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Strictly nilpotent elements and bispectral operators in the Weyl algebra

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Abstract

In this paper we give another characterization of the strictly nilpotent elements in the Weyl algebra, which (apart from the polynomials) turn out to be all bispectral operators with polynomial coefficients. This also allows to reformulate in terms of bispectral operators the famous conjecture, that all the endomorphisms of the Weyl algebra are automorphisms (Dixmier, Kirillov, etc).

Keywords: Weyl algebra; Bispectral operators; Automorphism group; Strictly nilpotent elements

0. Introduction

In a recent series of papers [13–15] there has been made an attempt to broaden the classification of bispectral operators that was started by Duistermaat and Grünbaum in [9] and continued by Wilson in [18]. The present paper could be considered as yet another step in this direction. I believe, however, that the results could be of interest also to specialists in other areas of research, in particular connected to the Weyl algebra A_1. For this reason I will try to present the material without using any specific knowledge on bispectral operators. To explain the main message of the paper let me first recall some definitions and results.

In what follows we will consider the algebra A_1 in its standard realization – i.e. as the algebra of differential operators with polynomial coefficients. An element $M \in A_1$ is said to *act nilpotently* on a non-constant element $H \in A_1$, when there exists a positive integer m such that $\mathrm{ad}_M^m(H) = 0$. An element is *strictly nilpotent* if it acts nilpotently on all elements of A_1. Slightly paraphrasing Dixmier the strictly nilpotent elements are characterized as

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those that belong to the orbits of the operators with constant coefficients under the action of the automorphism group $\mathrm{Aut}(A_1)$ *of* A_1.

Now I pass to the other main object of the present paper – the bispectral operators. They have been introduced by Grünbaum (cf. [10]) in his studies on applications of spectral analysis to medical imaging. Later it turned out that they are connected to several actively developing areas of mathematics and physics – the KP-hierarchy, infinite-dimensional Lie algebras and their representations, particle systems, automorphisms of algebras of differential operators, non-commutative geometry, etc. (see e.g. [1–3,5–7,11,16–19], as well as the papers in the proceedings volume of the conference in Montréal [12]).

An ordinary differential operator $L(x, \partial_x)$ is called bispectral if it has an eigenfunction $\psi(x, z)$, depending also on the spectral parameter z, which is at the same time an eigenfunction of another differential operator $\Lambda(z, \partial_z)$ now in the spectral parameter z. In other words we look for operators L, Λ and a function $\psi(x, z)$ satisfying equations of the form:

$$L\psi = f(z)\psi, \tag{0.1}$$

$$\Lambda\psi = \theta(x)\psi, \tag{0.2}$$

where the functions are defined in some open sets of $\mathbb{C}$ and $\mathbb{C}^2$. A simple consequence of the above definition is that the bispectral operator L *acts nilpotently* on the function $\theta(x)$. This is the well known ad-condition from [9] which is widely believed to be not only necessary but also a sufficient condition for the bispectrality of L, provided that L is normed as follows:

$$L = \partial^N + \sum_{j=0}^{N-2} V_j \partial^j, \tag{0.3}$$

i.e. $V_N = 1$ and $V_{N-1} = 0$. In fact for operators in A_1 the first condition, i.e. $V_N = 1$ suffices as by conjugating L by $\exp(Q(x))$ with an appropriate polynomial $Q(x)$ the second condition can be achieved remaining in A_1. In what follows we will also use this relaxed norming condition, i.e. we will assume that L has the form:

$$L = \partial^N + \sum_{j=0}^{N-1} V_j \partial^j, \tag{0.4}$$

We are ready to formulate our main results.

Theorem 0.1. *A differential operator from* A_1 *of the form* (0.4) *is bispectral if and only if it is strictly nilpotent.*

This result will be an easy consequence of the following theorem which will be formulated purely in terms of the Weyl algebra, i.e. without referring to bispectral operators.

Theorem 0.2. *An element* $L \in A_1$ *is strictly nilpotent if and only if it is either a polynomial in* x *or has the form* (0.4) *and acts nilpotently on some non-constant polynomial* $\theta(x)$.

In other words the strictly nilpotent operators (of non-zero order) are exactly those that satisfy the ad-condition and (0.4).

The theorems announced above constitute the main body of the paper.

Next I will explain some further connections between the Weyl algebra and bispectral operators.

In the fundamental paper [9] Duistermaat and Grünbaum classified the bispectral operators of order two. Roughly speaking they are Darboux transformations of some Bessel operators the only exception being the Airy operator. Another important result, due to Wilson [18], is that all bispectral operators of rank 1 are Darboux transformations of operators with constant coefficients. (This is not the original form of Wilson's theorem but a well known reformulation, cf. e.g. [14].) In [3] we suggested a general scheme for producing bispectral operators by application of Darboux transformations out of "simple" ones which apparently works for all differential operators (see also [14]). Now it seems that the most difficult problem in the classification of bispectral operators is not to perform Darboux transformations but to find a reasonable class of operators that could be considered "simple". The results of the present paper show that in A_1 all "simple" operators are those that satisfy the *canonical commutation relation* (CCR for brevity)

$$[L, P] = 1; \tag{0.5}$$

One can reinterpret the main results also as follows.

Proposition 0.3. *The centralizer $C(L)$ of any bispectral operator in A_1 is generated by an element L', which together with some other element P satisfy the CCR* (0.5)*.*

In view of this re-interpretation it is tempting to conjecture the opposite:

Conjecture 0.4. *If two elements L_0 and P satisfy the CCR* (0.5) *they are bispectral.*

The results of the present paper allow to easily show that the above conjecture is equivalent to the famous conjecture of Dixmier–Kirillov:

Conjecture 0.5. *If two operators L and P satisfy CCR then they generate A_1. In other words every endomorphism of the Weyl algebra is automorphism.*

This equivalence is demonstrated at the end of the last section.

To make the presentation less dependent on [8] we recall some of the results that are needed in Section 1. Then in the next section we give the proofs of the above results.

1. Preliminaries on the Weyl algebra

Before proceeding with the main results we briefly recollect some of the notions and results from [8]. The first Weyl algebra A_1 is an associative algebra generated over a field F by two elements p and q, subject to the canonical commutation relation (CCR) $[p, q] = 1$. In this paper F will be always $\mathbb{C}$, although most of the results can be reformulated for more general fields. In most of the paper A_1 will be realized as the algebra of differential operators of one variable x with polynomial coefficients, where $p = \partial$ and $q = x$. A major tool in the the study of A_1 is the introduction of suitable filtrations. Let $G = \sum a_{i,j} q^i p^j$ and let $E(G)$ be the set of all pairs (i, j), such that $a_{i,j} \neq 0$. If ρ and σ are two real numbers put

$$v_{\rho,\sigma}(G) = \sup_{(i,j)\in E(G)} (\rho i + \sigma j).$$

Denote by $E(G, \rho, \sigma)$ the set of pairs (i, j), such that $\rho i + \sigma j = v_{\rho,\sigma}(G)$. With each G, ρ and σ we associate a polynomial $f(X, Y)$ (in the commuting variables X and Y) as follows:

$$f(X,Y) = \sum_{a_{i,j}\in E(G,\rho,\sigma)} a_{i,j} X^i Y^j. \tag{1.1}$$

f will be called *the polynomial ρ, σ-associated with G*. Now we ready to recall Lemma 7.3 from [8] on the "normal form" of a polynomial f associated with an element G. To avoid unnecessary for us terminology we recall it for the particular case that we need. Namely we consider that $G \in A_1$ act on $M \in A_1$ nilpotently. Let ρ, σ be positive integers and let f and g be the polynomials (ρ, σ)-associated with G and M correspondingly and put $v = v_{\rho,\sigma}(G)$ and $w = v_{\rho,\sigma}(M)$.

Lemma 1.1. *Assume that $v + w > \rho + \sigma$ and that f is not a monomial. Then one of the following cases holds*:

(i) *f^w is proportional to g^v*;

(ii) *$\sigma > \rho$, ρ divides σ and*

$$f(X,Y) = \lambda X^\alpha \left(X^{\sigma/\rho} + \mu Y\right)^\beta; \tag{1.2}$$

(iii) *$\rho > \sigma$, σ divides ρ and*

$$f(X,Y) = \lambda Y^\alpha \left(Y^{\rho/\sigma} + \mu X\right)^\beta; \tag{1.3}$$

(iv) *$\sigma = \rho$, and*

$$f(X,Y) = \lambda(\mu X + \nu Y)^\alpha (\mu' X + \nu' Y)^\beta, \tag{1.4}$$

where $\lambda, \mu, \mu', \nu, \nu' \in \mathbb{C}$ and α, β are non-negative integers.

At the end we introduce (after Dixmier) the following notations. Let $S(\partial)$ be a polynomial in ∂. Then the automorphism Φ_S, given by:

$$\Phi_S = \mathrm{e}^{\mathrm{ad}_{S(\partial)}} \tag{1.5}$$

is well defined. In the same way for a polynomial $R(x)$ one defines:

$$\Psi_R = \mathrm{e}^{\mathrm{ad}_{R(x)}}. \tag{1.6}$$

A fundamental result from [8] is the following theorem, which will be used in the present paper.

Theorem 1.2. *The group* $\mathrm{Aut}(A_1)$ *of automorphisms of A_1 is generated by the automorphisms* (1.5) *and* (1.6).

2. Proofs

In this section we are going to give the proofs of the results from the Introduction. No doubt the central one is Theorem 0.2 from which the rest are easy consequences.

The proof of Theorem 0.2 uses an induction reminiscent of "Fermat's method of infinite descent", the main step of which in our case reduces the number of the factors of the order N of L. This will be done as follows. We can suppose that L depends on x, i.e. it is not a polynomial in ∂. First choose appropriately ρ and σ in such a way that the polynomial f, (ρ,σ)-associated with L has two terms, the first one being Y^N. Then show that f has the form from Lemma 1.1, (iii) with $\alpha = 0$. The next step is to apply an appropriate automorphism of A_1, sending our operator L to another one with similar properties but reducing the number of factors of the order N.

We need some preparations for the proof. Write L in the form:

$$L = \sum_{(i,j)\in E(L)} a_{i,j} x^i \partial^j \tag{2.1}$$

with $a_{0,N} = 1,\ a_{i,N} = 0,\ i > 0$. We would like to consider now the non-trivial cases when at least one point (i, j) with $i > 0$ belongs to $E(L)$, i.e. we assume that L depends non-trivially on x. Assuming that we will explain how to choose the weights ρ and σ to fit our purposes. Draw the line in the plane $\mathbb{R}^2$ passing through the point $(0, N)$ and at least one other point, say (k,m) with $k > 0$ and such that all other points remain below or on the line. Then one can choose ρ and σ to be one non-zero solution in integers of the equation $N\sigma = k\rho + m\sigma$. The solution does not depend on the specific (k,m). This gives that the polynomial f, (ρ,σ)-associated with L has the form:

$$f(X,Y) = Y^N + a_{k,m}X^kY^m + \cdots, \quad a_{k,m} \neq 0. \tag{2.2}$$

Here we have chosen the pair (k,m) so that k is the greatest possible. Our main concern will be to study the polynomial f associated with L. Introduce also the following object. Let M be an element from the orbit of x, which does not commute with L and has the form:

$$M = \Psi_{R_1} \circ \Phi_{S_1} \circ \cdots \circ \Psi_{R_l} \circ \Phi_{S_l}(x), \tag{2.3}$$

where R_j, S_j are polynomials with $\deg R_j \geqslant 3$, $\deg S_j \geqslant 3$. The number l could be zero. In this case $M = x$.

Lemma 2.1. *Assume that L is given as in* (0.3) *and that it acts nilpotently on a non-constant polynomial $\theta(M)$ in M, where M is given in* (2.3). *Let $k > 1$ in the above expression* (2.2) *of f. Then $v_{\rho,\sigma}(L) > \rho + \sigma$.*

Proof. Writing f in the form:

$$f = Y^m\big(Y^{N-m} + a_{k,m}X^k + \cdots\big), \quad a_{k,m} \neq 0 \tag{2.4}$$

we can choose $\sigma = k$ and $\rho = N - m$. If $m = 0$ and $k = N = 2$ then according to [8], Lemma 7.4 the element L is strictly semisimple and hence acts nilpotently only on elements of its centralizer $C(L)$, which cannot be true since M does not commute with L.

(A simple independent proof is also possible, cf. e.g. [13].) Hence we can assume that either $m > 0$ or $\max(N, k) \geqslant 3$. Then we have

$$v_{\rho,\sigma}(L) = N\sigma = Nk \geqslant N + k \geqslant (N - m) + k = \rho + \sigma.$$

In the case of $m > 0$ the second inequality in the above chain is strict, while in the case of $\max(N, k) \geqslant 3$ the first inequality is strict (recall that both $N \geqslant 2$ and $k \geqslant 2$). □

Next find a normal form for the polynomial associated with L.

Lemma 2.2. *Assume that L has at least one nonconstant coefficient $V_j(x)$ and satisfies the conditions of Lemma* 2.1. *Then there exist numbers ρ and σ, such that the polynomial (ρ, σ)-associated with L has the form*

$$f = \left(Y^r - \lambda X\right)^k, \quad \lambda \neq 0. \tag{2.5}$$

Proof. We choose the integers ρ and σ as explained above so that the f has the form (2.4). This is possible due to the assumption that L has at least one nonzero coefficient. First assume that in (2.3) we have $k > 1$. In this case according to Lemma 2.1 $v_{\rho,\sigma}(L) > \rho + \sigma$. Hence we can apply Lemma 1.1. Note that the polynomial (ρ, σ)-associated with $\theta(M)$ has the form $g = \gamma X^l$, hence the case (i) is ruled out. If we assume that the case (ii) of Lemma 1.1 holds than expanding f we see that the second coefficient in the expansion of L in ∂, i.e. the coefficient a_{N-1} is not zero which contradicts (0.3). By the same reason the case (iv) is possible only with $\lambda = \nu = \nu' = 1$, $\mu = -\mu' \neq 0$ and $\alpha = \beta$ or equivalent to it. Then applying the a linear automorphism Ψ, defined by $\Psi(\partial) = \partial + \mu x$, $\Psi(x) = x$ we can bring the polynomial f into the form $f = Y^\alpha (Y + 2\mu X)^\alpha$, keeping g untouched (but not $\theta(M)$). Finally, if f has the form (2.3) with $k = 1$ then it is exactly $f = Y^m(Y^{N-m} + \lambda X)$. Summing up the above two cases as well as (ii) we get that in general f has the form:

$$f = Y^n \left(Y^r - \lambda X\right)^k, \quad k \geqslant 1,\ \lambda \neq 0. \tag{2.6}$$

Now we want to show that $n = 0$. Perform the automorphism $\Phi = \Phi_{S_0}$ where $S_0(\partial) = \lambda^{-1} \frac{\partial^{r+1}}{r+1}$. This automorphism maps L into a new element $\Phi(L)$ with a new polynomial f_0 (ρ, σ)-associated with L of the form:

$$f_0 = (-\lambda)^k X^k Y^n, \tag{2.7}$$

while the polynomial associated with $\Phi(\theta(M))$ will become

$$g_0 = \gamma_0 \left(X^l + \lambda^{-1} Y^r\right)^l, \quad \gamma_0 \neq 0. \tag{2.8}$$

We are going to use that $\Phi(L)$ acts nilpotently on $\Phi(\theta(M))$. In what follows we will drop the non-essential coefficients γ' and $(-\lambda)^k$. Let us compute consecutively $\mathrm{ad}^s_{\Phi(L)}(\Phi(\theta(M)))$ with $s = 1, \ldots$. As we will be interested only on the terms with highest weight we will drop the rest. Then we have

$$\Phi(L) = \partial^n x^k + \cdots, \tag{2.9}$$

$$\Phi(\theta(M)) = \left(x + \lambda^{-1}\partial^r\right)^k + \cdots. \tag{2.10}$$

Expand the highest weight terms (2.10) as

$$\left(x+\lambda^{-1}\partial^r\right)^l=\sum c_j^l x^j\lambda^{-l+j}\partial^{r(l-j)}+\cdots. \tag{2.11}$$

By linearity we have

$$\begin{aligned}&\operatorname{ad}^s_{\Phi(L)}\left(\Phi(\theta(M))\right)\\&\quad=\operatorname{ad}^s_{(\partial^n x^k)}\left(x+\lambda^{-1}\partial^r\right)^l=\sum_{j=0}^{l}c_j^l\lambda^{-l+j}\operatorname{ad}^s_{(\partial^n x^k)}\left(x^j\partial^{r(l-j)}\right)+\cdots.\end{aligned} \tag{2.12}$$

We will consider separately two cases: with $k\geqslant n$ and $n\geqslant k$.

(1) Let $k\geqslant n$. Simple computation gives that

$$\left(\operatorname{ad}^s_{(\partial^n x^k)}(x^l)\right)=\prod_{j=0}^{s-1}[nl+(k-n)j]\partial^{s(n-1)}x^{l+s(k-1)}+\cdots.$$

Having in mind that $n\geqslant 1$ and $k-n\geqslant 0$ we get that the coefficient at the highest power of x is always positive for any $s\geqslant 1$, which shows that (2.11) cannot be zero. This contradicts the fact that L acts nilpotently on $\theta(M)$.

(2) Let $n\geqslant k$. We have

$$\operatorname{ad}^s_{(\partial^n x^k)}\left(\partial^{lr}\right)=(-1)^s\prod_{j=0}^{s-1}[lrk+(n-k)j]\partial^{s(n-1)+lr}x^{s(k-1)}+\cdots.$$

By the same argument the coefficient at the highest power in ∂ is not zero for any s. This shows that either $n=0$ or $k=0$. But from the assumption (2.3) it follows that k cannot be zero. □

Now we perform the main induction step.

Lemma 2.3. *Assume the conditions of the above lemma. Then there exists a polynomial $S(\partial)$ with $\deg S(\partial)=r+1\geqslant 3$, such that the image L_1 of L under the action of the corresponding automorphism Ψ_R has the form*:

$$L_1=\Psi_S(L)=(-\lambda)^k x^k+\sum_{j<k}c_j(\partial)x^j,\quad c_j(\partial)\in\mathbb{C}[\partial],\ c_{k-1}\equiv 0. \tag{2.13}$$

Proof. Use the obvious fact that the elements ∂ and $\partial^r-\lambda x$ are generators of A_1. Then Lemma 2.2 shows that the element L can be written in the form:

$$L=\left(\partial^r-\lambda x\right)^k+\sum_{j=0}^{k-1}b_{i,j}\left(\partial^r-\lambda x\right)^j\partial^i. \tag{2.14}$$

Apply the automorphism Ψ from the proof of the previous lemma, i.e. $\Psi(\partial)=\partial$, $\Psi(x)=x+\lambda^{-1}\partial^r$. Put $L_1=\Psi(L)$, $\theta_1=\Psi(\theta(M))$. Then one can write L_1 (dropping the non-essential constant factor) in the form:

$$L_1=x^k+\sum_{j=0}^{k-1}b_j(\partial)x^j,\quad b_j(\partial)\in\mathbb{C}[\partial]. \tag{2.15}$$

Notice that in the above expression all the terms after x^k have weights less than N in the chosen filtration. In particular for $b_{k-1} \neq 0$ we have:

$$v_{\rho,\sigma}\left(b_{k-1}(\partial)x^{k-1}\right) < N = kr. \tag{2.16}$$

Assume that $b_{k-1} \neq 0$. Having in mind that our filtration can be chosen so that $\rho = r$, $\sigma = 1$ the inequality (2.16) can be rewritten as $\deg b_{k-1} + r(k-1) < kr$. This shows that the degree of b_{k-1} is less than r. By an appropriate automorphism Ψ_0 we can kill b_{k-1}. The composition $\Psi_0 \circ \Psi$ is the automorphism Ψ_R we are looking for. Notice that $S(\partial) = c\partial^{r+1} + S_0(\partial)$, where $\deg S_0 \leqslant r$ and $c \neq 0$. Hence the degree of S is exactly $r+1$. □

Let us give the proofs of the main results.

Proof of Theorem 0.2. If the second coefficient V_{N-1} of L is not zero then apply appropriate automorphism Ψ_R, where $R' = -V_{N-1}$. This will bring our operator L into the situation of Lemma 2.1 with $M = x$. If the number k from (2.5) is equal to 1 then L is the generalized Airy operator, hence in the orbit of ∂. So assume that the number $k > 1$. If we assume that all the coefficients of L are constant then the theorem is again proven. Now assume that at least one coefficient of L is not constant. Then according to Lemma 2.3 we can find an automorphism Ψ_R which sends L into (2.13). Notice that the operator L_1 from (2.13) has the properties of L required by Lemma 2.2 (with x and $-\partial$ exchanging their places) but its order k is strictly less than the order N of L. This shows that after a finite number of steps we will come to either a polynomial in x or in ∂, thus proving that L is in their orbits. □

The next corollary follows from the proof of the last theorem (but not from the theorem as stated).

Corollary 2.4. *Let the operator L satisfy the conditions of Theorem* 0.1. *Then it has the form similar to* (2.2). *More precisely L is a polynomial in the element K of the form*

$$K = \Phi_1 \circ \Psi_1 \circ \cdots \circ \Psi_l \circ \Phi_{l+1}(x), \tag{2.17}$$

where $\Psi_j = \Psi_{R_j}$, $\Phi_j = \Phi_{S_j}$, $j = 1, \ldots, l$, and the polynomials have degrees $\geqslant 3$. The automorphism Φ_{l+1} is either of the same form or is defined by $\Phi_{l+1}(x) = \partial$, $\Phi_{l+1}(\partial) = -x$.

Proof of Theorem 0.1. If L is normalized as in (0.3) and bispectral it acts nilpotently on some nonconstant polynomial $\theta(x)$. Hence by Theorem 0.2 it is strictly nilpotent. The opposite also follows easily. Suppose that L belongs to the orbit of some nonconstant polynomial in ∂, say $Q(\partial)$. We need to consider only the case when L is not a polynomial in x. Then there exists an automorphism ϕ such that $L = Q(\phi^{-1}(\partial))$. Denote by L_0 the operator $\phi(\partial)$.

Let b_0 be the standard anti-involution:

$$b_0(x) = \partial_z, \quad b_0(\partial_x) = z. \tag{2.18}$$

(As usually treating bispectral operators we use different variables – x and z for the two copies of A_1.) Now define (cf. [3]) the anti-involution $b = b_0 \circ \phi$. It is enough to show that L_0 is bispectral. Then the bispectrality of L will follow immediately as L is a polynomial of L_0. We have

$$L_0 = b^{-1}(z) = \phi \circ b_0^{-1}(z) = \phi(\partial_x). \tag{2.19}$$

Define

$$\Lambda = b(x) = b_0 \circ \phi^{-1}(x). \tag{2.20}$$

We have only to exhibit the wave function $\psi(x, z)$, so that (0.1) and (0.2) are satisfied with $L_0, \Lambda,\ f(z) = z$ and $\theta(x) = x$. We can always assume that L_0 is normalized as in (0.3). Otherwise we can apply appropriate automorphism as explained above and bring it to this form. The point is that we would like to use Corollary 2.4, which assures that the the polynomials, defining the automorphism ϕ are of degree 3 or more except for Φ_{l+1}. Then we can apply the theorem from [4] which gives the wave function in explicit form. □

In what follows it would be convenient to consider the polynomials of x also bispectral. (In fact allowing the wave function to be distribution they are, cf. [3].)

In view of Theorem 0.1 it is obvious that the centralizer of each bispectral operator L is generated by $\phi(\partial)$, where ϕ is the automorphism, defining L from the proof of Corollary 2.4. Introduce also the operator $\phi(x)$. Then obviously they satisfy the CCR

$$[\phi(\partial), \phi(x)] = 1. \tag{2.21}$$

This gives the proof of Proposition 0.3. □

It is tempting to try to prove the opposite, i.e. Conjecture 0.4. This conjecture seems to be difficult to prove. The results of the present paper allow to show that it is equivalent to Conjecture 0.5.

We will give the simple proof of the equivalence of the two conjectures in the following form.

Proposition 2.5. *Let L, P be two operators from A_1 that satisfy the CCR* (0.5). *The following two statements are equivalent*:

(1) *L and P generate A_1*;

(2) *L and P are bispectral.*

Proof. Let L and P be bispectral. According to Theorem 0.1 L is in the orbit of some $Q(\partial)$, i.e. $L = \phi(Q(\partial))$. Put $M = \phi^{-1}(P)$. Then the pair $(Q(\partial), M)$ also satisfies (0.5). Obviously M has at least one term depending on x. This automatically give that Q is a polynomial of degree one, i.e. $Q = a\partial + b,\ a \neq 0$. Hence M has the form $M = a^{-1}x + R(\partial)$ with some polynomial R. This shows that the pair $(Q(\partial), M)$ generate A_1. The same is true for their images L, P under the automorphism ϕ^{-1}, thus proving $(2) \to (1)$.

The opposite is obvious. Really. Let L and P generate A_1. Then they are strictly nilpotent, hence bispectral. □

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