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Isochronicity conditions for some planar polynomial systems II

Magali Bardet [a], Islam Boussaada [b], A. Raouf Chouikha [c], Jean-Marie Strelcyn [b,c,*]

[a] *LITIS, Université de Rouen, Avenue de l'Université BP 12, F-76801 Saint Etienne du Rouvray, France*
[b] *Laboratoire de Mathématiques Raphaël Salem, CNRS, Université de Rouen, Avenue de l'Université BP 12, 76801 Saint Etienne du Rouvray, France*
[c] *Laboratoire Analyse Géometrie et Aplications, UMR CNRS 7539, Institut Gallilée, Université Paris 13, 99 Avenue J.-B. Clément, 93430 Villetaneuse, France*

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Abstract

We study the isochronicity of centers at $O \in \mathbb{R}^2$ for systems

$$\dot{x} = -y + A(x, y), \qquad \dot{y} = x + B(x, y),$$

where $A, B \in \mathbb{R}[x, y]$, which can be reduced to the Liénard type equation. When $\deg(A) \leqslant 4$ and $\deg(B) \leqslant 4$, using the so-called C-algorithm we found 36 new multiparameter families of isochronous centers. For a large class of isochronous centers we provide an explicit general formula for linearization. This paper is a direct continuation of a previous one with the same title [Islam Boussaada, A. Raouf Chouikha, Jean-Marie Strelcyn, Isochronicity conditions for some planar polynomial systems, Bull. Sci. Math. 135 (1) (2011) 89–112], but it can be read independently.

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* Corresponding author at: Laboratoire de Mathématiques Raphaël Salem, CNRS, Université de Rouen, Avenue de l'Université BP 12, 76801 Saint Etienne du Rouvray, France.

E-mail addresses: magali.bardet@univ-rouen.fr (M. Bardet), islam.boussaada@gmail.com (I. Boussaada), chouikha@math.univ-paris13.fr (A.R. Chouikha), Jean-Marie.Strelcyn@univ-rouen.fr, strelcyn@math.univ-paris13.fr (J.-M. Strelcyn).

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1. Introduction

Let us consider the system of real differential equations of the form

$$\frac{dx}{dt} = \dot{x} = -y + A(x, y), \qquad \frac{dy}{dt} = \dot{y} = x + B(x, y), \tag{1.1}$$

where (x, y) belongs to an open connected subset $U \subset \mathbb{R}^2$ containing the origin $O = (0, 0)$, with $A, B \in C^1(U, \mathbb{R})$ such that A and B as well as their first partial derivatives vanish at O. An isolated singular point $p \in U$ of system (1.1) is a *center* if there exists a punctured neighborhood $V \subset U$ of p such that every orbit of (1.1) lying in V is a closed orbit surrounding p. A center p is *isochronous* if the period is constant for all closed orbits in some neighborhood of p.

The simplest example is the linear system with an isochronous center at the origin O:

$$\dot{x} = -y, \qquad \dot{y} = x. \tag{1.2}$$

The problem of characterization of couples (A, B) such that O is an isochronous center (even for a center) for the system (1.1) is largely open.

The well-known Poincaré theorem asserts that when A and B are real analytic, a center of (1.1) at the origin O is isochronous if and only if in some real analytic coordinate system it takes the form of the linear center (1.2) (see for example [1, Theorem 13.1], and [20, Theorem 4.2.1]).

An overview [6] presents the basic results concerning the problem of the isochronicity, see also [1,10–12,20]. As this paper is a direct continuation of [3], we refer the reader to it for general introduction to the subject. Here we will recall only the strictly necessary facts.

In some circumstances system (1.1) can be reduced to *the Liénard type equation*

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0 \tag{1.3}$$

with $f, g \in C^1(J, \mathbb{R})$, where J is some neighborhood of $0 \in \mathbb{R}$ and $g(0) = 0$. In this case, system (1.1) is called *reducible*. Eq. (1.3) is associated to the equivalent, two-dimensional, Liénard type system

$$\left.\begin{aligned} &\dot{x} = y, \\ &\dot{y} = -g(x) - f(x)y^2. \end{aligned}\right\} \tag{1.4}$$

For reducible systems considered in this paper, the nature (center and isochronicity) of the singular point O for both systems (1.1) and (1.4) is the same.

Let us return now to the Liénard type equation (1.3). Let us define the following functions

$$F(x) := \int_0^x f(s)\, ds, \qquad \phi(x) := \int_0^x e^{F(s)}\, ds. \tag{1.5}$$

The first integral of the system (1.4) is given by the formula [21, Theorem 1]

$$I(x, \dot{x}) = \frac{1}{2}\big(\dot{x}e^{F(x)}\big)^2 + \int_0^x g(s)e^{2F(s)}\, ds. \tag{1.6}$$

When $xg(x) > 0$ for $x \neq 0$, define the function X by

$$\frac{1}{2}\xi(x)^2 = \int_0^x g(s)e^{2F(s)}\, ds \tag{1.7}$$

and $x\xi(x) > 0$ for $x \neq 0$.

Theorem 1.1. *(See [21, Theorem 2].) Let $f, g \in C^1(J,\mathbb{R})$. If $xg(x) > 0$ for $x \neq 0$, then the system* (1.4) *has a center at the origin O. When f and g are real analytic, this condition is also necessary.*

Theorem 1.2. *(See [12, Theorem 2.1].) Let f and g be real analytic functions defined in a neighborhood J of $0 \in \mathbb{R}$, and let $xg(x) > 0$ for $x \neq 0$. Then system* (1.4) *has an isochronous center at O if and only if there exists an odd function $h \in C^1(J,\mathbb{R})$ which satisfies the following conditions*

$$\frac{\xi(x)}{1 + h(\xi(x))} = g(x)e^{F(x)}, \tag{1.8}$$

the function $\phi(x)$ satisfying

$$\phi(x) = \xi(x) + \int_0^{\xi(x)} h(t)\,dt, \tag{1.9}$$

and $\xi(x)\phi(x) > 0$ for $x \neq 0$.

In fact by (1.7), it is easy to see that (1.8) and (1.9) are equivalent. When those equivalent conditions are satisfied, then the function h is analytic in the neighborhood J. h is called the *Urabe function* of system (1.4), or of its equivalent (1.3).

Corollary 1.3. *(See [12, Corollary 2.4].) Let f and g be real analytic functions defined in a neighborhood of $0 \in \mathbb{R}$, and $xg(x) > 0$ for $x \neq 0$. The origin O is an isochronous center of system* (1.4) *with Urabe function $h = 0$ if and only if*

$$g'(x) + g(x)f(x) = 1 \tag{1.10}$$

for x in a neighborhood of 0.

In the sequel we shall call the Urabe function of the isochronous center of reducible system (1.1) the Urabe function of the corresponding Liénard type equation.

In [12] the third author used Theorem 1.2 in order to build an algorithm (called C-algorithm, see appendix of [3] for more details) to look for isochronous centers at the origin for reducible system (1.1). He applied it to the case where A and B are polynomials of degree 3. This work was continued in [13] and in [3].

In this paper we apply the so-called *Rational C-Algorithm* introduced in [2] which is an adaptation of the C-algorithm for the case of rational function f and g (see (1.3), (1.4)).

The aim of the present paper is to extend these studies to the following real multiparameter family of polynomial system of differential equations:

$$\left.\begin{aligned} \dot{x} &= -y + a_{1,1}xy + a_{2,0}x^2 + a_{2,1}x^2y + a_{3,0}x^3 + a_{3,1}x^3y + a_{4,0}x^4, \\ \dot{y} &= x + b_{0,2}y^2 + b_{1,1}xy + b_{2,0}x^2 + b_{1,2}xy^2 + b_{2,1}x^2y + b_{3,0}x^3 \\ &\quad + b_{2,2}x^2y^2 + b_{3,1}x^3y + b_{4,0}x^4. \end{aligned}\right\} \tag{1.11}$$

The reported results, which are obtained by Maple computations are reproduced without almost any change to avoid misprints.

The paper is organized as follows. In Section 2 we report the necessary background and describe the investigated subfamilies of system (1.11). In Sections 3 and 4 we describe the obtained new isochronous centers. In total we provide 36 new families of isochronous centers. Among them two *Monsters* (4.29) and (4.30) of extreme complexity, never encountered before.

Let us stress that when describing the Urabe functions of the isochronous centers from Section 3, for the first time we encounter the *non-standard* examples of it. Indeed, up to now all identified Urabe functions were always of the form $h(\xi) = \frac{a\xi^{2n+1}}{\sqrt{b+c\xi^{4n+2}}}$ where $a, b, c \in \mathbb{R}$, $b > 0$ and n a non-negative integer. (See [12,13,3].)

Finally, in Section 5, when Urabe function $h = 0$, we describe the explicit general formula for linearizing change of coordinates whose existence is insured by the Poincaré theorem. We report also 5 examples of such linearization.

2. Preliminaries

2.1. Choudhury–Guha reduction

Let us consider the real polynomial system

$$\left.\begin{aligned} \dot{x} &= p_0(x) + p_1(x)y, \\ \dot{y} &= q_0(x) + q_1(x)y + q_2(x)y^2, \end{aligned}\right\} \tag{2.1}$$

where $p_0, p_1, q_0, q_1, q_2 \in \mathbb{R}[x]$.

We will always assume that $O = (0,0) \in \mathbb{R}^2$ is a singular point of (2.1), that is $p_0(0) = q_0(0) = 0$. Let us assume also that $p_1(0) \neq 0$.

Let us note that the system (1.11) is a particular case of (2.1) when

$$\left.\begin{aligned} p_0(x) &= a_{2,0}x^2 + a_{3,0}x^3 + a_{4,0}x^4, \\ p_1(x) &= -1 + a_{1,1}x + a_{2,1}x^2 + a_{3,1}x^3, \\ q_0(x) &= x + b_{2,0}x^2 + b_{3,0}x^3 + b_{4,0}x^4, \\ q_1(x) &= b_{1,1}x + b_{2,1}x^2 + b_{3,1}x^3, \\ q_2(x) &= b_{0,2} + b_{1,2}x + b_{2,2}x^2. \end{aligned}\right\} \tag{2.2}$$

The following change of coordinates $x = x$, $z = p_0(x) + p_1(x)y$ transforms the system (2.1) to the system

$$\left.\begin{aligned} \dot{x} &= z, \\ \dot{z} &= \left(\frac{q_2(x)}{p_1(x)} + \frac{p_1'(x)}{p_1(x)}\right)z^2 + \left(-\frac{(p_1'(x))p_0(x)}{p_1(x)} + q_1(x) + p_0'(x) - 2\frac{q_2(x)p_0(x)}{p_1(x)}\right)z \\ &\quad + \frac{q_2(x)(p_0(x))^2}{p_1(x)} - q_1(x)p_0(x) + p_1(x)q_0(x). \end{aligned}\right\} \tag{2.3}$$

If

$$-\frac{(p_1'(x))p_0(x)}{p_1(x)} + q_1(x) + p_0'(x) - 2\frac{q_2(x)p_0(x)}{p_1(x)} = 0, \tag{2.4}$$

the system (2.1) is of Liénard type (1.4), with

$$\left.\begin{aligned} f(x) &= -\left(\frac{q_2(x)}{p_1(x)} + \frac{p_1'(x)}{p_1(x)}\right),\\ g(x) &= -\frac{q_2(x)(p_0(x))^2}{p_1(x)} + q_1(x)p_0(x) - p_1(x)q_0(x). \end{aligned}\right\} \tag{2.5}$$

To the best of our knowledge, the above reduction of the system (2.1) to Liénard type system (1.4) was proposed for the first time in a preliminary and never published version of [9]. It is then natural to name it Choudhury–Guha reduction.

In all considered cases (see (2.2)) it is easy to see that for $|x|$ small enough $g(x) = x + x^2\tilde{g}(x)$ where $\tilde{g}$ is a real analytic function. Thus $xg(x) > 0$ for $x \neq 0$, $|x|$ small enough and Theorem 1.2 insures that the origin O is a center for the system (1.4). Our aim is to decide when this center is isochronous.

When p_0 and q_1 identically vanish, (2.4) is satisfied and we recover the *standard reduction* from [3] (see Case 1 from Section 1 of [3]). Many particular cases of system (1.11) where studied, using this standard reduction.

1. In [12]: $a_{1,1} = a_{2,0} = a_{3,0} = a_{3,1} = a_{4,0} = b_{2,2} = b_{4,0} = b_{3,1} = b_{2,1} = b_{1,1} = 0$. That means that $p_0(x) = q_1(x) = 0$, $p_1(x) = -1 + a_{2,1}x^2$ and $\deg(q_0(x)) \leqslant 3$, $\deg(q_2(x)) \leqslant 1$.
2. In [13]: $a_{2,0} = a_{3,0} = a_{3,1} = a_{4,0} = b_{2,2} = b_{4,0} = b_{3,1} = b_{2,1} = b_{1,1} = 0$. That means that $p_0(x) = q_1(x) = 0$ and we consider only cubic systems.
3. In [3] three families are studied:
   (a) $a_{2,0} = a_{3,0} = a_{4,0} = b_{3,1} = b_{2,1} = b_{1,1} = 0$ with zero Urabe function.
   (b) $a_{1,1} = b_{3,0} = a_{2,0} = a_{3,0} = a_{4,0} = b_{3,1} = b_{2,1} = b_{1,1} = 0$.
   (c) $a_{1,1} = a_{2,1} = a_{2,0} = a_{3,0} = a_{4,0} = b_{3,1} = b_{2,1} = b_{1,1} = 0$.

2.2. *Investigated families*

The exhaustive study of all isochronous center at the origin for the system (1.11) is hopeless at the present. Even for cubic system when all quartic terms vanish this problem is not yet solved.

Let us note that the condition (2.4) is equivalent to the following system of equations:

$$\left.\begin{aligned} &2a_{2,0} + b_{1,1} = 0,\\ &-b_{2,1} + a_{1,1}b_{1,1} + a_{1,1}a_{2,0} - 2b_{0,2}a_{2,0} - 3a_{3,0} = 0,\\ &a_{1,1}b_{2,1} - 4a_{4,0} - b_{3,1} + 2a_{1,1}a_{3,0} - 2b_{0,2}a_{3,0} - 2b_{1,2}a_{2,0} + a_{2,1}b_{1,1} = 0,\\ &-a_{3,1}a_{2,0} + 3a_{1,1}a_{4,0} + a_{2,1}b_{2,1} - 2b_{0,2}a_{4,0} - 2b_{2,2}a_{2,0} + a_{2,1}a_{3,0}\\ &\quad + a_{1,1}b_{3,1} - 2b_{1,2}a_{3,0} + a_{3,1}b_{1,1} = 0,\\ &a_{3,1}b_{2,1} + a_{2,1}b_{3,1} + 2a_{2,1}a_{4,0} - 2b_{1,2}a_{4,0} - 2b_{2,2}a_{3,0} = 0,\\ &a_{3,1}b_{3,1} - 2b_{2,2}a_{4,0} + a_{3,1}a_{4,0} = 0. \end{aligned}\right\} \tag{2.6}$$

In the present paper we determine all isochronous centers of the system (1.11) in each of the following three cases.

1. When the standard reduction is possible (i.e. $p_0(x) = 0$ and $q_1(x) = 0$, that is $a_{2,0} = a_{3,0} = a_{4,0} = b_{3,1} = b_{2,1} = b_{1,1} = 0$). We provide all candidates for isochronous centers in the cases

where either $a_{1,1} = 0$, or $b_{2,0} = -3b_{0,2}$. In all the cases but one (a subcase of $b_{2,0} = -3b_{0,2}$), we prove the isochronicity. The general case is not yet completely explored.
2. When Choudhury–Guha reduction for the cubic case is possible (i.e. conditions (2.6) are satisfied and $a_{3,1} = a_{4,0} = b_{3,1} = b_{2,2} = b_{4,0} = 0$). For this case we obtain the exhaustive list of all isochronous centers at the origin.
3. When Choudhury–Guha reduction is possible and the Urabe function is null. That means that condition (2.6) and condition (1.10) for f and g defined by (2.5) are simultaneously satisfied. In this case we provide 25 examples of new isochronous centers and our analysis is not exhaustive.

Moreover, when the Urabe function $h = 0$, we give the explicit formulas for linearizing coordinates from Poincaré theorem. We report 5 examples where such coordinates are explicitly computed.

2.3. *Time-reversible systems*

The general notion of time-reversible system of ordinary differential equations goes back to [15] where the motivations and general discussion can be found. Here we follow Section 1 of [5] (see also Section 3.5 [20]).

The planar system (1.1) of ordinary differential equation is *time-reversible* if there exists at least one straight line passing through the origin which is a symmetry axis of the phase portrait of the system under consideration. By appropriate rotation this straight line is mapped on the x-axis and the phase portrait of the rotated system is invariant with respect to symmetry $(x, y) \to (x, -y)$ if only one changes time t into $-t$.

Note that a system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y),$$

is time-reversible system with respect to x-axis if and only if $P(x, -y) = -P(x, y)$ and $Q(x, -y) = Q(x, y)$. When P and Q are polynomials, this means that the variable y appears in all monomials of P in odd power and in all monomials of Q in even power (0 included).

Consequently, to decide if a polynomial center for system (1.1) is time-reversible or not, we consider the rotated system in coordinates (x_α, y_α), where $x_\alpha = x\cos\alpha - y\sin\alpha$ and $y_\alpha = x\sin\alpha + y\cos\alpha$ and we examine the parity of the powers of the variable y_α for all angles α.

This notion plays an essential role in our topics. Indeed, for system (1.1) the origin is either a center or a focus. Thus, if such system is time-reversible the focus case is excluded and the origin is necessarily a center.

To the best of our knowledge the majority of already known isochronous centers for polynomial system (1.1) are time-reversible. For instance, all systems studied in [22,5,12,13,3,8] are time reversible. Moreover, among 27 polynomial isochronous centers presented in Tables 3–29 of [6] only 7 are not time-reversible; indeed, those from Tables 17 and 23–28. In what concerns the cubic isochronous centers for system (1.1) the complete enumeration of those which are time reversible was obtained in [8]; there are 17 such cases. In [6] one find 4 non-time-reversible isochronous centers (Tables 25–28) and in the present paper we present three new such cases which are described in Theorem 4.1. In [7] a complete list of quartic homogeneous time-reversible isochronous centers is provided, there are 9 such cases. In the present paper we provide 33 new cases of quartic (non-homogeneous) isochronous centers. Among them 8 are time reversible and at least 23 are not time reversible.

2.4. Background on Gröbner bases

The use of the Rational C-Algorithm leads to a system of polynomial equations

$$f_1 = 0, \dots, f_m = 0 \tag{2.7}$$

with $f_i \in \mathbb{R}[x_1, \dots, x_n]$. To solve this system we consider the ideal $\langle f_1, \dots, f_m\rangle \subset \mathbb{R}[x_1, \dots, x_n]$. For this aim, we use Gröbner bases computations. In this section, we recall the basic facts about Gröbner bases, and refer the reader to [14] for details.

A *monomial ordering* is a total order on monomials that is compatible with the product and such that every nonempty set has a smallest element for the order. The leading term of a polynomial is the greatest monomial appearing in this polynomial.

A *Gröbner basis* of an ideal $\mathcal{I}$ for a given monomial ordering is a set G of generators of $\mathcal{I}$ such that the leading terms of G generate the ideal of leading terms of polynomials in $\mathcal{I}$. A polynomial is *reduced* with respect to the Gröbner basis G when its leading term is not a multiple of those of G. The basis is *reduced* if each element $g \in G$ is reduced with respect to $G \setminus \{g\}$. For a given monomial ordering, the reduced Gröbner basis of a given set of polynomials exists and is unique, and can be computed using one's favorite general computer algebra system, like Maple, Magma or Singular. The most efficient Gröbner basis algorithm is currently F_4 [18], which is implemented in the three above cited systems. For our computations, we use the FGb implementation of F_4 available in Maple [17].

The complexity of a Gröbner basis computation is well known to be generically exponential in the number of variables, and in the worst case doubly exponential in the number of variables. Moreover, the choice of the monomial ordering is crucial for time of the computation.

The *grevlex* ordering is the most suited ordering for the computation of the (reduced) Gröbner basis. The monomials are first ordered by degree, and the order between two monomials of the same degree $x_\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $x_\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ is given by $x_\alpha \succ x_\beta$ when the last nonzero element of $(\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n)$ is negative. Thus, among the monomials of degree d, the order is

$$x_1^d \succ x_1^{d-1}x_2 \succ x_1^{d-2}x_2^2 \succ \cdots \succ x_2^d \succ x_1^{d-1}x_3 \succ x_1^{d-2}x_2x_3 \succ x_1^{d-2}x_3^2 \succ \cdots \succ x_n^d.$$

However, a Gröbner basis for the grevlex ordering is not appropriate for the computation of the solutions of the system (2.7). The most suited ordering for this computation is the *lexicographical* ordering (or *lex* ordering for short). The monomials are ordered by comparing the exponents of the variables in lexicographical order. Thus, any monomial containing x_1 is greater than any monomial containing only variables $x_2, \dots, x_n$.

Under some hypotheses (radical ideal with a finite number of solutions, and up to a linear change of coordinates), the Gröbner basis of an ideal $\langle f_1, \dots, f_m\rangle$ for the lexicographical order $x_1 > \cdots > x_n$ has the shape

$$\{x_1 - g_1(x_n), x_2 - g_2(x_n), \dots, x_{n-1} - g_{n-1}(x_{n-1}), g_n(x_n)\}, \tag{2.8}$$

where the g_i's are univariate polynomials. In this case, the computation of the solutions of the system follows easily. In the general case, the shape of the Gröbner basis for the lexicographical ordering is more complicated, but it is equivalent to several triangular systems for which the computation of the solutions are easy.

An important point is that a Gröbner basis for the lex order is in general hard to compute directly. It is much faster to compute first a Gröbner basis for the grevlex order, and then to make a change of ordering to the lex order.

The precise ordering we use to compute the Gröbner bases of the polynomial systems occurring in this paper is a weighted order: we fix a weight $i + j - 1$ for the variables $a_{i,j}$ and $b_{i,j}$ (see (1.11)), and use the weighted grevlex or lex ordering. For those orderings, the polynomials are `homogeneous`, which simplifies the computation. Indeed, without loss of generality, we can pick a variable $a_{i,j}$ and split the computations into two cases $a_{i,j} = 0$ and $a_{i,j} = 1$ (the same concerns $b_{i,j}$). The entire set of solutions can then be recovered in the standard way. For instance, all solutions with $a_{1,1} \neq 0$ for system (1.11) are obtained from solutions with $a_{1,1} = 1$ by the change of variables $X = a_{1,1}x$ and $Y = a_{1,1}y$. This trick reduces by one the number of variables for the Gröbner basis computation and improves the time of the computations. **In what follows, all results are presented up to such homogenization**.

Finally, we use repeatedly the *Radical Membership Theorem*:

Theorem 2.1. *(See [14].) Let* $I = \langle f_1, \dots, f_s \rangle$ *be an ideal of* $k[x_1, \dots, x_n]$*, then* f *belongs to* $\sqrt{I}$ *if and only if* $\langle f_1, \dots, f_s, 1 - yf \rangle = \langle 1 \rangle = k[x_1, \dots, x_n, y]$.

3. The standard reduction

In this section we are concerned by system (1.11) with $a_{2,0} = a_{3,0} = a_{4,0} = b_{3,1} = b_{2,1} = b_{1,1} = 0$ which gives

$$\left.\begin{aligned} \dot{x} &= -y + a_{1,1}xy + a_{2,1}x^2y + a_{3,1}x^3y, \\ \dot{y} &= x + b_{2,0}x^2 + b_{3,0}x^3 + b_{0,2}y^2 + b_{1,2}xy^2 + b_{2,2}x^2y^2 + b_{4,0}x^4. \end{aligned}\right\} \tag{3.1}$$

Recall that all cases when the origin O is an isochronous center of the system (3.1) with zero Urabe function are described in [3]. In the following theorem we omit all isochronous centers with zero Urabe function, as well as all cubic isochronous centers that where all described in [13,12].

For each case we prove the isochronicity by determining explicitly its Urabe function. For system (k) we will denote it by $h_{(k)}$.

Theorem 3.1. *The following particular cases of system* (3.1) *have an isochronous center at the origin* O.

$$\left.\begin{aligned} \dot{x} &= -y + 3x^2y \pm \sqrt{2}x^3y, \\ \dot{y} &= x \pm \sqrt{2}x^2 \mp \frac{\sqrt{2}}{2}y^2 + x^3 + 4xy^2 \pm 2\sqrt{2}x^2y^2 \pm \frac{\sqrt{2}}{4}x^4, \end{aligned}\right\}$$

$$\text{where } h_{(3.2)} = -\frac{\pm\xi}{\sqrt{2 + 9\xi^2}}, \tag{3.2}$$

$$\left.\begin{aligned} \dot{x} &= -y + x^3y, \\ \dot{y} &= x + \tfrac{1}{2}x^2y^2 - \tfrac{1}{2}x^4, \end{aligned}\right\} \quad \text{where } h_{(3.3)} = \frac{\xi^3}{\sqrt{4 + \xi^6}}, \tag{3.3}$$

$$\left.\begin{aligned}\dot{x} &= -y - \frac{x^2 y}{2} + \frac{x^3 y}{8} + xy,\\ \dot{y} &= x - \frac{3x^2}{4} + \frac{y^2}{4} + \frac{5x^3}{24} + \frac{3xy^2}{8} - \frac{x^2y^2}{16} - \frac{x^4}{48},\end{aligned}\right\}$$
$$\text{where } h_{(3.4)} = \frac{3\xi}{\sqrt{16+9\xi^2}}, \tag{3.4}$$

$$\left.\begin{aligned}\dot{x} &= -y + 9x^2y + 6x^3y + xy,\\ \dot{y} &= x + \frac{3x^2}{2} - \frac{y^2}{2} + x^3 + 12xy^2 + 12x^2y^2 + \frac{x^4}{2},\end{aligned}\right\}$$
$$\text{where } h_{(3.5)} = -\frac{\xi}{\sqrt{4+49\xi^2}}, \tag{3.5}$$

$$\left.\begin{aligned}\dot{x} &= -y - \left(3a_{3,1} + \frac{2}{9}\right)x^2y + a_{3,1}x^3y + xy,\\ \dot{y} &= x + \left(-3a_{3,1} + \frac{1}{9}\right)xy^2,\end{aligned}\right\}$$
$$\text{where } h_{(3.6)} = \frac{\xi}{\sqrt{(1-27a_{3,1})\xi^2+9}}, \tag{3.6}$$

$$\left.\begin{aligned}\dot{x} &= -y + xy,\\ \dot{y} &= x - \frac{3x^2}{2} + y^2 + x^3 - \frac{x^4}{4},\end{aligned}\right\} \quad \text{where } h_{(3.7)} = \frac{\xi}{\sqrt{1+\xi^2}}. \tag{3.7}$$

Moreover, all other possible isochronous centers at O for non-cubic system (3.1)*, where either* $a_{1,1} = 1$ *or* $b_{2,0} = -3b_{0,2}$*, and with non-vanishing Urabe functions, belong to the family*

$$\left.\begin{aligned}\dot{x} &= -y + \left(-\frac{3}{8} - 2b_{2,2}\right)x^2y + \left(\frac{1}{16} + b_{2,2}\right)x^3y + xy,\\ \dot{y} &= x - \frac{3x^2}{4} + \frac{y^2}{4} + \frac{3x^3}{8} - 2b_{2,2}xy^2 + b_{2,2}x^2y^2 - \frac{x^4}{16}.\end{aligned}\right\} \tag{3.8}$$

In particular, when $b_{2,2} \in \{-\frac{1}{16}, 0, \frac{1}{16}\}$*, the origin O is an isochronous center with non-standard Urabe functions*:

$$h_{\{b_{2,2}=-\frac{1}{16}\}} = \frac{\sqrt{2}\sqrt{2L(\frac{\xi^2}{4})+8}\sqrt{\frac{\xi^2}{L(\frac{\xi^2}{4})}}(L(\frac{\xi^2}{4})+3)L(\frac{\xi^2}{4})}{2\xi(L(\frac{\xi^2}{4})+4)(L(\frac{\xi^2}{4})+1)},$$

where $L = \mathrm{Lambert}W$ *is the Lambert function* (*see* [16]),

$$h_{\{b_{2,2}=0\}} = \frac{\sqrt{2}\sqrt{\frac{-4+\xi^2+2\sqrt{4+2\xi^2}}{\xi^2}}\,\xi(\xi^2+2\sqrt{4+2\xi^2}+2)}{(2+\xi^2)(\sqrt{4+2\xi^2}+6)},$$
$$h_{\{b_{2,2}=\frac{1}{16}\}} = \frac{\sqrt{2}\xi\sqrt{2\xi^2+32}(\xi^2+12)}{2(\xi^2+4)(\xi^2+16)}.$$

Proof. Necessary conditions. This part of the proof is based on the C-algorithm. Indeed, 19 steps are necessary to find the algebraic conditions of isochronicity (see Appendix A of [3]). We did not succeed in computing the full grevlex Gröbner basis of the corresponding system of polynomial equations. We restricted ourselves to the cases $a_{1,1}=0$, which gives the cases (3.2)–(3.3), and $\{a_{1,1}=1, b_{2,0}=-3b_{0,2}\}$ which gives the cases (3.4)–(3.6) and (3.8). We also get case (3.7) as a particular solution.

Sufficient conditions. For each case we determine its Urabe function. For systems (3.2)–(3.7) the procedure in Section 2 of [3] is applied.

The search of the Urabe function for system (3.8) is more subtle. Indeed, we verified that for all values of parameters the first 20 necessary conditions of isochronicity given by C-algorithm are satisfied. This strongly suggests that for all values of parameters the system (3.8) has an isochronous center at the origin O. For this system

$$f(x)=-\frac{20-96xb_{2,2}+64x^2b_{2,2}-12x+3x^2}{-16-6x^2-32x^2b_{2,2}+16x^3b_{2,2}+x^3+16x}$$

and

$$g(x)=\frac{1}{256}\left(-16-6x^2-32x^2b_{2,2}+16x^3b_{2,2}+x^3+16x\right)x\left(-16+12x-6x^2+x^3\right),$$

from formula (1.7) one obtains

$$\xi^2(x)=2x^2(2-x)^{-2(16b_{2,2}+1)^{-1}}\left(4x^2b_{2,2}+1/4x^2-x+2\right)^{-\frac{16b_{2,2}-1}{16b_{2,2}+1}}. \tag{3.9}$$

From formula (1.8) one deduces that

$$h\big(\xi(x)\big)=-\frac{x(12-6x+x^2)}{(x-4)(x^2-2x+4)}.$$

Now the problem is to find the reciprocal function $x=x(\xi)$. Unfortunately we succeeded in finding it only for $b_{2,2}\in\{-\frac{1}{16},0,\frac{1}{16}\}$ because in those cases Eq. (3.9) takes a sufficiently simple form. □

Note that the system (3.3) was already identified in Theorem 2.2 of [3].

4. The Choudhury–Guha reduction

4.1. Cubic isochronous centers

Choudhury–Guha reduction is more general than the standard one used in preceding papers [12,13,3]. Here we provide the complete enumeration of all cubic systems from (1.11) and we find three new cases of isochronous centers at the origin. The system we consider is

$$\left.\begin{aligned}\dot x&=-y+a_{1,1}xy+a_{2,0}x^2+a_{2,1}x^2y+a_{3,0}x^3,\\ \dot y&=x+b_{2,0}x^2+b_{1,1}xy+b_{0,2}y^2+b_{2,1}yx^2+b_{1,2}y^2x+b_{3,0}x^3.\end{aligned}\right\} \tag{4.1}$$

Condition (2.4) is equivalent to the following system of equations:

$$\left.\begin{aligned}&2a_{2,0}+b_{1,1}=0,\\ &a_{1,1}a_{2,0}-3a_{3,0}-b_{2,1}+a_{1,1}b_{1,1}-2b_{0,2}a_{2,0}=0,\\ &a_{1,1}b_{2,1}-2b_{0,2}a_{3,0}+2a_{1,1}a_{3,0}+a_{2,1}b_{1,1}-2b_{1,2}a_{2,0}=0,\\ &a_{2,1}a_{3,0}+a_{2,1}b_{2,1}-2b_{1,2}a_{3,0}=0.\end{aligned}\right\} \tag{4.2}$$

Theorem 4.1. *Under the assumptions* (4.2) *the origin O is an isochronous center of system* (4.1) *only in one of the following cases*:

1. *The standard reduction is possible, that means $a_{3,0}=a_{2,0}=b_{1,1}=b_{2,1}=0$ and the system is one of those from Theorem* 3 *of* [13].
2. *We are in one of the following cases*:

$$\left.\begin{aligned}\dot{x}&=-y-2b_{2,0}xy+x^2+2b_{2,0}x^3,\\ \dot{y}&=x-4b_{2,0}y^2-2xy+b_{2,0}x^2+4b_{2,0}x^2y+2x^3,\end{aligned}\right\}\tag{4.3}$$

$$\left.\begin{aligned}\dot{x}&=-y\pm 2\sqrt{2}xy+x^2\mp 2\sqrt{2}x^3,\\ \dot{y}&=x\pm 8\sqrt{2}y^2-2xy\mp 3\sqrt{2}x^2\mp 12\sqrt{2}x^2y+10x^3,\end{aligned}\right\}\tag{4.4}$$

$$\left.\begin{aligned}\dot{x}&=-y-\frac{1}{2}b_{2,0}xy+x^2+\frac{1}{2}b_{2,0}x^3,\\ \dot{y}&=x-b_{2,0}y^2-2xy+b_{2,0}x^2+b_{2,0}x^2y+\left(2+\frac{1}{4}{b_{2,0}}^2\right)x^3.\end{aligned}\right\}\tag{4.5}$$

Proof. The necessary conditions are given by the solutions of the polynomial system of equations consisting of Eqs. (4.2) (called $C_1,\dots,C_4$) and the 8 equations obtained from the Rational C-Algorithm (15 steps), called $C_5,\dots,C_{12}$. Let us denote by I the ideal generated by $C_1,\dots,C_{12}$.

We exclude the standard reduction by adding to I the variable T and the polynomial

$$C_{13}=(Ta_{3,0}-1)(Ta_{2,0}-1)(Tb_{1,1}-1)(Tb_{2,1}-1).$$

For $a_{2,0}=0$, a Gröbner basis of $\langle C_1,\dots,C_{12},C_{13},a_{2,0}\rangle$ is $\langle 1\rangle$ (i.e. there is no solution), which implies that we can take $a_{2,0}=1$. We use the weighted order $b_{1,1}>b_{2,1}>b_{3,0}>b_{1,2}>a_{2,1}>b_{0,2}>a_{1,1}>b_{2,0}>a_{3,0}$.

First, a Gröbner basis of system $\langle C_1,\dots,C_6\rangle$ for the weighted lex order contains the polynomial

$$P=(a_{1,1}+2b_{0,2})\left(a_{2,1}-a_{1,1}a_{3,0}-{a_{3,0}}^2\right).$$

We split our problem into two subcases according to this factorization:

- For $a_{1,1}+2b_{0,2}=0$, we get only one real solution

$$\left.\begin{aligned}\dot{x}&=-y+x^2,\\ \dot{y}&=x-2xy+2x^3,\end{aligned}\right\}$$

  which is a particular case of (4.5) with $b_{2,0}=0$.
- For $a_{2,1}-a_{1,1}a_{3,0}-{a_{3,0}}^2=0$, we eliminate the solutions that are not real by adding to the system the polynomials $P_i\cdot T_i-1$ for each P_i in

$$\left\{16{a_{3,0}}^2+1,4{a_{3,0}}^2+9,4{a_{3,0}}^2+1,{a_{3,0}}^2+4,{a_{3,0}}^2+1,{a_{3,0}}^2+16,\right.$$
$$\left.{a_{3,0}}^2+9-4b_{2,0}a_{3,0}+4{b_{2,0}}^2\right\}$$

  that have no real solution. Then, the solutions of the resulting system are those quoted in the theorem.

Sufficiency. For the cases (4.3) and (4.4) we have $g'(x) + f(x)g(x) = 1$. Hence by Corollary 1.3 the origin is an isochronous center. Moreover we easily check that $h_{(4.5)}(\xi) = -\frac{1}{2}b_{2,0}\xi$. □

4.2. *Quartic isochronous centers*

Our first target was to identify all isochronous centers at the origin with zero Urabe function for the system (1.11)

$$\left.\begin{aligned}\dot{x} &= -y + a_{1,1}xy + a_{2,0}x^2 + a_{2,1}x^2y + a_{3,0}x^3 + a_{3,1}x^3y + a_{4,0}x^4,\\ \dot{y} &= x + b_{0,2}y^2 + b_{1,1}xy + b_{2,0}x^2 + b_{1,2}xy^2 + b_{2,1}x^2y + b_{3,0}x^3\\ &\quad + b_{2,2}x^2y^2 + b_{3,1}x^3y + b_{4,0}x^4,\end{aligned}\right\}$$

under the condition (2.6). That means finding all values of the 15 parameters for which the equation $g'(x) + f(x)g(x) = 1$ is satisfied where f and g are defined by (2.5) (see Corollary 1.3).

When the standard reduction is possible, that means $a_{4,0} = a_{3,0} = a_{2,0} = b_{1,1} = b_{2,1} = b_{3,1} = 0$ all the 6 isochronous centers with zero Urabe function were described in Theorem 3.1 of [3]. Otherwise when the Choudhury–Guha reduction needs to be applied the problem becomes substantially more complicated.

Taking in account the great complexity of the problem we did not succeed in solving it completely. Nevertheless, during our investigations we obtained 25 new isochronous centers for the system (1.11), two of them of extreme complexity, called Monsters. We are convinced that our list is not exhaustive.

The procedure to obtain the isochronous centers listed below consists in solving by Gröbner method the system (2.6) simultaneously with the set of equations on parameters which corresponds the equation $g'(x) + f(x)g(x) = 1$. First, one applies the Solve routine of Maple (based on Gröbner basis technic) which splits the variety of solutions into 37 subvarieties. The cases (4.6)–(4.29) were obtained by detailed inspection of some of them. The remaining 7 isochronous centers (4.23)–(4.30) were obtained by restricting ourselves to $b_{2,2} = a_{3,1} = 0$ and by application of the standard Gröbner basis technique.

We verified also that all above isochronous centers are not time-reversible, except perhaps the two Monsters (4.29) and (4.30).

Theorem 4.2. *The following quartic systems have an isochronous center at the origin O with zero Urabe function.*

$$\left.\begin{aligned}\dot{x} &= -y + b_{0,2}xy + x^2 - b_{0,2}x^3,\\ \dot{y} &= x + b_{0,2}y^2 - 2xy + 2x^3 - b_{0,2}x^4,\end{aligned}\right\} \tag{4.6}$$

$$\left.\begin{aligned}\dot{x} &= -y + xy - a_{3,0}x^2 + a_{3,0}x^3,\\ \dot{y} &= x + y^2 + 2a_{3,0}xy + 2{a_{3,0}}^2x^3 - {a_{3,0}}^2x^4,\end{aligned}\right\} \tag{4.7}$$

$$\left.\begin{aligned}\dot{x} &= -y + xy - a_{3,0}x^2 + a_{3,0}x^3,\\ \dot{y} &= x + 3y^2 + 2a_{3,0}xy - x^2 + 4a_{3,0}x^2y + \left(\frac{1}{3} + 2{a_{3,0}}^2\right)x^3 + {a_{3,0}}^2x^4,\end{aligned}\right\} \tag{4.8}$$

$$\left.\begin{aligned}\dot{x} &= -y + xy - a_{3,0}x^2 + a_{3,0}x^3,\\ \dot{y} &= x + 4y^2 + 2a_{3,0}xy - \frac{3x^2}{2} + 6a_{3,0}x^2y + \left(1 + 2a_{3,0}{}^2\right)x^3\\ &\quad + \left(-\frac{1}{4} + 2a_{3,0}{}^2\right)x^4,\end{aligned}\right\} \tag{4.9}$$

$$\left.\begin{aligned}\dot{x} &= -y + \frac{b_{0,2}xy}{3} + x^2 - \frac{b_{0,2}x^3}{3},\\ \dot{y} &= x + b_{0,2}y^2 - 2xy - \frac{b_{0,2}x^2}{3} - \frac{4b_{0,2}x^2y}{3} + \left(\frac{b_{0,2}{}^2}{27} + 2\right)x^3 + \frac{b_{0,2}x^4}{3},\end{aligned}\right\} \tag{4.10}$$

$$\left.\begin{aligned}\dot{x} &= -y + \frac{b_{0,2}xy}{4} + x^2 - \frac{b_{0,2}x^3}{4},\\ \dot{y} &= x + b_{0,2}y^2 - 2xy - \frac{3b_{0,2}x^2}{8} - \frac{3b_{0,2}x^2y}{2} + \left(\frac{b_{0,2}{}^2}{16} + 2\right)x^3\\ &\quad + \left(-\frac{1}{256}b_{0,2}{}^3 + \frac{b_{0,2}}{2}\right)x^4,\end{aligned}\right\} \tag{4.11}$$

$$\left.\begin{aligned}\dot{x} &= -y - \frac{45}{8}x^2y + x^2 + \frac{45}{8}x^4,\\ \dot{y} &= x - 2xy - \frac{225}{8}xy^2 + \frac{19}{2}x^3 + 45x^3y,\end{aligned}\right\} \tag{4.12}$$

$$\left.\begin{aligned}\dot{x} &= -y + \frac{b_{4,0}{}^2x^2y}{2} + x^2 - \frac{b_{4,0}{}^2x^4}{2},\\ \dot{y} &= x + b_{4,0}y^2 - 2xy - \frac{b_{4,0}x^2}{2} + b_{4,0}{}^2xy^2 - 2b_{4,0}x^2y\\ &\quad + 2x^3 - b_{4,0}{}^2x^3y + b_{4,0}x^4,\end{aligned}\right\} \tag{4.13}$$

$$\left.\begin{aligned}\dot{x} &= -y + (-2 + 2\sqrt{19})x^2y + x^2 - (-2 + 2\sqrt{19})x^4,\\ \dot{y} &= x \pm \alpha_1y^2 - 2xy \mp \frac{\alpha_1x^2}{2} + \alpha_2xy^2 \mp 2\alpha_1x^2y + \alpha_4x^3 + \alpha_3x^3y \pm 4\alpha_1x^4,\end{aligned}\right\} \tag{4.14}$$

where $\alpha_1 = \sqrt{-106 + 34\sqrt{19}}$, $\alpha_2 = -10 + 10\sqrt{19}$, $\alpha_3 = 16 - 16\sqrt{19}$, $\alpha_4 = -13 + 3\sqrt{19}$.

$$\left.\begin{aligned}\dot{x} &= -y + a_{1,1}xy + \frac{15}{8}a_{1,1}{}^2x^2y + x^2 - x^3a_{1,1} - \frac{15}{8}x^4a_{1,1}{}^2,\\ \dot{y} &= x - \frac{a_{1,1}y^2}{2} - 2xy + \frac{3a_{1,1}x^2}{4} + \frac{15}{4}a_{1,1}{}^2xy^2 + 3a_{1,1}x^2y\\ &\quad + 2x^3 - \frac{15}{4}a_{1,1}{}^2x^3y - \frac{5a_{1,1}x^4}{2},\end{aligned}\right\} \tag{4.15}$$

$$\left.\begin{aligned}\dot{x} &= -y \mp \frac{2}{35}\alpha_5xy + \alpha_6x^2y + x^2 \pm \frac{2}{35}\alpha_5x^3 - \alpha_6x^4,\\ \dot{y} &= x \pm \frac{\alpha_5y^2}{35} - 2xy \mp \frac{3}{70}\alpha_5x^2 + 5\alpha_6xy^2 \mp \frac{6}{35}\alpha_5x^2y\\ &\quad + \alpha_7x^3 - 8\alpha_6x^3y \pm \frac{38}{35}\alpha_5x^4,\end{aligned}\right\} \tag{4.16}$$

where $\alpha_5 = \sqrt{-77\,798 + 1162\sqrt{4691}}$, $\alpha_6 = -\frac{354}{5} + \frac{6\sqrt{4691}}{5}$, $\alpha_7 = -\frac{2183}{35} + \frac{27}{35}\sqrt{4691}$.

$$\left.\begin{aligned}\dot{x}&=-y+xy-\frac{3a_{3,0}x^2}{4}+a_{3,0}x^3-\frac{a_{3,0}x^4}{4},\\ \dot{y}&=x+3y^2+\frac{3a_{3,0}xy}{2}-x^2+\frac{9a_{3,0}x^2y}{4}+\left(\frac{1}{3}+\frac{9}{8}{a_{3,0}}^2\right)x^3\\ &\quad-\frac{3a_{3,0}x^3y}{4}-\frac{3{a_{3,0}}^2x^4}{8},\end{aligned}\right\} \tag{4.17}$$

$$\left.\begin{aligned}\dot{x}&=-y+xy\pm\frac{\sqrt{2}}{2}x^2\mp\frac{2\sqrt{2}}{3}x^3\pm\frac{\sqrt{2}}{6}x^4,\\ \dot{y}&=x+6y^2\mp\sqrt{2}xy-\frac{5}{2}x^2\mp\frac{9}{2}\sqrt{2}x^2y+\frac{13}{3}x^3\pm\frac{3\sqrt{2}}{2}x^3y-\frac{4}{3}x^4,\end{aligned}\right\} \tag{4.18}$$

$$\left.\begin{aligned}\dot{x}&=-y\mp 2\sqrt{2}xy+x^2\pm 2\sqrt{2}x^3,\\ \dot{y}&=x\mp 6\sqrt{2}y^2-2xy\pm 2\sqrt{2}x^2\pm 8\sqrt{2}x^2y+\frac{14}{3}x^3\mp 2\sqrt{2}x^4,\end{aligned}\right\} \tag{4.19}$$

$$\left.\begin{aligned}\dot{x}&=-y+a_{1,1}xy+{a_{1,1}}^2x^2y+x^2-a_{1,1}x^3-{a_{1,1}}^2x^4,\\ \dot{y}&=x+3a_{1,1}y^2-2xy-a_{1,1}x^2+2{a_{1,1}}^2xy^2\\ &\quad-4a_{1,1}x^2y+2x^3-2{a_{1,1}}^2x^3y+a_{1,1}x^4,\end{aligned}\right\} \tag{4.20}$$

$$\left.\begin{aligned}\dot{x}&=-y+a_{1,1}xy+3{a_{1,1}}^2x^2y+x^2-a_{1,1}x^3-3{a_{1,1}}^2x^4,\\ \dot{y}&=x+4a_{1,1}y^2-2xy-\frac{3}{2}a_{1,1}x^2+6{a_{1,1}}^2xy^2-6a_{1,1}x^2y\\ &\quad+2x^3-6{a_{1,1}}^2x^3y+2a_{1,1}x^4,\end{aligned}\right\} \tag{4.21}$$

$$\left.\begin{aligned}\dot{x}&=-y+a_{1,1}xy+\left({a_{1,1}}^2-\frac{3}{2}b_{0,2}a_{1,1}+\frac{1}{2}{b_{0,2}}^2\right)x^2y\\ &\quad+x^2-a_{1,1}x^3-\left({a_{1,1}}^2-\frac{3}{2}b_{0,2}a_{1,1}+\frac{1}{2}{b_{0,2}}^2\right)x^4,\\ \dot{y}&=x+b_{0,2}y^2-2xy+\left(\frac{1}{2}a_{1,1}-\frac{1}{2}b_{0,2}\right)x^2\\ &\quad+\left(2{a_{1,1}}^2-3b_{0,2}a_{1,1}+{b_{0,2}}^2\right)xy^2\\ &\quad+\left(2\left({a_{1,1}}^2-\frac{3}{2}b_{0,2}a_{1,1}+\frac{1}{2}{b_{0,2}}^2\right)\right.\\ &\quad\left.-2\left(2{a_{1,1}}^2-3b_{0,2}a_{1,1}+{b_{0,2}}^2\right)\right)x^3y\\ &\quad+(-2a_{1,1}+b_{0,2})x^4+(2a_{1,1}-2b_{0,2})x^2y+2x^3,\end{aligned}\right\} \tag{4.22}$$

$$\left.\begin{aligned}\dot{x}&=-y+a_{1,1}xy+x^2-a_{1,1}x^3,\\ \dot{y}&=x+4a_{1,1}y^2-2xy-\frac{3}{2}a_{1,1}x^2-6a_{1,1}x^2y+\left(2+{a_{1,1}}^2\right)x^3\\ &\quad+\left(2a_{1,1}-\frac{1}{4}{a_{1,1}}^3\right)x^4,\end{aligned}\right\} \tag{4.23}$$

$$\left.\begin{aligned}\dot{x}&=-y+a_{1,1}xy+x^2-a_{1,1}x^3,\\ \dot{y}&=x+3a_{1,1}y^2-2xy-a_{1,1}x^2-4a_{1,1}x^2y+\left(\frac{1}{3}{a_{1,1}}^2+2\right)x^3+a_{1,1}x^4,\end{aligned}\right\} \tag{4.24}$$

$$\left.\begin{aligned}\dot{x} &= -y + 2\beta xy + x^2 - 2\beta x^3,\\ \dot{y} &= x + 8\beta y^2 - 2xy - 3\beta x^2 - 12\beta x^2 y + 14x^3 - 2\beta x^4,\end{aligned}\right\} \tag{4.25}$$

where $\beta = \pm\sqrt{3}$.

$$\left.\begin{aligned}\dot{x} &= -y + \alpha xy + x^2 - 4/3\alpha x^3 + 2/3x^4,\\ \dot{y} &= x + 6\alpha y^2 - 2xy - 5/2\alpha x^2 - 9\alpha x^2 y + \frac{26}{3}x^3 + 6x^3 y - 8/3\alpha x^4,\end{aligned}\right\} \tag{4.26}$$

$$\left.\begin{aligned}\dot{x} &= -y + \alpha xy + x^2 - \frac{4}{3}\alpha x^3 + \frac{2}{3}x^4,\\ \dot{y} &= x + 3\alpha y^2 - 2xy - \alpha x^2 - 3\alpha x^2 y + \frac{8}{3}x^3 + 2x^3 y - \frac{2}{3}\alpha x^4,\end{aligned}\right\} \tag{4.27}$$

where $\alpha = \pm\sqrt{2}$.

$$\left.\begin{aligned}\dot{x} &= -y + a_{1,1}xy + x^2 + \left({a_{1,1}}^2 - \frac{3}{2}b_{0,2}a_{1,1} + \frac{1}{2}{b_{0,2}}^2\right)x^2 y - a_{1,1}x^3\\ &\quad + \left(-{a_{1,1}}^2 + \frac{3}{2}b_{0,2}a_{1,1} - \frac{1}{2}{b_{0,2}}^2\right)x^4,\\ \dot{y} &= x + b_{0,2}y^2 - 2xy + \left(\frac{1}{2}a_{1,1} - \frac{1}{2}b_{0,2}\right)x^2\\ &\quad + \left(2{a_{1,1}}^2 - 3b_{0,2}a_{1,1} + {b_{0,2}}^2\right)xy^2 + (2a_{1,1} - 2b_{0,2})x^2 y + 2x^3\\ &\quad + \left(-2{a_{1,1}}^2 + 3b_{0,2}a_{1,1} - {b_{0,2}}^2\right)x^3 y + (-2a_{1,1} + b_{0,2})x^4.\end{aligned}\right\} \tag{4.28}$$

Finally, the two Monsters mentioned in the Introduction are of the form

$$\left.\begin{aligned}\dot{x} &= -y + Txy + Mx^2 y + x^2 - Tx^3 - Mx^4,\\ \dot{y} &= x + Py^2 - 2xy - \frac{P}{2}x^2 + 5Mxy^2 - 2Px^2 y + Sx^3 - 8Mx^3 y + 4Px^4,\end{aligned}\right\} \tag{4.29}$$

$$\left.\begin{aligned}\dot{x} &= -y + \alpha xy + Mx^2 y + x^2 - \alpha x^3 - Mx^4,\\ \dot{y} &= x + 5Mxy^2 - 8Mx^3 y - 2xy + \frac{B_{0,2}}{12((M+9)\alpha\beta)}y^2 - \frac{\delta}{12(M+9)\alpha\beta}x^2\\ &\quad - \frac{\delta}{3(M+9)\alpha\beta}x^2 y - \frac{B_{3,0}}{2(M+9)\beta}x^3 - \frac{B_{4,0}}{12(M+9)\alpha\beta}x^4.\end{aligned}\right\} \tag{4.30}$$

The exact description of their coefficients is too cumbersome to be reproduced here. They are written down in arXiv variant of the present paper (arXiv:1005.5048, isochronous centers (4.23) and (4.30) respectively).

We are puzzled by the algebraic features of the isochronous centers here presented, as (4.14), (4.16), (4.18), (4.19), (4.25), (4.27), (4.29), (4.30).

5. Explicit linearization

5.1. Linearization formulas

Let us consider the Liénard type system (1.4)

$$\left.\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -g(x) - f(x)y^2,\end{aligned}\right\}$$

with a center at the origin (0, 0) where f and g are real analytic in a neighborhood of zero.

It is known by Sabatini formula (1.6) that the first integral associated to the system (1.4) can be written

$$I(x,\dot{x}) = \int_0^x g(s)e^{2F(s)}\,ds + \frac{1}{2}\big(\dot{x}e^{F(x)}\big)^2 \tag{5.1}$$

where $F(x) = \int_0^x f(s)\,ds$.

Following [9] (see also [19]), let us perform the following change of variables

$$\left.\begin{aligned}p(x,\dot{x}) &= \dot{x}e^{F(x)},\\ q(x) &= \int_0^x e^{F(s)}\,ds.\end{aligned}\right\} \tag{5.2}$$

As $\frac{\partial(p,q)}{\partial(x,\dot{x})} = -e^{2F(x)} < 0$ and $p(0,0) = q(0) = 0$ then this is an analytic change of variables preserving the origin and well defined around it. Moreover, $q'(x) = e^{F(x)} > 0$ and thus the function $x \mapsto q(x)$ is strictly increasing. In the (p,q) coordinates the first integral (5.1) becomes

$$I(x,\dot{x}) = H(p,q) = \frac{1}{2}p^2 + U(q), \tag{5.3}$$

where U is some uniquely defined real analytic function, $U(0) = 0$. Now it is easy to see that the system (1.4) in (p,q) coordinates can be written as

$$\left.\begin{aligned}\dot{q} &= \frac{\partial H}{\partial p} = p,\\ \dot{p} &= -\frac{\partial H}{\partial q} = -\frac{d}{dq}U\end{aligned}\right\} \tag{5.4}$$

that is as a Hamiltonian system corresponding to the Hamiltonian (5.3).

The main result of this section is

Theorem 5.1. *Let us consider the Liénard type system* (1.4) *with real analytic functions f and g such that $xg(x) > 0$ for $x \neq 0$. Then the origin O is an isochronous center with Urabe function $h = 0$ if and only if $U(q) = \frac{q^2}{2}$.*

Proof. It is easy to see that O is a center. Now, from Corollary 1.3, one knows that O is an isochronous center with Urabe function $h = 0$ if and only if $g'(x) + g(x)f(x) = 1$ or equivalently $g'(x)e^{F(x)} + g(x)f(x)e^{F(x)} = e^{F(x)}$. The last equality is nothing else $(g(x)e^{F(x)})' = (\int_0^x e^{F(s)}\,ds)'$. As $g(0)e^{F(0)} = 0$, when integrating one obtains $g(x)e^{F(x)} = \int_0^x e^{F(s)}\,ds$ or equivalently $U'(q) = q$ because $(\frac{dU}{dq})(q(x)) = g(x)e^{F(x)}$. Since $U(0) = 0$ one has $U(q) = \frac{1}{2}q^2$. □

Consequently when the Urabe function h identically vanishes, the system of coordinate (p,q) defined by (5.2) is the linearizing system of coordinates for system (1.4). Indeed,

$$\left.\begin{aligned}\dot{q} &= p,\\ \dot{p} &= -q.\end{aligned}\right\} \tag{5.5}$$

It is interesting to compare Theorem 5.1 with Chalykh–Veselov theorem that we formulate only for potential U without pole at 0.

Theorem 5.2. *(See [4, Theorem 1].) Let us consider the Hamiltonian system with the Hamiltonian* $H(P,Q)=\frac{1}{2}P^2+U(Q)$ *where* U *is a rational function without pole at* 0. *Then* O *is an isochronous center for the associated Hamiltonian system* $\dot{Q}=\frac{\partial H}{\partial P}$, $\dot{P}=-\frac{\partial H}{\partial Q}$ *if and only if up to a shift* $Q\to Q+a$ *and adding a constant,* $U(Q)=kQ^2$ *for some* $k\in\mathbb{R}-\{0\}$.

5.2. Examples

Now applying formula (5.2) we provide 5 examples of linearization of isochronous centers with zero Urabe function. The reduction to Liénard type system (1.4) is always obtained by standard or Choudhury–Guha reduction. To compute variables (p,q) (see (5.2)) we use Maple, and the identity (5.5) was verified in all cases. Our choice is somewhere random, because all reported examples with zero Urabe function are good for such purpose.

5.2.1. Cubic examples

1. Consider the case 6 of Theorem 3 of [13], that is the system

$$\left.\begin{aligned}\dot{x} &= -y+a_{1,1}xy+\left(a_{1,1}{}^2-\frac{3a_{1,1}b_{0,2}}{2}+\frac{b_{0,2}{}^2}{2}\right)x^2y,\\ \dot{y} &= x+\left(-\frac{b_{0,2}}{2}+\frac{a_{1,1}}{2}\right)x^2+b_{0,2}y^2+\left(2a_{1,1}{}^2-3a_{1,1}b_{0,2}+b_{0,2}{}^2\right)xy^2.\end{aligned}\right\} \tag{5.6}$$

In this case the functions f and g are

$$\left.\begin{aligned}f(x) &= -\frac{b_{0,2}+(2a_{1,1}{}^2-3a_{1,1}b_{0,2}+b_{0,2}{}^2)x+a_{1,1}+2(a_{1,1}{}^2-\frac{3a_{1,1}b_{0,2}}{2}+\frac{b_{0,2}{}^2}{2})x}{-1+a_{1,1}x+(a_{1,1}{}^2-\frac{3a_{1,1}b_{0,2}}{2}+\frac{b_{0,2}{}^2}{2})x^2},\\ g(x) &= \left(1-a_{1,1}x-\left(a_{1,1}{}^2-\frac{3a_{1,1}b_{0,2}}{2}+\frac{b_{0,2}{}^2}{2}\right)x^2\right)\left(x+\left(-\frac{b_{0,2}}{2}+\frac{a_{1,1}}{2}\right)x^2\right),\end{aligned}\right\} \tag{5.7}$$

for which we obtain the following linearizing change of coordinates

$$\left.\begin{aligned}q(x) &= -\frac{(2+a_{1,1}x-b_{0,2}x)xe^{2(a_{1,1}-b_{0,2})\frac{(A(x)-A(0))}{\sqrt{-5a_{1,1}{}^2+6a_{1,1}b_{0,2}-2b_{0,2}{}^2}}}}{(-2+2a_{1,1}x+2a_{1,1}{}^2x^2-3x^2a_{1,1}b_{0,2}+x^2b_{0,2}{}^2)},\\ p(x,y) &= \frac{-2yq(x)}{(2+a_{1,1}x-b_{0,2}x)x},\end{aligned}\right\} \tag{5.8}$$

where

$$A(x)=\arctan\left(\frac{2a_{1,1}{}^2x-3xa_{1,1}b_{0,2}+xb_{0,2}{}^2+a_{1,1}}{\sqrt{-5a_{1,1}{}^2+6a_{1,1}b_{0,2}-2b_{0,2}{}^2}}\right). \tag{5.9}$$

2. Consider system (4.3) from Theorem 4.1. In this case the functions f and g are

$$\left.\begin{aligned} f(x) &= -6\frac{b_{2,0}}{1+2b_{2,0}x}, \\ g(x) &= x\left(2{b_{2,0}}^2x^2+3b_{2,0}x+1\right), \end{aligned}\right\} \tag{5.10}$$

for which we obtain the following linearizing change of coordinates

$$\left.\begin{aligned} q(x) &= \frac{x(b_{2,0}x+1)}{(1+2b_{2,0}x)^2}, \\ p(x,y) &= \frac{-y+x^2}{(1+2b_{2,0}x)^2}. \end{aligned}\right\} \tag{5.11}$$

5.2.2. Quartic examples

1. As a quartic example we consider the system III of Theorem 3.1 of [3]. We choose the following restrictions on the parameters $a_{21} = -3$, $b_{0,2} = -4$, $b_{20} = \frac{1}{2}$ to obtain simple, presentable expressions for linearizing variables.

$$\left.\begin{aligned} \dot{x} &= -y-3xy-3x^2y-x^3y, \\ \dot{y} &= x+\frac{1}{2}x^2-4y^2-4xy^2-2x^2y^2. \end{aligned}\right\} \tag{5.12}$$

In this case the functions f and g are

$$\left.\begin{aligned} f(x) &= -\frac{7+10x+5x^2}{1+3x+3x^2+x^3}, \\ g(x) &= \frac{1}{2}\left(1+3x+3x^2+x^3\right)x(2+x), \end{aligned}\right\} \tag{5.13}$$

for which we obtain the following linearizing change of coordinates

$$\left.\begin{aligned} q(x) &= \frac{1}{2}x(2+x)e^{-\frac{x(2+x)}{(1+x)^2}}(1+x)^{-2}, \\ p(x,y) &= -ye^{-\frac{x(2+x)}{(1+x)^2}}(1+x)^{-2}. \end{aligned}\right\} \tag{5.14}$$

2. We consider the system (4.29) of Theorem 4.2.

In this case the functions f and g are

$$\left.\begin{aligned} f(x) &= \frac{4{a_{1,1}}^2x-6xb_{0,2}a_{1,1}+2x{b_{0,2}}^2+a_{1,1}+b_{0,2}}{1-a_{1,1}x-{a_{1,1}}^2x^2+3/2x^2b_{0,2}a_{1,1}-1/2x^2{b_{0,2}}^2}, \\ g(x) &= -1/4x(a_{1,1}x-xb_{0,2}+2)\left(-2+2a_{1,1}x+2{a_{1,1}}^2x^2 \right. \\ &\quad \left. -3x^2b_{0,2}a_{1,1}+x^2{b_{0,2}}^2\right), \end{aligned}\right\} \tag{5.15}$$

for which we obtain the following linearizing change of coordinates

$$\left.\begin{aligned} q(x) &= -\frac{x(a_{1,1}x-xb_{0,2}+2)S(x)}{-2+2a_{1,1}x+2{a_{1,1}}^2x^2-3x^2b_{0,2}a_{1,1}+x^2{b_{0,2}}^2}, \\ p(x,y) &= 2\frac{(x^2-y)S(x)}{-2+2a_{1,1}x+2{a_{1,1}}^2x^2-3x^2b_{0,2}a_{1,1}+x^2{b_{0,2}}^2}, \end{aligned}\right\} \tag{5.16}$$

where

$$S(x) = e^{\frac{2(a_{1,1}-b_{0,2})\left(\arctan\left(\frac{2{a_{1,1}}^2x-3xb_{0,2}a_{1,1}+x{b_{0,2}}^2+a_{1,1}}{\sqrt{-5{a_{1,1}}^2+6b_{0,2}a_{1,1}-2{b_{0,2}}^2}}\right)-\arctan\left(\frac{a_{1,1}}{\sqrt{-5{a_{1,1}}^2+6b_{0,2}a_{1,1}-2{b_{0,2}}^2}}\right)\right)}{\sqrt{-5{a_{1,1}}^2+6b_{0,2}a_{1,1}-2{b_{0,2}}^2}}}.$$

5.2.3. A rational example

Let us consider the system

$$\left.\begin{aligned} \dot{x} &= -y + \frac{yx}{1+x}, \\ \dot{y} &= x + \frac{y^2}{1+x}. \end{aligned}\right\} \tag{5.17}$$

In this case the functions f and g are

$$\left.\begin{aligned} f(x) &= \frac{2+x}{1+x}, \\ g(x) &= \frac{x}{1+x}, \end{aligned}\right\} \tag{5.18}$$

for which we obtain the following linearizing change of coordinates

$$\left.\begin{aligned} & q(x) = xe^x, \\ & p(x,y) = ye^x(1+x). \end{aligned}\right\} \tag{5.19}$$

5.3. Comments

It is really astonishing that in all above cases the linearizing variables (p,q) are always expressed in “finite terms”. This follows from the fact that if $g'(x) + f(x)g(x) = 1$ then $f = \frac{1-g'}{g}$. Moreover, as $g(0) = 0$ and $g'(0) = 1$ the singularity of f at zero is spurious. In all examples considered in this and related papers [12,13,3] f and g always are rational functions. Then $F(x) = \int_0^x f(s)\,ds$ is expressed in “finite terms” and thus also $p(x,\dot{x})$. The problem is slightly more delicate in what concerns $q(x)$. But $\int e^{F(s)}\,ds = \int e^{\int f(s)\,ds}\,ds = \int e^{\int \frac{1-g'(s)}{g(s)}\,ds}\,ds = \int \frac{1}{g(s)} e^{\int \frac{1}{g(s)}\,ds}\,ds = e^{\int \frac{1}{g(s)}\,ds} + const.$ g being a rational function, $\int \frac{ds}{g(s)}$ is obtained in “finite terms” and thus also $\int e^{F(s)}\,ds$.

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