Criteria for Generalized Diagonally Dominant Matrices and $M$-matrices

Yi-ming Gao and Xiao-hui Wang

Mathematics Department
Northeast Normal University
Changchun, Jinlin, 130024, People’s Republic of China

Submitted by Richard A. Brualdi

ABSTRACT

We provide new necessary and sufficient conditions for identifying generalized diagonally dominant matrices and obtain some criteria for judging nonsingular $M$-matrices.

1. INTRODUCTION

Let $A$ be an $n \times n$ complex matrix, $N = \{1, 2, \ldots, n\}$ $J = \{i \mid |a_{ii}| > \sum_{j \neq i} |a_{ij}| = \Lambda_i, i \in N\} \neq \emptyset$. When $J = N$, $A$ is a strictly diagonally dominant matrix; then $\det A \neq 0$ by the Lévy-Desplanques theorem. O. Taussky [1] once proved that if $A$ is an irreducible matrix and $|a_{ii}| > \Lambda_i, i \in N$, then $\det A \neq 0$. P. N. Shivakumar and K. H. Chew [2] showed that if $|a_{ii}| > \Lambda_i, i \in N$, and

(*) there exists a nonzero element chain $a_{i_1}, a_{i_1}, \ldots, a_{i_k}$ for any $i \in N - J$ where $p \in J$,

then $\det A \neq 0$. Since an irreducible diagonally dominant matrix satisfies the above conditions, Shivakumar and Chew [2] generalized the results of [1].

Let $A = (a_{ij})$ be an $n \times n$ complex matrix. If there exists a positive diagonal matrix $D$ such that $AD$ is a strictly diagonally dominant matrix, then $A$ is a generalized diagonally dominant matrix (GDDM).

Yi-ming Gao [3] proved that if $A$ is a GDDM, or $B = \frac{1}{2}(A + A^*)$ is a GDDM with positive (or negative) diagonal elements, then $\det A \neq 0$. Moreover, he proved that if $|a_{ii}|a_{kk} > \Lambda_i \Lambda_k, 1 \leq i, k \leq n$, or if $A$ satisfies the
condition (*) stated in [2], then $A$ is a GDDM. Thus Gao [3] revealed the common role of $\det A \neq 0$ in [1] and [2]. Gao [4] also proved that if $N_1 = J$, $N_2 = N - J$ and if

$$ (|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) \geq \beta_i \alpha_j $$

(1.1)

for any $i \in N_1$, $j \in N_2$, where

$$ \alpha_i = \sum_{j \in N_1, j \neq i} |a_{ij}|, \quad \beta_i = \sum_{j \in N_2, j \neq i} |a_{ij}|, $$

then one has the following results: If all strict inequalities for any pair of indices hold in (1.1) or $A$ is an irreducible matrix and strict inequality holds in (1.1) for at least one pair of indices, then $A$ is a GDDM, i.e. $\det A \neq 0$. If all "\(\geq\)" are changed to "\(<\)" in (1.1), then $A$ is not a GDDM.

In this paper let $A = (a_{ij})$ be an $n \times n$ complex matrix, $N = \{1, 2, \ldots, n\}$, $J = \{i \mid |a_{ii}| > \sum_{j \neq i} |a_{ij}|, i \in N\} \neq \emptyset$. If there exist $N_1, N_2$ such that $N_1 \cup N_2 = N$ and

$$ (|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) \geq \beta_i \alpha_j $$

(1.2)

for any $i \in N_1$, $j \in N_2$, where

$$ \alpha_i = \sum_{j \in N_1, j \neq i} |a_{ij}|, \quad \beta_i = \sum_{j \in N_2, j \neq i} |a_{ij}|, $$

then we get the following results:

(a) If all strict inequalities hold in (1.2) for any pair of indices or $A$ is an irreducible matrix and strict inequality holds in (1.2) for at least one pair of indices, then $A$ is a GDDM.

(b) If all "\(\geq\)" are changed to "\(<\)" in (1.2), then $A$ is not a GDDM.

This paper is organized as follows. Section 2 contains the main results and their proofs. It is also shown in the remarks of Section 2 that the theorems in this paper are extensions of the main theorems in [1–4]. In Section 3 we give two examples to further illustrate the generalizations; moreover, we provide a method to choose the diagonal matrix $D$ which makes $AD$ a strict diagonally dominant matrix.
2. THE MAIN RESULTS

Theorem 1. Let $A = (a_{ij})$ be an $n \times n$ complex matrix, $N = \{1, 2, \ldots, n\}$, $J = \{i \mid |a_{ii}| > \sum_{j \neq i} |a_{ij}| = A_i, i \in N\} \neq \emptyset$, and $M(A) = (m_{ij})$ with

$$m_{ii} = |a_{ii}|, \quad m_{ij} = -|a_{ij}|, \quad i \neq j,$$

$i, j \in N$. If there exist $N_1, N_2$ such that $N_1 \cup N_2 = N$ and

$$\left(|a_{ii}| - \alpha_i\right)\left(|a_{jj}| - \beta_j\right) > \beta_i \alpha_j$$

(2.1)

for any $i \in N_1$, $j \in N_2$, where

$$\alpha_i = \sum_{j \in N_1 \setminus \{i\}} |a_{ij}|, \quad \beta_i = \sum_{j \in N_2 \setminus \{i\}} |a_{ij}|,$$

then $A$ is a GDDM and $M(A)$ is a nonsingular $M$-matrix.

Proof. Let $A_1 = \sum_{j \neq i} |a_{ij}|$, $N_1 \subseteq J$, $N_2 = N - N_1$, $H_j = (|a_{jj}| - \beta_j) / \alpha_j$, $j \in N_2$, $h_i = \beta_i / (|a_{ii}| - \alpha_i)$, $i \in N_1$. From (2.1) we know that $H_j > h_i$ for any $i \in N_1$, $j \in N_2$, so we choose $d$ such that

$$\max_{i \in N_1} h_i < d < \min_{j \in N_2} H_j,$$

construct

$$D = \text{diag}(d_i, d_i = d, i \in N_1; d_i = 1, i \in N_2)$$

and write

$$A_1 = AD = \left(a^{(1)}_{ij}\right).$$

Then we have

$$|a^{(1)}_{ii}| - A^{(1)}_i = d(|a_{ii}| - \alpha_i) - \beta_i$$

$$> h_i(|a_{ii}| - \alpha_i) - \beta_i = 0$$
for any $i \in N_1$, and

$$\Lambda_j^{(1)} = d\alpha_j + \beta_j < H_j \alpha_j + \beta_j = |a_{jj}| = |a_j^{(1)}|$$

for any $j \in N_2$, so $A$ is a GDDM. From Chapter 6 (M35) in [5] we can get the result: $M(A)$ is a nonsingular $M$-matrix.

**Corollary 1.** If there exist two positive diagonal matrices $D$ and $E$ such that $DAE$ satisfies the condition (2.1), then $A$ is a GDDM and $M(A)$ is a nonsingular $M$-matrix.

**Corollary 2.** If there exist two nonsingular matrices $P$ and $Q$ such that $PAQ$ satisfies the condition (2.1), then $\det A \neq 0$.

**Corollary 3.** If $A$ is a matrix with positive (or negative) diagonal elements, and $B = \frac{1}{2}(A + A^*)$ satisfies the condition (2.1), then $\det A \neq 0$.

**Proof.** By the lemma in [3], we know that any eigenvalue $\mu_i$ of $A$ satisfies inequalities

$$\min \lambda_B \leq \Re \mu_i \leq \max \lambda_B,$$

since $B$ satisfies the condition (2.1), so there exists a diagonal matrix $D$ such that $C = D^{-1}BD$ is a strictly diagonally dominant matrix. Since $B$ has positive (negative) diagonal elements, we get

$$\min \lambda_B > 0 \ (\max \lambda_B < 0), \quad \text{i.e.,} \quad \Re \mu_i > 0 \ (\ < 0),$$

from which we get $\det A \neq 0$.

**Corollary 4.** If there exist two nonsingular matrices $P$ and $Q$ such that $B = PAQ$ is a matrix with positive (or negative) diagonal elements and $C = \frac{1}{2}(B + B^*)$ satisfies the condition (2.1), then $\det A \neq 0$.

**Proof.** By Corollary 3, we have

$$\det B = \det P \det A \det Q \neq 0,$$
GENERALIZED DIAGONALLY DOMINANT MATRICES

so

\[ \det A \neq 0. \]

**Remark 2.1.** From [3] we know that if \( A \) satisfies the condition of the theorem in [2], then there exists a positive diagonal matrix \( D \) such that \( B = AD \) is a strictly diagonally dominant matrix. Clearly \( B \) satisfies the condition of Theorem 1 in this paper.

If \( A \) satisfies the condition \( |a_{ii}| |a_{jj}| > \Lambda_i \Lambda_j, 1 \leq i, j \leq n, \) we can see that there exists at most one row \( p \) such that \( |a_{pp}| < \Lambda_p \) and \( |a_{ii}| > \Lambda_i, i \neq p, i \in N. \) We choose \( N_2 = \{p\}, N_1 = N - N_2; \) then

\[
(a_{ii} - \alpha_i)(|a_{pp}| - \beta_p) - \alpha_p \beta_i = |a_{ii}| |a_{pp}| - |a_{pp}| \alpha_i - \alpha_p \beta_i \geq |a_{ii}| |a_{pp}| - \Lambda_p \Lambda_i > 0,
\]

so the condition (2.1) in this paper is satisfied. Thus we extend the main results in [1], [2], and [3].

**Theorem 2.** Let \( A = (a_{ij}) \) be an irreducible matrix, \( J = \{i \mid |a_{ii}| > \sum_{j \neq i} |a_{ij}| = \Lambda_i, i \in N\} \neq \emptyset. \) If there exist \( N_1, N_2 \) such that \( N_1 \cup N_2 = N, \) and

\[
(a_{ii} - \alpha_i)(|a_{jj}| - \beta_j) \geq \beta_i \alpha_j \tag{2.2}
\]

for any \( i \in N_1, j \in N_2, \) and if there exists strict inequality for at least one pair of indices in (2.2), then \( A \) is a GDDM and \( M(A) \) is a nonsingular \( M \)-matrix.

**Proof.** Let \( N_1 \subseteq J, J = \{i \mid |a_{ii}| > \Lambda_i, i \in N\}, N_2 = N - N_1, \)

\[
h_i = \frac{\beta_i}{|a_{ii}| - \alpha_i}, \quad i \in N_1, \quad H_j = \frac{|a_{jj}| - \beta_j}{\alpha_j}, \quad j \in N_2.
\]

From (2.2) we know that \( H_j \geq h_i \) for any \( i \in N_1, j \in N_2, \) and there exists strict inequality for at least one pair of indices, so we suppose \( \max_{i \in N_1} h_i > h_p, \) choose \( d \) such that

\[
\max_{i \in N_1} h_i = d = \min_{j \in N_2} H_j.
\]
and construct

\[ D_1 = \text{diag}\{d_i | d_i = d, \ i \in N_1; \ d_i = 1, \ i \in N_2\} \]

and write \( A_1 = \Delta D_1 = (a_{ij}^{(1)})\).

When \( i \in N_1, \ i \neq p, \) we have

\[ |a_{ii}^{(1)}| - \Lambda_i^{(1)} = d(|a_{ii}| - \alpha_i) - \beta_i \geq h_i(|a_{ii}| - \alpha_i) - \beta_i = 0; \]

when \( i = p \in N_1, \) we have

\[ |a_{pp}^{(1)}| - \Lambda_p^{(1)} = d(|a_{pp}| - \alpha_p) - \beta_p \geq h_p(|a_{pp}| - \alpha_p) - \beta_p = 0; \]

when \( j \in N_2, \) we have

\[ \Lambda_j^{(1)} = d\alpha_j + \beta_j \leq H_j\alpha_j + \beta_j = |a_{jj}| = |a_{jj}^{(1)}|. \]

So \( A_1 = \Delta D_1 = (a_{ij}^{(1)}) \) is an irreducible diagonally dominant matrix. From Theorem 3 in [4], we know that there exists \( N = \bigcup_{t=1}^{k} M_t \) in which

\[ M_1 = \{ i | |a_{ii}^{(1)}| > \Lambda_i^{(1)} = \sum_{j \neq i} |a_{ij}^{(1)}|, \ i \in N \}, \quad M_t = \{ i | |a_{ij}^{(1)}| \neq 0, \ j \in M_{t-1} \}, \]

\[ 2 \leq t \leq k. \]

We choose

\[ \max_{i \in M_1} \frac{\Lambda_i^{(1)}}{|a_{ii}^{(1)}|} < \delta_1 < 1, \quad i \in M_1, \]

\[ \max_{i \in M_t} \frac{|a_{ii}^{(1)}| - (1 - \delta_{t-1}) r_i^{(1)}}{|a_{ii}^{(1)}|} < \delta_t < 1, \quad 2 \leq t \leq k - 1, \]

where \( r_i^{(1)} = \sum_{j \in M_{t-1}} |a_{ij}^{(1)}| \),
construct

\[ D_2 = \text{diag} \{ d_i \mid d_i = \delta_t, \ i \in M_t, \ 1 \leq t \leq k - 1; \ d_i = 1, \ i \in M_k \} , \]

and write \( A_2 = A_1 D_2 = (a_{ij}^{(2)}) \).

When \( i \in M_1 \), we have

\[ |a_{ii}^{(2)}| - \lambda_i^{(2)} \geq \delta_1 |a_{ii}^{(1)}| - \lambda_i^{(1)} > \lambda_i^{(1)} - \lambda_i^{(1)} = 0; \]

when \( i \in M_t \), we have

\[ |a_{ii}^{(2)}| - \lambda_i^{(2)} \geq \delta_t |a_{ii}^{(1)}| - \lambda_i^{(1)} + (1 - \delta_{t-1}) r_i^{(1)} \]

\[ > |a_{ii}^{(1)}| - \lambda_i^{(1)} = 0, \quad t = 2, \ldots, k - 1; \]

when \( i \in M_k \), we have

\[ |a_{ii}^{(2)}| - \lambda_i^{(2)} = |a_{ii}^{(1)}| - \delta_{k-1} \lambda_i^{(1)} > |a_{ii}^{(1)}| - \lambda_i^{(1)} = 0. \]

So if we choose the positive diagonal matrix \( D \) such that

\[ D = D_1 D_2 = \text{diag} \{ d_i \mid d_i = d \delta_t, \ i \in M_t \cap N_1 \}; \]

\[ d_i = \delta_t, \ i \in M_t \cap N_2 \ (1 \leq t \leq k - 1); \]

\[ d_i = d, \ i \in M_k \cap N_1, \ d_i = 1, \ i \in M_k \cap N_2 \}, \]

then \( A_2 = A_1 D_2 = AD_1 D_2 = AD \) is a strictly diagonally dominant matrix, i.e., \( A \) is a GDDM. From Chapter 6 (M35) in [5], we can get the result: \( M(A) \) is a nonsingular \( M \)-matrix.

**Corollary 1.** If there exist two positive diagonal matrices \( E \) and \( D \) such that \( B = EAD \) satisfies the condition (2.2), then \( A \) is a GDDM and \( M(A) \) is a nonsingular \( M \)-matrix.

**Corollary 2.** If there exist two nonsingular matrices \( P \) and \( Q \) such that \( PAQ \) satisfies the condition (2.2), then \( \det A \neq 0. \)
COROLLARY 3. If A is a matrix with positive (or negative) diagonal elements, and \( B = \frac{1}{2}(A + A^*) \) satisfies the condition (2.2), then \( \det A \neq 0 \).

Proof. By the lemma in [3], we know that any eigenvalue \( \mu_i \) of A satisfies the inequalities

\[
\min \lambda_B \leq \Re \mu_i \leq \max \lambda_B.
\]

Since \( B \) satisfies the condition (2.2), there exists a diagonal matrix \( D \) such that \( C = D^{-1}BD \) is an irreducible diagonally dominant matrix. Because \( B \) has positive (or negative) diagonal elements, we get

\[
\min \lambda_B > 0 \quad (\max \lambda_B < 0), \quad \text{i.e.,} \quad \Re \mu_i > 0 \quad (\, < 0); \]

hence \( \det A \neq 0 \).

COROLLARY 4. If there exist two nonsingular matrices \( P \) and \( Q \) such that \( B = PAQ \) is a matrix with positive (or negative) diagonal elements, and \( C = \frac{1}{2}(B + B^*) \) satisfies the condition (2.2), then \( \det A \neq 0 \).

Proof. By Corollary 3, we can get \( \det B = \det P \det A \det Q \neq 0 \), so \( \det A \neq 0 \).

THEOREM 3. Let \( A = (a_{ij}) \) be an \( n \times n \) complex matrix, and \( J = \{ i \mid |a_{ii}| > \Lambda, \ i \in N \} \neq \emptyset \). If there exist \( N_1, N_2 \) such that \( N_1 \cup N_2 = N \), and

\[
(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) < \beta_i \alpha_j
\]

for any \( i \in N_1, \ j \in N_2 \), then \( A \) is not a GDDM and \( M(A) \) is not a nonsingular \( M \)-matrix.

Proof. Just as in Theorem 1, from (2.3), for any \( i \in N_1, \ j \in N_2 \) we have \( H_j \leq h_i \). We choose \( d \) such that

\[
\max_{j \in N_2} H_j \leq d \leq \min_{i \in N_1} h_i.
\]
construct

\[ D = \text{diag}\{d_i \mid d_i = d, \ i \in N_1; \ d_i = 1, \ i \in N_2\}, \]

and write \( A_1 = AD = (a^{(1)}_{ij}) \). Then for any \( i \in N_1 \), we have

\[ |a^{(1)}_{ii}| - \Lambda^{(1)}_i = d(|a_{ii}| - \alpha_i) - \beta_i \leq h_i(|a_{ii}| - \alpha_i) - \beta_i = 0. \]

For any \( j \in N_2 \), we have

\[ A^{(1)}_j = d\alpha_j + \beta_j \geq H_j\alpha_j + \beta_j = |a_{jj}| = |a^{(1)}_{jj}|; \]

thus \( A_1 \) cannot be in a dominant row, so \( A_1 \) and \( A \) are not GDDMs, and from [5], we find that \( M(A) \) is not a nonsingular \( M \)-matrix.

\[ \square \]

Remark 2.2. We obviously generalize the condition

\[ N_1 = \{i \mid |a_{ii}| > \Lambda_i, \ i \in N\} \]

in [4] to

\[ N_1 \subseteq \{i \mid |a_{ii}| > \Lambda_i, \ i \in N\} \]

in this paper.

3. EXAMPLES

Example 1. Let

\[ A = \begin{pmatrix} 8 & 0 & 2 & 2 \\ 0 & 8 & 2 & 2 \\ 0 & 3 & 8 & 4 \\ 0 & 7 & 4 & 8 \end{pmatrix}. \]
Obviously $A$ is a reducible matrix which satisfies neither $|a_{ii}| > \lambda_i$ $(1 < i < n)$ nor $|a_{ii}|/|a_{kk}| > \lambda_i \lambda_k$ $(1 < i, k < n)$, that is, $A$ fails to satisfy the conditions of main theorems in [1], [2], and [3]. Since $A$ fails to satisfy the condition $(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) > \beta_i \alpha_j$ for any $i \in N_1 = \{1, 2, 3\}$, $j \in N_2 = \{4\}$, $A$ does not satisfy the conditions of the main theorems in [4] either. But $A$ satisfies the condition of Theorem 1 in this paper, where $N_1 = \{1, 2\}$, $N_2 = \{3, 4\}$.

We choose

$$\max_{i \in N_1} \frac{\beta_i}{|a_{ii}| - \alpha_i} = \frac{1}{2} < d = \frac{15}{28} < \min_{j \in N_2} \frac{|a_{jj}| - \beta_j}{\alpha_j} = \frac{4}{7}$$

and construct $D = \text{diag}(\frac{15}{28}, \frac{15}{28}, 1, 1)$. Then

$$A_1 = AD = \begin{pmatrix}
8 & 0 & 2 & 2 \\
0 & 8 & 2 & 2 \\
0 & 3 & 8 & 4 \\
0 & 7 & 4 & 8 \\
\end{pmatrix}
\begin{pmatrix}
\frac{15}{28} & \\
\frac{15}{28} & 1 \\
1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
30 \\
30 \\
45 \\
15 \\
\end{pmatrix}
= \begin{pmatrix}
0 & 2 & 2 \\
0 & 30 & 2 & 2 \\
0 & 45 & 8 & 4 \\
0 & 15 & 4 & 8 \\
\end{pmatrix}$$

is a strictly diagonally dominant matrix, so $A$ is a GDDM, $\det A \neq 0$.

**Example 2.** Let

$$A = \begin{pmatrix}
6 & 1 & 2 & 1 \\
3 & 6 & 1 & 1 \\
0 & 3 & 3 & 1 \\
0 & 0 & 3 & 3 \\
\end{pmatrix}.$$  

Clearly $A$ is an irreducible matrix satisfying the conditions of Theorem 2 in
this paper, where \( N_1 = \{1, 2\}, N_2 = \{3, 4\} \). We choose

\[
d = \max_{i \in N_1} \frac{\beta_i}{|a_{ij}| - \alpha_i} = \min_{j \in N_2} \frac{|a_{ij}| - \beta_j}{\alpha_j} = \frac{2}{3};
\]

then \( M_1 = \{1\}, M_2 = \{2\}, M_3 = \{3\}, M_4 = \{4\} \). We choose

\[
\Lambda_1^{(1)}/|a_{11}^{(1)}| = \frac{11}{15} < \delta_1 = \frac{23}{24},
\]

\[
\frac{|a_{22}^{(1)}| - (1 - \delta_1)\beta_2^{(1)}}{|a_{22}^{(1)}} = 4 - \left(1 - \frac{23}{24}\right) \times 2 = \frac{47}{48} < \delta_2 = \frac{95}{96},
\]

\[
\frac{|a_{33}^{(1)}| - (1 - \delta_2)\beta_3^{(1)}}{|a_{33}^{(1)}} = 3 - \left(1 - \frac{95}{96}\right) \times 2 = \frac{143}{144} < \delta_3 = \frac{287}{288},
\]

\( \delta_4 = 1 \)

and construct \( D = \text{diag}\{\frac{23}{36}, \frac{95}{144}, \frac{287}{288}, 1\} \). Then

\[
A_2 = AD = \begin{pmatrix} 6 & 1 & 2 & 1 \\ 3 & 6 & 1 & 1 \\ 0 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \end{pmatrix} \begin{pmatrix} \frac{23}{36} & \frac{95}{144} & \frac{287}{288} & 1 \\ \frac{23}{6} & \frac{95}{144} & \frac{287}{288} & 1 \\ \frac{23}{12} & \frac{95}{24} & \frac{287}{288} & 1 \\ \frac{23}{48} & \frac{95}{96} & \frac{287}{96} & 3 \end{pmatrix}
\]

is a strictly diagonally dominant matrix, so \( A \) is a GDDM, \( \det A \neq 0 \).

The authors are much indebted to Professor Richard A. Brualdi and the referees for their valuable suggestions and kind help in revising this paper to make it a more substantial one.
REFERENCES


Received 12 March 1990; final manuscript accepted 1 May 1991