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A CLASS OF SYMMETRIC POLYTOPES

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Abstract—In E^3 a polytope which possesses two facets which can be interchanged always possesses a second pair. However, this is not so in E^n, $n \geqslant 4$.

INTRODUCTION

This paper is the outcome of an empirical observation of one of us (A.H.) that whenever a 3-polytope possesses a pair of facets which could be interchanged then it always seemed to possess a second pair. Our first theorem proves that this must always be the case. However, we shall also show that in E^d, $d \geqslant 4$, there exists a d-polytope which has exactly two facets which can be interchanged. It would be interesting to characterize all such polytopes which seems to belong, as the proof of Theorem 1 shows, to a somewhat limited class.

If we dualize our observations in E^3 we are led to consider 3-polytopes with pairs of vertices which can be interchanged. Using Steinitz's theorem, Theorem 1 also holds for 3-connected planar graphs. However, we shall show that the result does not hold for all 3-connected graphs with at least 7 vertices.

Theorem 1

Let P be a polytope in E^3 which possesses a pair of facets which can be interchanged by an isomorphism of the face lattice. Then P possesses at least two such pairs.

Example 1

There exists in E^d, $d \geqslant 4$, a d-polytope P which possesses only one pair of facets which can be interchanged by an isomorphism of the face lattice of P.

The first part of Theorem 2 is the dual version of Theorem 1 (using Steinitz's theorem).

Theorem 2

Let G be a 3-connected planar graph which possesses a pair of vertices which can be interchanged by an isomorphism of the graph. Then there are at least two such pairs. Further, the result holds for all 3-connected graphs with at most 6 vertices, but there exists a 3-connected graph with 7 vertices with only one pair of interchangeable vertices.

Proof of Theorem 1. Let P^* be a 3-polytope and let ϕ^* be an isomorphism which interchanges two facets A, B of P^*. Let P be the dual of P^* and let ϕ be the isomorphism of the face lattice which interchanges the corresponding two vertices $\mathbf{a}$, $\mathbf{b}$ of P. We shall suppose, for the moment, that P is a d-polytope, $d \geqslant 3$, and specialize to $d = 3$ only when necessary.

We claim that we can assume that any vertex $\mathbf{c}$ of P which is joined to $\mathbf{a}$ by an edge is also joined to $\mathbf{b}$ by an edge. If not, then $\phi(\mathbf{c})$ is joined to $\mathbf{b}$ by an edge but not to $\mathbf{a}$. In particular $\phi(\mathbf{c}) \neq \mathbf{c}$. Indeed, repeating this process, we have that $\phi^{2m}(\mathbf{c})$ is joined to $\mathbf{a}$ by an edge but not to $\mathbf{b}$, and that $\phi^{2m+1}(\mathbf{c})$ is joined to $\mathbf{b}$ by an edge but not to $\mathbf{a}$. Eventually there exists m, n; $n > m$ such that $\phi^{2n}(\mathbf{c}) = \phi^{2m}(\mathbf{c})$. If we suppose that $n - m$ is minimal then $\phi^{n-m}(\mathbf{c}) \neq \mathbf{c}$. For, if $\phi^{n-m}(\mathbf{c}) = \mathbf{c}$ then $n - m$ must be odd (by the minimality of $n - m$) but then $\phi^{n-m}(\mathbf{c})$ is joined to $\mathbf{b}$ by an edge but $\mathbf{c}$ is not, i.e. $\phi^{n-m}(\mathbf{c}) \neq \mathbf{c}$. So ϕ^{n-m} is an isomorphism of P which interchanges $\mathbf{c}$ and $\phi^{n-m}(\mathbf{c})$ as required. Hence we may assume that any vertex $\mathbf{c}$ of P which is joined to $\mathbf{a}$ by an edge is also joined to $\mathbf{b}$ by an edge.

Let $\mathbf{c}_1, \ldots, \mathbf{c}_k$, $k \geqslant 2$ be a list of the vertices joined to both $\mathbf{a}$ and $\mathbf{b}$ by an edge. Let F be a facet of P containing $\mathbf{a}$ and let $[\mathbf{a}, \mathbf{c}_1], \ldots, [\mathbf{a}, \mathbf{c}_l]$ (possibly together with $[\mathbf{a}, \mathbf{b}]$) be a list of the edges containing **a** and contained in F. We claim that we may suppose that $[\mathbf{b}, \mathbf{c}_1], \ldots, [\mathbf{b}, \mathbf{c}_t]$ (possibly together with $[\mathbf{a}, \mathbf{b}]$) form the list of edges containing **b** and contained in some facet of G of P.

If not, then F cannot contain **b** for, if it did, then $[\mathbf{b}, \mathbf{c}_1], \ldots, [\mathbf{b}, \mathbf{c}_t]$ are the edges of F emanating from **b**. Hence $\phi(F) \neq F$. Eventually, there exists m, n, $n > m$, $n - m$ minimal such that $\phi^{2(n-m)}F = F$. So

$$\{\mathbf{c}_1, \ldots, \mathbf{c}_t\} = \{\phi^{2(n-m)}(\mathbf{c}_1), \ldots, \phi^{(n-m)}(\mathbf{c}_t)\}.$$

So for each j, $j = 1, \ldots, t$ there will be a least positive integer k_j such that $\phi^{2k_j}(\mathbf{c}_j) = \mathbf{c}_j$. If k_j were even, then $\phi^{k_j}(\mathbf{c}_j) \neq \mathbf{c}_j$ and so ϕ^{k_j} would be the required isomorphism. So each k_j can be supposed odd.

If there exists j, $1 \leqslant j \leqslant t$, with $\phi^{k_j}(\mathbf{c}_j) \neq \mathbf{c}_j$ then ϕ^{k_j} is the required isomorphism interchanging $\mathbf{c}_j$ and $\phi^{k_j}(\mathbf{c}_j)$. So we may suppose that $\phi^{k_j}(\mathbf{c}_j) = \mathbf{c}_j, j = 1, \ldots, t$. So $\phi^S(F)$ is a facet of P containing **b**, as required. Hence, we may suppose that $[\mathbf{b}, \mathbf{c}_1], \ldots, [\mathbf{b}, \mathbf{c}_t]$ (possibly together with $[\mathbf{a}, \mathbf{b}]$) form the list of edges containing **b** and containing in some facet G of P. If F and G are distinct then F is the convex hull of $\mathbf{a}, \mathbf{c}_1, \ldots, \mathbf{c}_t$ and G is the convex hull of $\mathbf{a}, \mathbf{b}, \mathbf{c}_1, \ldots, \mathbf{c}_t$. If $F = G$ then F is the convex hull of $\mathbf{a}, \mathbf{b}, \mathbf{c}_1, \ldots, \mathbf{c}_t$. From above, it also follows that P is the convex hull of $\mathbf{a}, \mathbf{b}, \mathbf{c}_1, \ldots, \mathbf{c}_t$ and the facets of P take one of the three forms $\mathrm{conv}\{\mathbf{a}, \mathbf{c}_1, \ldots, \mathbf{c}_t\}$, $\mathrm{conv}\{\mathbf{b}, \mathbf{c}_1, \ldots, \mathbf{c}_t\}$ and $\mathrm{conv}\{\mathbf{a}, \mathbf{c}_1, \ldots, \mathbf{c}_t\}$.

If we now dualize the situation there are two possible cases arising.

*(i) [**a** **b**] is not an edge of P*

In this case P^* is a cylinder with bottom facet A and top facet B, where A and B are the duals of **a** and **b**. For P^* in E^3 any two consecutive side facets are interchangeable under an isomorphism of the face lattice of P^*. An isomorphism will be the (combinatorial) reflection in their common edge.

*(ii) [**a** **b**] is an edge of P*

In this case P^* is the convex hull of A and B (where A and B are similar facets) with $A \cap B$ a $d-2$ face of P^*, i.e. P^* is almost a cylinder with bottom facet A and top facet B except that A meets B in a $d-2$ face $A \cap B$.

For P^* in E^3 the two side facets adjacent to $A \cap B$ are interchangeable under an isomorphism of the face lattice of P^*. An isomorphism will be the (combinatorial) reflection in $A \cap B$.

This completes the proof of Theorem 1.

Construction of Example 1

In order to construct the example let us continue the analysis of Theorem 1. An obvious candidate would be a cylinder P^* in E^d whose bottom facet A (and hence top face B) have no isomorphisms of their face lattice which interchanges $d-2$ faces. Then we could interchange A and B but there remains the possibility that there is some other isomorphism λ which interchanges some other pair of facets (and necessarily $\lambda A \neq A$ or B, or $\lambda B \neq B$ or A). To prevent this occurring we ensure that the number of $d-2$ faces of A (and B) exceed those of the side facets.

So dualizing this argument we need to construct in E^{d-1}, $d \geqslant 4$, a $d-1$ polytope Q^{d-1} with no interchangeable vertices and whose total number of vertices exceed the maximal valence of any one vertex by at least 3. This we do by induction. For E^3, Q^3 is as in Fig. 1.

We show that no pair of vertices can be interchanged (by exhaustion).

***b** is fixed.* **b** is the only six valent vertex.

***a**, **d** are fixed.* The only two five valent vertices are **a** and **d**. However, **d** lies on a facet with 7 vertices and **a** does not. Hence **a** and **d** cannot be interchanged.

***g** is fixed.* **g** is the only vertex with edges to **a**, **d** and **b**.

***c** is fixed.* **g** is the only other vertex with edges to **a** and **d**.

***j** is fixed.* **d** is the only other vertex with edges to **a** and **c**.

***i** is fixed.* **i** is the only vertex with edges to **a**, **b**, **j**.

***h** is fixed.* **h** is the only vertex amongst **h**, **e**, **f** with an edge to **i**.

***f** is fixed.* **f** is the only vertex amongst **e**, **f** with an edge to **h**.

***e** is fixed.* All the other vertices of Q^3 have now been shown to be fixed.

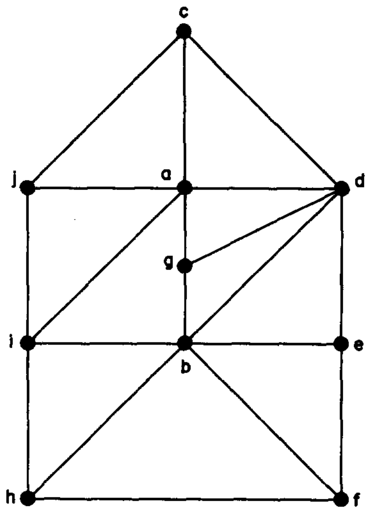

Fig. 1

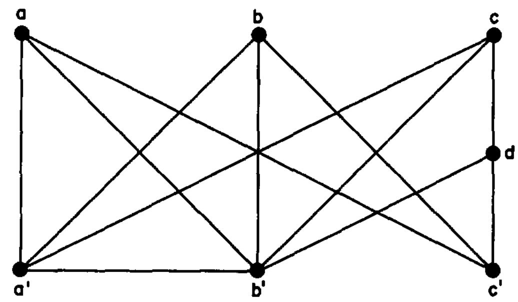

Fig. 2

So no pair of vertices of Q^3, which is a 3-polytope with 10 vertices and the maximum valence of its vertices is 6, can be interchanged by an isomorphism of the face lattice of Q^3.

Suppose now that a $d-1$ polytope Q^{d-1} has been constructed, $d \geqslant 4$, with no interchangeable vertices. Suppose further that Q^{d-1} has X_{d-1} vertices with maximum valence Y_{d-1}, where $Y_{d-1}+3 \leqslant X_{d-1}$. As $X_3 = 10$ and $Y_3 = 6$, this is true for $d = 4$.

For the construction of Q^d we suppose that Q^{d-1} has centroid $\mathbf{O}$ and lies in the coordinate hyperplane $x_d = 0$ of E^d. Now, if $\mathbf{e}_d$ is the dth unit vector let

$$Q^d = \operatorname{conv}\left\{\mathbf{O}, \frac{1}{2}\mathbf{e}_d + \frac{1}{\sqrt{2}} Q^{d-1}, \mathbf{e}_d + Q^{d-1}\right\}.$$

The vertices of Q^d are $\mathbf{O}$ and the vertices of the two copies of Q^{d-1}, i.e. Q^d has $2X_{d-1}+1$ vertices. The valence of $\mathbf{O}$ is X_{d-1} and the valences of any other vertex is at most two more than its valence in (the copy of) Q^{d-1}. So $Y_d = X_{d-1}$, $d \geqslant 4$ and hence $Y_d + 3 \leqslant X_d$. Thus, Q^d has been constructed inductively. We claim that there is no isomorphism of the face lattice which interchanges two vertices. Firstly the vertex $\mathbf{O}$ has uniquely the maximum valence and hence is fixed. There are no edges joining $\mathbf{O}$ to any vertex within $\mathbf{e}_d + Q^{d-1}$. Consequently any such isomorphisms would have to permute the vertices of

$$\frac{1}{2}\mathbf{e}_d + \frac{1}{\sqrt{2}} Q^{d-1}$$

and $\mathbf{e}_d + Q^{d-1}$ separately. By the inductive assumption on Q^{d-1} this is only possible if all the vertices remain fixed.

Finally taking the dual of Q^{d-1}, say $(Q^{d-1})^*$ and taking the cylinder in E^d over $(Q^{d-1})^*$ we obtain an example of a d-polytope, $d \geqslant 4$ in which exactly two facets can be interchanged by an isomorphism of the face lattice.

Proof of Theorem 2. The first part of Theorem 2 follows from Theorem 1 using Steinitz's theorem. So if a 3-connected graph G has just one interchangeable pair of vertices then it must be non-planar. Consequently G must contain a refinement of at least one of the two Kuratowski graphs.

If G has 5 vertices it is the complete 5-graph which has all pairs of vertices interchangeable. If G has 6 vertices then it is either the complete (3,3) bipartite graph, in which case all pairs of vertices are interchangeable, or G contains a refinement of the completed 5-graph C_5. In this latter case we consider G as a vertex $\mathbf{v}$ being added (with edges) to C_5. There are two possibilities.

(i) $\mathbf{v}$ does not lie on the edges of C_5 and hence $\mathbf{v}$ is joined by edges to at least 3 of the 5 vertices of C_5. Then all the vertices of C_5 joined by an edge to $\mathbf{v}$ are interchangeable as are all the vertices of C_5 which are not joined by an edge to $\mathbf{v}$. This yields at least two interchangeable pairs.

(ii) **v** lies on one of the edges, [**a**, **b**] say and so **v** is joined to at least one other vertex of C_5. In this case **a** and **b** are interchangeable. Also amongst the other three vertices **c**, **d**, **e** there will be a pair which are either both joined to **v** or both are not joined to **v**. Such a pair is also interchangeable. So again there are two interchangeable pairs.

To complete the proof of Theorem 2, we construct a 3-connected graph G with 7 vertices which possesses exactly one pair of interchangeable vertices. This graph is illustrated in Fig. 2.

Clearly **a**, **b** are interchangeable and we claim that there are no other pairs of interchangeable vertices.

Firstly any isomorphism ϕ of the graph must fix **a**′ and **b**′ since they are respectively the only 4 and 5 valent vertices.

Since of the five remaining vertices **a**, **b**, **c**, **d**, **c**′ only **c**′ is not joined to **b**′, ϕ must also fix **c**′.

We next show that ϕ fixes **d**. If not then ϕ**d** must be one of **a**, **b**, **c**. If ϕ**d** = **a** (or **b**) then ϕ**c**, which also has to be one of **a**, **b**, **c**, **d** is not joined by an edge to ϕ**d**, whereas [**c**, **d**] is an edge; which is impossible. If ϕ**d** = **c** then again since [**c**, **d**] is an edge, ϕ**c** = **d**. However, [**c**, **a**′] is an edge but [ϕ**c**, ϕ**a**′] = [**d**, **a**′] is not. So ϕ fixes **d**.

It remains to show that ϕ**c** = **c**. If ϕ**c** = **a** then [**c**, **d**] is an edge but [ϕ**c**, ϕ**d**] = [**a**, **d**] is not, which is impossible. So ϕ**c** = **c** which completes the proof of Theorem 2.

REFERENCES

1. B. Grünbaum, *Convex Polytopes*. Interscience, London (1967).
2. A. F. Hawkins, A. C. Hill, J. E. Reeve and J. A. Tyrrell, On certain polyhedra. *Math. Gaz.* L (372), 140–144 (1966).