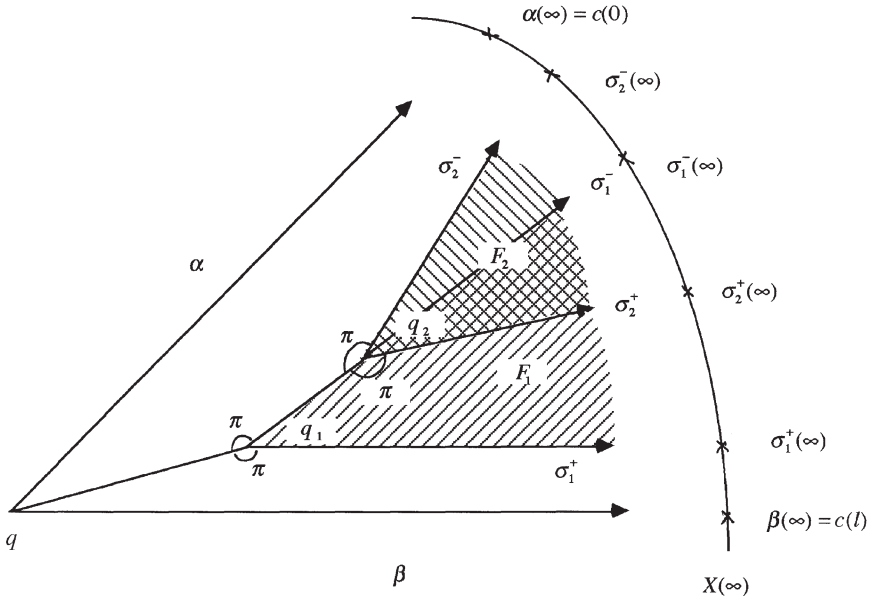

Fig. 4.

The next result corresponds to proposition in Section 2 of [9] and is a key step for the proof of Theorem 3.5.

Proposition 3.7. *Under the same notation as Lemma* 3.6, *we have that*

$$e(F_0) = \angle_q^{F_0}(\alpha, \beta) - \mathrm{Td}\big(\alpha(\infty), \beta(\infty)\big) - \sum_{\partial F_0 \setminus \{q\}} \theta^{F_0},$$

whenever $\mathrm{Td}(\alpha(\infty), \beta(\infty)) < \pi$.

Proof. Let $\xi(t) := \angle_{\alpha(t)}(\sigma_{\alpha(t)p}, \sigma_{\alpha(t)\beta(t)})$ and $\eta(t) := \angle_{\beta(t)}(\sigma_{\beta(t)p}, \sigma_{\beta(t)\alpha(t)})$. By Exercise 4.3 of Chapter II in [2], we have

$$\overline{\angle}\big(\alpha(\infty), \beta(\infty)\big) = \lim_{t\to\infty} \big(\pi - \xi(t) - \eta(t)\big). \tag{1}$$

(See [3, Lemma 4.3, p. 34] for a proof of Riemannian case. The proof of the general case proceeds in the same way.) Since $\mathrm{Td}(\alpha(\infty), \beta(\infty)) < \pi$, $\mathrm{Td}(\alpha(\infty), \beta(\infty)) = \overline{\angle}(\alpha(\infty), \beta(\infty))$. Let $F_t := \bigcup_{t_0 \leqslant s \leqslant t} \sigma_{\alpha(s)\beta(s)}$, which is homeomorphic to the closed disk. Applying Theorem 2.1 to F_t, we obtain

$$e(F_t) = \angle_q^{F_0}(\alpha, \beta) - \big(\pi - \xi(t) - \eta(t)\big) - \sum_{\alpha((t_0,t))\cup\beta((t_0,t))} \theta^{F_t} - \sum_{\overset{\circ}{\sigma}_{\alpha(t)\beta(t)}} \theta^{F_t}. \tag{2}$$

Here we note that

$$0 \geqslant \theta^{F_t}(x) = \pi - L(\Sigma_x^{F_t}) = k(x) + \big(L(\Sigma_x^{X\setminus F_t}) - \pi\big) \geqslant k(x)$$

for each $x \in \mathring{\sigma}_{\alpha(t)\beta(t)}$, where the last inequality follows from the fact that $\sigma_{\alpha(t)\beta(t)}$ is a geodesic. Since $\sum_{\mathring{F}_0} k$ is finite by Lemma 3.6, and $\sigma_{\alpha(t)\beta(t)}$ is divergent as $t \to \infty$, we see that $\sum_{\mathring{\sigma}_{\alpha(t)\beta(t)}} k$ tends to 0 as $t \to \infty$. Therefore taking the limit in (2) with the use of (1) and the above notice, we have the desired equality. □

Now we prove Theorem 3.5.

Proof of Theorem 3.5. The proof is basically the same as the one for a Hadamard 2-manifold in [9] or [11] via Propositions 3.3 and 3.7, Lemma 3.6, Theorem 3.5 above, and we give a detail here for completeness.

Assume first that $\operatorname{diam}_{\mathrm{Td}} X(\infty) < \infty$. By Theorem 3.3, $(X(\infty), \mathrm{Td})$ is homeomorphic to $\mathbb{S}^1$. Since $\operatorname{Sing}(X)$ is a countable set, we can choose a sequence $\alpha_1, \ldots, \alpha_n$ of geodesic rays emanating from a point $p \in X \setminus \operatorname{Sing}(X)$ such that

(1) there exists no singular points on α_i for each i,

(2) $\mathrm{Td}(\alpha_i(\infty), \alpha_{i+1}(\infty)) < \pi$ for each i and

$$\sum_{i=1}^{n} \mathrm{Td}\big(\alpha_i(\infty), \alpha_{i+1}(\infty)\big) = 2 \operatorname{diam}_{\mathrm{Td}} X(\infty),$$

where $\alpha_{n+1} := \alpha_1$. From the condition (1), it follows that $\alpha_i \cap \alpha_j = \{p\}$ for any $i \neq j$.

Let $F_i := \bigcup_{t \geqslant 0} \sigma_{\alpha_i(t)\alpha_{i+1}(t)}$. Applying Proposition 3.7 to each F_i and noticing that the exterior angle term vanishes on $\partial F_i \setminus \{p\}$ by (1), we have that

$$e(F_i) = \angle_p^{F_i}(\alpha_i, \alpha_{i+1}) - \mathrm{Td}\big(\alpha_i(\infty), \alpha_{i+1}(\infty)\big) \quad \text{for each } i = 1, \ldots, n.$$

Summing up the above equalities, and noticing that p is a nonsingular point, we obtain

$$e(X) = 2\big(\pi - \operatorname{diam}_{\mathrm{Td}} X(\infty)\big),$$

which completes the proof when $\operatorname{diam}_{\mathrm{Td}} X(\infty) < \infty$.

Next we assume that $\operatorname{diam}_{\mathrm{Td}} X(\infty) = \infty$. Take two points $\alpha(\infty)$ and $\beta(\infty)$ such that $\mathrm{Td}(\alpha(\infty), \beta(\infty)) > \pi$. The boundary $(X(\infty), st) \approx \mathbb{S}^1$ is divided into two intervals I_1 and J_1 by $\alpha(\infty)$ and $\beta(\infty)$. Since $\operatorname{diam}_{\mathrm{Td}} X(\infty) = \infty$, either $l_{\mathrm{Td}}(I_1) = \infty$ or $l_{\mathrm{Td}}(J_1) = \infty$, where l_{Td} is the one-dimensional Hausdorff measure on $(X(\infty), \mathrm{Td})$. If $l_{\mathrm{Td}}(I_1) = \infty$, there exists a point $\xi_1 \in I_1$ such that $\mathrm{Td}(\alpha(\infty), \xi_1) > \pi$ and $\mathrm{Td}(\beta(\infty), \xi_1) > \pi$. Let $I_1 = I_2 \cup J_2$ where I_2 and J_2 are closed intervals in $(X(\infty), st)$ with $\partial I_2 = \{\alpha(\infty), \xi_1\}$ and $\partial J_2 = \{\beta(\infty), \xi_1\}$. Then $l_{\mathrm{Td}}(I_2) = \infty$ or $l_{\mathrm{Td}}(J_2) = \infty$. Assuming the former case, take a point $\xi_2 \in I_2$ which is far from $\alpha(\infty)$ and ξ_1. Continuing this process and changing the indexes, we can choose, for each $l \in \mathbb{N}$, a sequence $\{\xi_i \mid i = 1, \ldots, l\}$ with $\xi_0 = \alpha(\infty) < \xi_1 < \cdots < \xi_{l+1} = \beta(\infty)$ with respect to the natural order on $(X(\infty), st)$ such that

$$\mathrm{Td}(\xi_i, \xi_{i+1}) > \pi \quad \text{for each } i = 0, \ldots, l. \tag{1}$$

Then we can choose a suitable non-singular point p such that, there exists no singular points on the geodesic γ_i from p such that, $\gamma_i(\infty) = \xi_i$ for each i. By the condition (1), there exists a straight line $\sigma_i : \mathbb{R} \to X$ such that $\sigma_i(\infty) = \gamma_i(\infty) = \xi_i$ and $\sigma_i(-\infty) = \gamma_{i+1}(\infty) = \xi_{i+1}$ (cf. Lemma 4.10 in [3]). Let F_i be the domain bounded by γ_i, γ_{i+1}

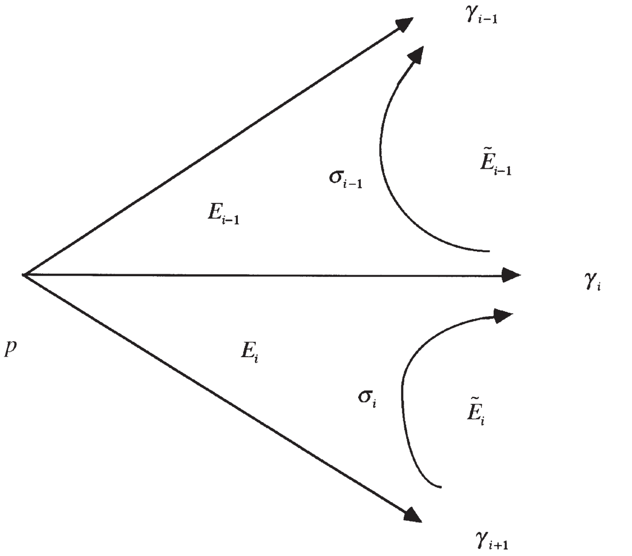

Fig. 5.

containing σ_i, and let E_i be the subdomain of F_i bounded by γ_i, γ_{i+1} and σ_i, and let $\widetilde{E}_i$ be the closure of $F_i \setminus E_i$ (see Fig. 5). For each $t > 0$, let $E_i(t)$ be the compact domain bounded by $\gamma_i|_{[0,t]}$, $\gamma_{i+1}|_{[0,t]}$ and $\sigma_{\gamma_i(t)\gamma_{i+1}(t)}$. Applying Theorem 2.1, we have that

$$\begin{aligned} e\bigl(E_i(t)\bigr) = {} & \angle_p^{F_i}(\gamma_i, \gamma_{i+1}) + \angle_{\gamma_i(t)}^{E_i}\bigl((\gamma_i|_{[0,t]})^{-1}, \sigma_{\gamma_i(t)\gamma_{i+1}(t)}\bigr) \\ & + \angle_{\gamma_{i+1}(t)}^{E_i}\bigl((\gamma_{i+1}|_{[0,t]})^{-1}, \sigma_{\gamma_{i+1}(t)\gamma_i(t)}\bigr) - \pi - \sum_{\overset{\circ}{\sigma}_{\gamma_i(t)\gamma_{i+1}(t)}} \theta^{E_i(t)}. \qquad (*) \end{aligned}$$

Let $q \in \sigma_i$. By Lemma 3.1 in [6], we have that

$$\angle_{\gamma_i(t_j)}^{E_i}\bigl((\gamma_i|_{[0,t_j]})^{-1}, \sigma_{q\gamma_i(t_j)}\bigr) \to 0$$

as $j \to \infty$ for some divergent sequence t_j. Hence

$$\angle_{\gamma_i(t_j)}^{E_i}\bigl((\gamma_i|_{[0,t_j]})^{-1}, \sigma_{\gamma_i(t_j)\gamma_{i+1}(t_j)}\bigr) \to 0$$

as $j \to \infty$ because of

$$\angle_{\gamma_i(t_j)}^{E_i}\bigl((\gamma_i|_{[0,t_j]})^{-1}, \sigma_{\gamma_i(t_j)\gamma_{i+1}(t_j)}\bigr) \leqslant \angle_{\gamma_i(t_j)}^{E_i}\bigl((\gamma_i|_{[0,t_j]})^{-1}, \sigma_{q\gamma_i(t_j)}\bigr).$$

Also the proof of Lemma 3.2 of [6] (the proof of Case 2) reveals that

$$\lim_{t\to\infty} \sum_{\overset{\circ}{\sigma}_{\gamma_i(t)\gamma_{i+1}(t)}} \theta^{E_i(t)} = \sum_{\sigma_i} \theta^{E_i}.$$

Hence tending t of $(*)$ to infinity,

$$e(E_i) = \angle_p^{F_i}(\gamma_i, \gamma_{i+1}) - \pi - \sum_{\sigma_i} \theta^{E_i}.$$

Repeating the proof of Theorem A in [6] with respect to $\widetilde{E}_i$ and σ_i, we have that $e(\widetilde{E}_i) \leqslant -\sum_{\sigma_i} \theta^{\widetilde{E}_i}$. Therefore, we have

$$\begin{aligned} e(F_i) &= e(E_i) + e(\widetilde{E}_i) + \sum_{\mathring{\sigma}_i} k \\ &\leqslant \angle_p^{F_i}(\gamma_i, \gamma_{i+1}) - \pi - \sum_{\mathring{\sigma}_i} \theta^{E_i} - \sum_{\mathring{\sigma}_i} \theta^{\widetilde{E}_i} + \sum_{\mathring{\sigma}_i} k \\ &= \angle_p^{F_i}(\gamma_i, \gamma_{i+1}) - \pi. \end{aligned}$$

Summing up all these inequalities, we have that $e(X) = \sum e(F_i) \leqslant (2-l)\pi$. Since l is arbitrary, $e(X) = -\infty$. This completes the proof. □

Remark. The last step of the proof above shows the following statement which will be used later; If there is a straight line from $\alpha(\infty)$ to $\beta(\infty)$, then the surface component F bounded by α and β with the vertex p satisfies that $e(F) \leqslant \angle_p^F(\alpha, \beta) - \pi$.

The following corollary is essentially proved in Proposition 2.1 in [10] for Hadamard 2-manifolds.

Corollary 3.8. *Let X be a simply connected nonpositively curved piecewise Riemannian 2-manifold without boundary. The following conditions are equivalent*:

(a) $(X(\infty), \mathrm{Td})$ *is homeomorphic to* $\mathbb{S}^1$.
(b) $(X(\infty), \mathrm{Td})$ *is compact.*
(c) $\mathrm{diam}_{\mathrm{Td}}\, X(\infty)$ *is finite.*
(d) $e(X)$ *is finite.*

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are obvious and (c) $\Rightarrow$ (a) follows from Proposition 3.4. Theorem 3.5 implies the equivalence (c) $\Leftrightarrow$ (d). □

Corollary 3.9. *If X has an infinite properly discontinuous group Γ of isometries on X and* $(X(\infty), \mathrm{Td})$ *is compact, then X is isometric to* $\mathbb{R}^2$.

Proof. Suppose that there exists a compact domain K of X such that $e(K) < 0$. Since the action is properly discontinuous, the orbit $\Gamma(K)$ tends to infinity. Also, since the elements of Γ act on X as isometries, $e(\gamma(K)) < 0$ for any $\gamma \in \Gamma$. Therefore $e(X) = -\infty$ and this contradicts Corollary 3.8. □

The next result shows a more precise connection between the total excess and the Tits topology of X and will be used in Section 4.

For a continuous curve $c:[0,1] \to (X(\infty), st)$, let $s_t := \sup\{s \geqslant 0 \mid \sigma_{pc(0)}(s) = \sigma_{pc(t)}(s)\}$. Also, let

$$F_t := \bigcup_{0 \leqslant s \leqslant t} \sigma_{pc(s)}\big([s_t, \infty)\big),$$

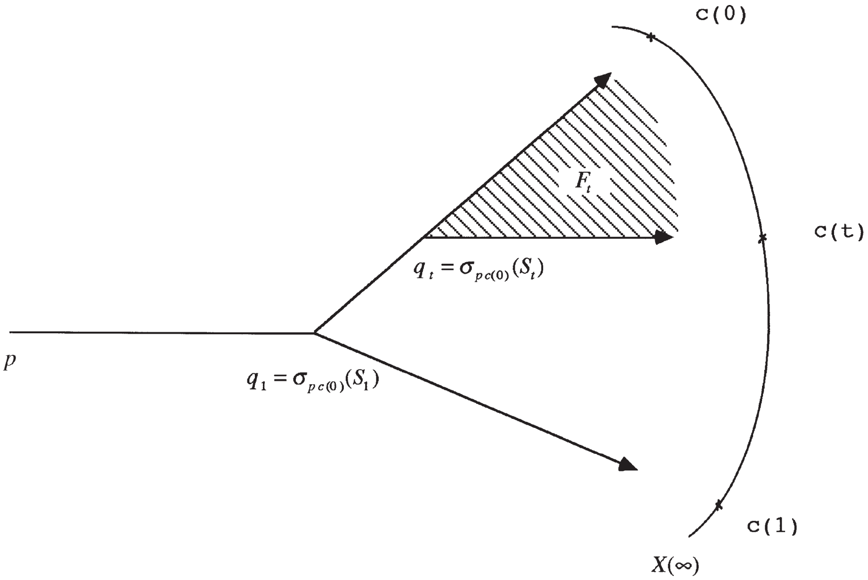

Fig. 6.

the surface component of the domain bounded by $\sigma_{pc(0)}$ and $\sigma_{pc(t)}$, and $q_t := \sigma_{pc(0)}(s_t)$, the vertex of F_t (see Fig. 6). Note that s_t is a monotone decreasing function on $(0,1]$ and may be divergent as $t \to 0$, and q_t is a negative singular point if $s_t > 0$. Note also that $L(\Sigma_{q_t}^{F_t})$ is not necessarily convergent to 0 as $t \to 0$. For notational convenience, let $\overline{e}(F_t) := e(F_t) + \sum_{\partial F_t \setminus \{q_t\}} \theta^{F_t}$.

Proposition 3.10. *Under the above notation, the following conditions are equivalent*:

(a) $\lim_{t\to 0} \mathrm{Td}(c(0), c(t)) = 0$,

(b) $\lim_{t\to 0} \overline{e}(F_t) = 0$,

(c) *for some* $t' > 0$, $\overline{e}(F_{t'})$ *is finite.*

Proof. We begin with the following claim: If $\sum_{\sigma_{q_t c(0)}} \theta^{F_t}$ is finite for some t, then $\lim_{t\to 0} \angle_{q_t}^{F_t}(\sigma_{q_t c(0)}, \sigma_{q_t c(t)}) = 0$.

Indeed, if the set $\{q_t \mid t > 0\}$ is contained in a bounded domain, then the conclusion follows easily from the fact that $\lim_{t\to 0} c(t) = c(0)$ with respect to the standard topology. Suppose that $q_t \to \infty$ as $t \to 0$. Since $\sum_{\sigma_{q_t c(0)}} \theta^{F_t}$ is finite, we have that $\lim_{t\to 0} \theta^{F_t}(q_t) = 0$, which clearly implies the conclusion in this case as well.

(a) $\Rightarrow$ (b) Under the present notation, Proposition 3.7 implies that, for small t,

$$\overline{e}(F_t) = \angle_{q_t}^{F_t}\big(\sigma_{q_t c(0)}, \sigma_{q_t c(t)}\big) - \mathrm{Td}\big(c(0), c(t)\big).$$

On the other hand, enumerating all singular points on $\sigma_{q_t c(0)}$ and proceeding as in the proof of Proposition 3.6, we obtain that

$$\mathrm{Td}\big(c(0), c(t)\big) \geqslant \angle_{q_t}^{F_t}\big(\sigma_{q_t c(0)}, \sigma_{q_t c(t)}\big) - \sum_{\sigma_{q_t c(0)} \setminus \{q_t\}} \theta^{F_t}.$$

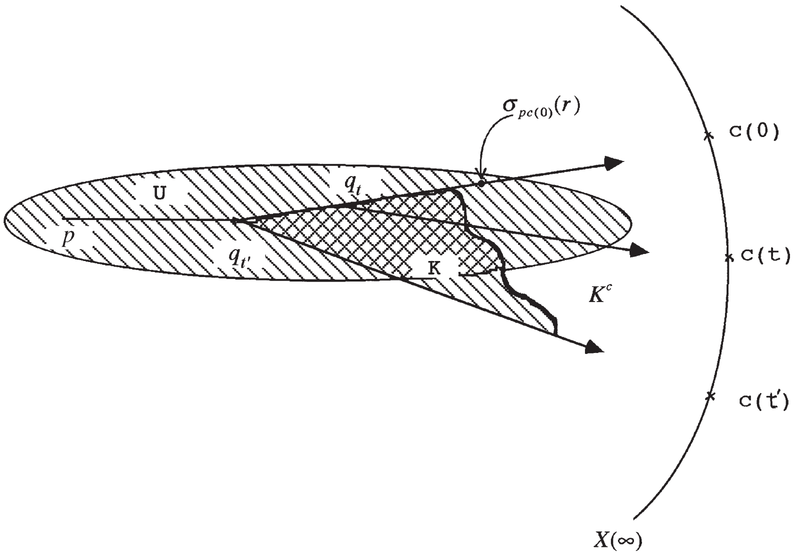

Fig. 7.

Hence $\sum_{\sigma_{q_t c(0)}} \theta^{F_t}$ is finite. Then, by the claim above, $\lim_{t\to 0} \angle_{q_t}^{F_t}(\sigma_{pc(0)}, \sigma_{pc(t)}) = 0$, which implies the condition (b).

(b) ⇒ (a) Since $\lim_{t\to 0} \sum_{\partial F_t \setminus \{q_t\}} \theta^{F_t} = 0$ by (b), we have that $\sum_{\sigma_{q_t c(0)}} \theta^{F_t}$ is finite for each t. Hence, by the claim above, $\lim_{t\to 0} \angle_{q_t}^{F_t}(\sigma_{pc(0)}, \sigma_{pc(t)}) = 0$. If there is a straight line from $c(0)$ to $c(t)$, then by remark after Theorem 3.5, $\overline{e}(F_t) \leqslant \angle_{q_t}^{F_t}(\sigma_{pc(0)}, \sigma_{pc(t)}) - \pi$. Hence $\lim_{t\to 0} \overline{e}(F_t) \leqslant -\pi$, which contradicts the assumption (b). Hence there is no straight line from $c(0)$ to $c(t)$ for small t. Now applying Proposition 3.7, we obtain the conclusion.

(b) ⇒ (c) This is trivial.

(c) ⇒ (b) Fix an $\varepsilon > 0$ arbitrarily. We need to find a positive constant t_1 such that, for any $t \in (0, t_1)$, $\overline{e}(F_t) > -\varepsilon$.

For a given $\varepsilon > 0$, there exists a compact domain K of $F_{t'}$ such that

$$-\frac{\varepsilon}{2} < e(K^c) + \sum_{\partial K^c \cap (\sigma_{pc(0)} \cup \sigma_{pc(t')})} \theta^{K^c} \leqslant 0,$$

where $K^c := F_{t'} \setminus \mathring{K}$. Note that

$$-\frac{\varepsilon}{2} < e(K_t^c) + \sum_{\partial K_t^c \cap (\sigma_{pc(0)} \cup \sigma_{pc(t)})} \theta^{K_t^c} \leqslant 0 \quad \text{for each } 0 < t \leqslant t', \tag{1}$$

where $K_t^c := K^c \cap F_t$. For a large number $r > 0$ such that $\sigma_{pc(0)}([0, r]) \supset \partial K \cap \sigma_{pc(0)}$, there exists a neighborhood U of $\sigma_{pc(0)}([0, r])$ such that $\mathrm{Sing}(X) \cap U \subset \sigma_{pc(0)}([0, r])$. See Fig. 7. Then we can take a small $t_0 > 0$ such that $K_t := F_t \cap K \subset U$ and $e(K_t) =$

$e_{\mathrm{reg}}(K_t) > -\varepsilon/2$ for any $t \in (0, t_0]$. There is no singular point on $\sigma_{pc(t)}((s_t, \infty]) \cap K$ and $\partial K_t \cap \partial K_t^c$, and hence for $t \in (0, t_0]$,

$$\sum_{\sigma_{pc(t)}((s_t,\infty])\cap K} \theta^{F_t} = 0 \quad \text{and} \quad \sum_{\partial K_t \cap \partial K_t^c} k = 0. \tag{2}$$

Suppose that there is no singular point on $\sigma_{pc(0)}((s_{t_0}, r])$. Since $\sigma_{pc(0)}((s_{t_0}, \infty)) \cap \partial K \subset \sigma_{pc(0)}((s_{t_0}, r])$, we have $\sum_{\sigma_{pc(0)}((s_{t_0},\infty))\cap\partial K} \theta^{F_{t_0}} = 0$. Let $t_1 := t_0 > 0$. Then for any $0 < t < t_1$,

$$\begin{aligned}
\overline{e}(F_t) &= e(F_t) + \sum_{\partial F_t \setminus \{q_t\}} \theta^{F_t} \\
&= e(K_t) + e(K_t^c) + \sum_{\partial K_t \cap \partial K_t^c} k + \sum_{\sigma_{pc(0)}((s_t,\infty))} \theta^{F_t} + \sum_{\sigma_{pc(t)}((s_t,\infty))} \theta^{F_t} \\
&= e(K_t) + e(K_t^c) + \sum_{(\sigma_{pc(0)}\cup\sigma_{pc(t)})\cap\partial K_t^c} \theta^{F_t} \\
&> -\varepsilon.
\end{aligned}$$

The third equality follows from (2).

If there is a singular point on $\sigma_{pc(0)}((s_{t_0}, r])$, then let $s_0 := \max\{s \mid \sigma_{pc(0)}(s) \in \mathrm{Sing}(X),\ s_{t_0} < s \leqslant r\}$. It is clear that there is a positive constant $t_1 < t_0$ such that $s_{t_1} \geqslant s_0$. Then $\sum_{\sigma_{pc(0)}((s_t,\infty))\cap\partial K} \theta^{F_t} = 0$ for any $t < t_1$. Then for any $0 < t < t_1$, $\overline{e}(F_t) \geqslant -\varepsilon$ as the above computation, which completes the proof. □

4. A construction of a 2-dimensional Hadamard space with the prescribed boundary at infinity

As was mentioned before, for any simply connected nonpositively curved piecewise Riemannian 2-manifold X without boundary, the identity map $id : (X(\infty), \mathrm{Td}) \to (X(\infty), st) \approx \mathbb{S}^1$ is continuous. It follows easily from this fact that each connected component of $(X(\infty), \mathrm{Td})$ is homeomorphic to either a point or an (open, closed or half-open) interval. These components form a decomposition of $\mathbb{S}^1$. A natural question arises as to whether there is some restriction on the "configuration" of the components. The following theorem states that there is no such restriction.

To state our result precisely, we introduce the following definition. A *decomposition of a topological space* A is a collection $\mathcal{D}$ of connected subsets of A such that $A_1 \cap A_2 = \emptyset$ for any $A_1 \neq A_2 \in \mathcal{D}$ and $\bigcup \mathcal{D} = A$. Let $\mathbf{A}_{\mathcal{D}}$ be the topological space with the weak topology with respect to the elements of $\mathcal{D}$. That is, a subset G of $\mathbf{A}_{\mathcal{D}}$ is open if and only if $G \cap D$ is open with respect to the relative topology of D for any $D \in \mathcal{D}$.

Theorem 4.1. *Let $\mathcal{D}$ be a decomposition of $\mathbb{S}^1$ into points and subsets homeomorphic to intervals. Then there exist a simply connected nonpositively curved piecewise Riemannian 2-manifold X and a homeomorphism $f : (X(\infty), st) \to \mathbb{S}^1$ such that $f : (X(\infty), \mathrm{Td}) \to \mathbb{S}^1_{\mathcal{D}}$ is also a homeomorphism.*

Construction of X. We identify $\mathbb{S}^1$ with $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. First we prepare some basic pieces of the construction. Let $\mathcal{J}$ be the set of all elements of $\mathcal{D}$ which are homeomorphic to intervals and let $A = \bigcup \mathcal{J}$ and $B = \mathbb{S}^1 \setminus A$. The set of all connected components of B on $\mathbb{S}^1$ which are homeomorphic to intervals is denoted by $\mathcal{B}$. Each connected component of the set $\mathbb{S}^1 \setminus \bigcup\{\mathcal{J} \cup \mathcal{B}\}$ is a point.

Note that the collection $\mathcal{J} \cup \mathcal{B}$ is countable and can be enumerated as $\{E_i \mid i = 1, 2, \ldots\}$. Let $\{a_i, b_i\}$ be the end points of $\overline{E_i}$, the closure of E_i.

Fix an $\varepsilon > 0$ arbitrarily and let $l(E_i)$ be the length of $E_i \subset \mathbb{S}^1$. For each E_i, we take a piecewise Riemannian 2-manifold F_i with boundary as follows.

Case 1. $E_i \in \mathcal{J}$ is homeomorphic to a closed interval. Let F_i be a sector in $\mathbb{R}^2$ with the vertex angle $l(E_i)$.

Case 2. $E_i \in \mathcal{J}$ is homeomorphic to a half-open interval with $a_i \in E_i$. Let S_0 be a sector with the vertex angle $l(E_i)$, bounded by the rays l_0^1 and l_0^2. Also, for each $j \in \mathbb{N}$, let S_j be a sector with the vertex angle ε, bounded by the rays l_j^1 and l_j^2. Let $f_0 : l_1^1([0, \infty)) \to l_0^2([i, \infty))$ be the obvious isometry with $f_0(l_1^1(0)) = l_0^2(i)$ and $f_j : l_{j+1}^1([0, \infty)) \to l_j^2([1, \infty))$ be the obvious isometry with $f_j(l_{j+1}^1(0)) = l_j^2(1)$ for each $j \geqslant 1$.

Let $L_0 := S_0 \cup_{f_0} S_1$ and inductively let $L_{j+1} = L_j \cup_{f_j} S_{j+1}$. Then we take F_i as the union $\bigcup_{j=0}^{\infty} L_j$, which is a piecewise flat Riemannian 2-manifold bounded by geodesic rays $\alpha = l_0^1([0, \infty))$ and $\beta = l_0^2([0, i]) \cup \bigcup_{j=1}^{\infty} l_j^2([0, 1])$. We say that the ray α (β, respectively) corresponds to the point at infinity a_i (b_i, respectively) and is denoted by γ_{a_i} (γ_{b_i}, respectively).

Case 3. $E_i \in \mathcal{J}$ is homeomorphic to an open interval. Let c be the midpoint of E_i. Then E_i is divided into two components E_i^1 and E_i^2 such that $E_i^1 \cap E_i^2 = \{c\}$, and for E_i^1 and E_i^2 we construct F_i^1 and F_i^2 as in Case 2. Let γ_1 and γ_2 be geodesic rays in F_i^1 and F_i^2, respectively, both of which correspond to the point c. Then F_i is obtained from F_i^1 and F_i^2 by gluing γ_1 and γ_2 by the obvious isometry.

Case 4. $E_i \in \mathcal{B}$. Note that an interval $E_i \in \mathcal{B}$ has the *discrete topology* in $\mathbb{S}^1_{\mathcal{D}}$. Let F_i be a simply connected Riemannian 2-manifold of nonpositive curvature bounded by geodesics γ_{a_i} and γ_{b_i} with $\gamma_{a_i} \cap \gamma_{b_i} = \{p_i\}$ such that

(1) $B(p_i, i)$ at p_i in F_i is isometric to the intersection of the sector with the vertex angle $l(E_i)$ and the i-ball at the vertex in $\mathbb{R}^2$, and
(2) there exists a compact set $K \supset B(p_i, i)$ such that $F_i \setminus K$ is isometric to a subset of the hyperbolic plane $\mathbb{H}^2$.

In above four cases, F_i is homeomorphic to a sector and bounded by two geodesic rays γ_{a_i} and γ_{b_i}. The point $p_i := \gamma_{a_i} \cap \gamma_{b_i}$ is called the vertex of F_i. Using these pieces, we will construct the space X.

Now let

$$\widehat{X} := \bigcup F_i / \sim,$$

where $p_i \sim p_j$ for each i, j and also, $x \sim y$ if $x = \gamma_{a_i}(t) \in F_i$ and $y = \gamma_{b_j}(t) \in F_j$ for $a_i = b_j \in \mathbb{S}^1$ and $t \geqslant 0$.

Note here that $\mathbb{S}^1 \setminus \bigcup \overline{E_i}$ may not be empty and for $x \in \mathbb{S}^1 \setminus \bigcup \overline{E_i}$, there exists a subsequence $\{E_{k_j}\}$ of $\{E_i\}$ which converges to x. Hence in order to construct the desired space X, we need to take the completion of $\widehat{X}$ with respect to an appropriate metric.

For a subset A of $\mathbb{S}^1$, let $c(A)$ be the infinite cone over the origin $o \in \mathbb{R}^2$: $c(A) = \{ta \mid a \in A,\ t \geqslant 0\}$. For each i, there exists a homeomorphism $h_i : F_i \to c(E_i)$ such that $h_i(p_i) = o$ and the restrictions

$$h_i : \gamma_{a_i} \to c(a_i) \quad \text{and} \quad h_i : \gamma_{b_i} \to c(b_i)$$

are isometries. Recall that p_i is the vertex of F_i and a_i, b_i are the end points of E_i. Then the map $\hat{h} = \bigcup h_i : \widehat{X} \to \mathbb{R}^2$ is a well-defined topological embedding of $\widehat{X}$ into $\mathbb{R}^2$. Pull back the standard metric of $\mathbb{R}^2$ to $\widehat{X}$ via $\hat{h}$ and the completion with respect to that metric is denoted by X. This metric is introduced only to define a topology on X.

Next we will show that X has a metric such that the inclusion map from F_i into X is an isometry for each i. The metric on X is defined as follows: For $x, y \in X$, let c be a continuous curve from x to y with respect to the above topology. The length $l(c)$ of c is defined by

$$l(c) := \sum_i l(c \cap F_i),$$

where $l(c \cap F_i)$ is the length of the curve $c \cap F_i$ on F_i. Then the distance $d(x, y)$ on X is defined as the infimum of the lengths of such curves. It clear that the inclusion map from F_i with the original metric into (X, d) is an isometry for each i.

Since F_i is flat on the i-ball centered at the vertex p_i for each i, it is easily seen that 1-ball centered at $p \in X$, the equivalence class of p_i, is isometric to the 1-ball in $\mathbb{R}^2$. Furthermore, by tending $i \to \infty$, we can show easily that any point $x \in X \setminus \widehat{X}$ has a flat neighborhood. This fact guarantees that the set of the singular points is contained in $\bigcup_i \partial F_i$ and X admits a structure of a simply connected nonpositively curved piecewise Riemannian 2-manifold without boundary which induces the metric d above as the natural metric.

Finally we show that, with respect to this metric, the resulting space X is the required piecewise Riemannian 2-manifold. From the construction of X, it is easily seen that there exists a homeomorphism $f : (X(\infty), st) \to \mathbb{S}^1$ such that $f(F_i(\infty)) = \overline{E_i}$, $f(\gamma_{a_i}(\infty)) = a_i$, and $f(\gamma_{b_i}(\infty)) = b_i$ for each i, where $F_i(\infty)$ denotes the set of all points at infinity defined by the equivalence classes of geodesic rays on F_i. Then we shall verify that $f : (X(\infty), \mathrm{Td}) \to \mathbb{S}^1_{\mathcal{D}}$ is a homeomorphism, which follows from the following two claims:

(1) The collection $f^{-1}(\mathcal{D})$ is exactly the collection of the components of $(X(\infty), \mathrm{Td})$.

(2) For each $D \in \mathcal{D}$, $f|_{f^{-1}(D)} : (f^{-1}(D), \mathrm{Td}) \to D$ is a homeomorphism.

To check the claims, we divide our consideration into several cases.

Case 1. $E_i \in \mathcal{J}$. Suppose that E_i is homeomorphic to a half-open interval with $a_i \in E_i$. From the construction of F_i, we see that $\sum_{\gamma_{b_i}} \theta^{F_i} = -\sum \varepsilon = -\infty$, while there is no singular point in $\mathring{F_i}$ and $\sum_{\gamma_{a_i}} \theta^{F_i} = 0$. Thus for each geodesic rays $\gamma \neq \gamma_{b_i}$ on F_i from p, $\overline{e}(F_i^{\gamma}) = -\infty$, where F_i^{γ} is the subdomain of F_i bounded by γ_{b_i} and γ. Hence Proposition 3.7 implies that $\gamma_{b_i}(\infty)$ is not "accessible" from $\gamma_{a_i}(\infty)$. Let $\widehat{F}_i^{\gamma}$ be the closure of $F_i \setminus F_i^{\gamma}$. Since $\overline{e}(\widehat{F}_i^{\gamma})$ is finite, all other points of $F_i(\infty)$ are joined with

$\gamma_{a_i}(\infty)$ by geodesics on $F_i(\infty)$. This means that $F_i(\infty)$ is isometric to $[0,\infty) \cup \{\infty\}$ in such a way that ∞ corresponds to $f^{-1}(b_i)$. Hence $f|_{f^{-1}(E_i)} : (f^{-1}(E_i), \mathrm{Td}) \to E_i$ is a homeomorphism.

Since $a_i \in E_i$, it is clear that there exists no $E_j \in \mathcal{J}$ $(i \neq j)$ such that $a_i \in E_j$. Hence a_i is not accessible from another side. This implies that $f^{-1}(E_i)$ is a connected component of $(X(\infty), \mathrm{Td})$.

When E_i is homeomorphic to an open or closed interval, a similar proof to the above shows that $f|_{f^{-1}(E_i)} : f^{-1}(E_i) \to E_i$ is a homeomorphism and $f^{-1}(E_i)$ is a connected component of $(X(\infty), \mathrm{Td})$ as well.

Case 2. $d \in \mathcal{D} \setminus \mathcal{J}$. Note that d is a point on $\mathbb{S}^1$. Then it is clear that $f|_{f^{-1}(d)} : f^{-1}(d) \to d$ is a homeomorphism. Hence it suffices to show that $f^{-1}(d)$ is an isolated point on $(X(\infty), \mathrm{Td})$.

Case 2.1. $d \in \mathring{E}_i$ for some $E_i \in \mathcal{B}$. In this case, F_i is isometric to a subdomain of $\mathbb{H}^2$ near infinity, so it is easy to see that $(F_i(\infty), \mathrm{Td})$ is discrete. Therefore $f^{-1}(d)$ is an isolated point.

Case 2.2. There is no $E_i \in \mathcal{B}$ such that $d \in \mathring{E}_i$. For $\xi := f^{-1}(d)$, take a neighborhood U of the point ξ with respect to the standard topology such that $U = I \cup J$, where I and J are half-open intervals which have ξ as their end points such that $I \cap J = \{\xi\}$. If ξ is "accessible" from the "I-side" with respect to Td, Proposition 3.10 implies that there is an interval $E \in \mathcal{D}$ on I-side such that $d \in E$, which contradicts the assumption $d \in \mathcal{D} \setminus \mathcal{J}$. Hence ξ is not accessible from either side of I or J. Hence ξ is an isolated point in $(X(\infty), \mathrm{Td})$.

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