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Symmetric matrix pencils

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Abstract

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A significant number of matrix eigenvalue problems, quadratic or linear, are best reformulated as pencils (A, M) in which both A and M are real and symmetric. Some examples are given and then the canonical forms are re-examined to explain the role of the sign characteristic attached to real eigenvalues. In addition we examine the limitations on the use of the Rayleigh quotient functional $(x, Ax)/(x, Mx)$ in describing the eigenvalues. This sheds new light on the class of definite pencils and the stability of their eigenvalues under perturbations. The reduction of indefinite pencils to useful sparse forms is mentioned.

Keywords: Generalized eigenvalue problem, matrix pencils, Rayleigh quotient, equivalence, congruence and rotation transformations.

1. Introduction and summary

This study is concerned with the generalized linear eigenvalue problem

$$(H - \lambda A)z = 0, \tag{1}$$

where H and A are symmetric matrices in $\mathbb{R}^{n\times n}$, $\det(H - \lambda A) \neq 0$ for some λ, $z \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. The ordered pair (H, A) defines a *regular symmetric pencil*. Many of the results extend, with some care, to regular complex Hermitian pencils. Moreover, (H, A) is really just a representative of the class of indistinguishable pencils $\{(\tau H, \tau A), \tau \neq 0\}$.

The nature of solutions to (1) is well understood. At one extreme, when A is symmetric positive definite (we write A is spd, hereafter), there exists a full set of linearly independent eigenvectors in $\mathbb{R}^n$ whereas, at the other end, when both H and A are indefinite, the full complications of complex eigenvalues and a nontrivial Jordan canonical form may be present.

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In the classical case, when A is spd, the leading role in the analysis is played by the Rayleigh quotient functional

$$\rho(x) := \frac{x^t H x}{x^t A x}, \quad x(\neq 0) \in \mathbb{R}^n. \tag{2}$$

The (ordered) eigenvalues λ of (1) may then be characterized by the Courant–Fischer minmax and maxmin values of ρ. Moreover the eigenvectors z of (1) determine the simultaneous diagonalization of the two quadratic forms $x^t H x$ and $x^t A x$; $Z^t H Z = \Lambda$, $Z^t A Z = I$.

The purpose of this study is to clarify what happens to the connection between simultaneous diagonalization, the Rayleigh quotient, and solutions to (1), as the class of pencils is broadened to include first definite pencils and, beyond that, indefinite pencils. These and other terms are defined in Sections 2–5, 7. For simplicity we use the noun symmetry to correspond to two adjectives: real symmetric and Hermitian.

It is well known that definite pencils can be diagonalized by congruence transformations but definiteness is not necessary. Our curiosity was aroused by the following result, see [8].

Theorem 1. *A symmetric pencil (H, A) with invertible A is definite if, and only if,*
(a) *$A^{-1}H$ is similar to a diagonal matrix,*
(b) *each eigenvalue of $A^{-1}H$ is real,*
(c) *the eigenvalues of positive type are separated from those of negative type.*

It is (c) that is puzzling. In our context a real eigenvalue λ is of negative type if its eigenvector x satisfies $x^t A x < 0$. Since A^{-1} exists, all eigenvalues are finite and so no eigenvector z has $z^T A z = 0$. When (a) and (b) hold but (c) fails, then we have a pencil that can be diagonalized but is not definite. Yet Gantmacher [5] in his treatment of pencils never mentions grouping real eigenvalues in this way.

In later sections we explain our view that for eigenvalue-eigenvector problems (and there are other instances of pencils) the sign characteristic is superfluous. However in a context in which congruence transformations are the only ones permitted, then the sign characteristic does have a place.

Let us turn next to the beautiful minmax and maxmin characterization of the eigenvalues of a symmetric matrix. This characterization can be adapted to definite pencils. This pleasing result seems to be due to Stewart [11], but the Crawford number utilized in his perturbation theory does not seem to be the most natural measure of the stability of the spectrum of a definite pencil. This is discussed in Section 5. Some work has been done on extending the characterization beyond definite pencils. Suppose a symmetric pencil (H, A) with invertible A has a mixture of real and complex eigenvalues. Might it be possible to describe the real eigenvalues, or some of them, as minmax or maxmin values of the Rayleigh quotient? Some ingenious results of this type are presented in [8]; the difficulty is that the domains over which the Rayleigh quotient varies must be limited in some way, because, for an indefinite pencil, the Rayleigh quotient can take on all possible values. Thus even if the eigenvalue with largest real part is actually real, it is not the maximum of the Rayleigh quotient. Even with restrictions to vectors of one type ($+$ or $-$) the actual index of an eigenvalue that can be characterized as a minmax is not easy to ascertain. We conclude that the class of definite Hermitian pencils seems to be the broadest extension of the class of Hermitian matrices that retains the classical properties. See [4,6,11] for good discussions of pencils.

In order to clarify the distinction between symmetric generalized eigenvalue problems and quadratic forms we present two different applications of matrix pencils in the next section. After that, we discuss equivalence, congruence, definite pencils and diagonalization. Householder conventions will be followed closely: capital letters for matrices, lower case for column vectors, Greek for scalars.

2. Contrasting applications

2.1. Viscous damping

The equations of motion governing small displacements $q(t)$ of a system, from an equilibrium position may be written in the form

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = f(t), \tag{3}$$

where M, C and K are the $n \times n$ mass, damping and stiffness matrices and $\ddot{q}$ is the acceleration vector. For solid structures attached to the Earth M will be symmetric positive definite. There are many possible forms for C but the only one of interest here is positive semi-definite C, often called viscous damping. In most coordinate representations the stiffness matrix K will be positive definite.

To understand the response q to a variety of external forces f it is useful to know the dominant modes under free vibration. These mode shapes w_i are solutions to the quadratic eigenvalue problem

$$(\lambda_i^2 M + \lambda C + K) w_i = 0, \quad i = 1, \dots, 2n. \tag{4}$$

In general λ_i will be complex but under certain conditions (completely overdamped systems) all λ_i may be real.

An attractive way to rewrite (4) as a linear eigenproblem is

$$\left\{\lambda_i \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} - \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}\right\} \begin{pmatrix} w_i \\ w_i\lambda_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5}$$

This equation is of the form $(\lambda A - H)z = 0$ where A and H are symmetric but indefinite. A single symmetric matrix A is *indefinite* if its quadratic form $x^t A x$ is not of one sign. Symmetry has practical advantages.

The important point about (5) is that it is not the only legitimate linearized version of (4). Another version is

$$\left\{\lambda_i \begin{bmatrix} -C & -M \\ M & 0 \end{bmatrix} - \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix}\right\} \begin{pmatrix} w_i \\ w_i\lambda_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6}$$

Equation (6) makes one matrix positive definite but at the expense of symmetry in the other.

2.2. Conservative systems

In classical mechanics the representations of potential energy $\langle Vq, q\rangle = q^t Vq$ and kinetic energy $\langle T\dot{q}, \dot{q}\rangle = \dot{q}^t T\dot{q}$ are of fundamental importance. In particular the ratio $\langle Vx, x\rangle / \langle Tx, x\rangle$

is an important instance of a (generalized) Rayleigh quotient. In this context linear reversible changes of variable lead naturally to congruence transformations because it is the quadratic forms that are fundamental. However, $\langle Tx, x\rangle$ is positive definite and in such cases congruence is the same as symmetry-preserving equivalence because the diagonal canonical form (under strict equivalence) can be achieved by congruence.

The moral is that in problems such as Section 2.2 congruence is the proper tool while in problems such as Section 2.1 there is no reason not to use symmetry-preserving equivalence transformations.

3. Equivalence

Let us recall some definitions.

Definition 2. If B and C are square matrices with entries that are polynomials over a field, then B is *equivalent* to C if there exist invertible matrices E and F, whose entries may be polynomials, such that

$$C = EBF.$$

Definition 3. If B and C are square matrices with entries that are polynomials over a field, then B is *strictly equivalent* to C if there exist invertible matrices E and F, with entries in the field, such that

$$C = EBF.$$

Definition 4. If a square matrix $A - \lambda B$ has a determinant that does not vanish for some values of λ, then the pencil (A, B) is *regular* (or nondegenerate).

Notice that if B is invertible, then the pencil is certainly regular. The basic result on strict equivalence has been known for many years. See [5, vol. 2].

Theorem 5. *Two regular pencils are strictly equivalent if, and only if, they have the same elementary divisors.*

Our interest is mainly in symmetric pencils whose second members are invertible. In such cases $(\hat{H}, \hat{A})$ is strictly equivalent to (H, A) only if $\hat{A}^{-1}\hat{H}$ is similar to $A^{-1}H$. When $A^{-1}H$ has a full set of eigenvectors, then the elementary divisors are linear and two such pencils are strictly equivalent if, and only if, they have the same set of eigenvalues. However, even when the pencil does not possess a full set of eigenvectors, the canonical form, under strict equivalence, is just the real Jordan normal form of $A^{-1}H$. Symmetry may be restored by allowing the second term to differ from I.

Here is an example of a real Jordan block corresponding to a complex conjugate pair of eigenvalues $\alpha + \mathrm{i}\beta$, $\mathrm{i}^2 = -1$,

$$J = \begin{bmatrix} \alpha & -\beta & 1 & 0 & 0 & 0 \\ \beta & \alpha & 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & -\beta & 1 & 0 \\ 0 & 0 & \beta & \alpha & 0 & 1 \\ 0 & 0 & 0 & 0 & \alpha & -\beta \\ 0 & 0 & 0 & 0 & \beta & \alpha \end{bmatrix}.$$

Let $\tilde{I}$ denote the reversing matrix; $\tilde{I}=(e_n,\dots,e_1)$ if $I=(e_1,\dots,e_n)$. Then $(\tilde{I}J,\ \tilde{I})$ is the regular symmetric canonical form corresponding to J under strict equivalence. In general the symmetric canonical form is a direct sum of blocks of the form $(\tilde{I}J,\ \tilde{I})$.

In [5, vol. 2] attention is given to complex symmetric matrices (not Hermitian) but these are of no concern here. Complex Hermitian pencils have Hermitian canonical forms as above.

To summarize: every symmetric pencil $(H,\ A)$ with invertible A is strictly equivalent to the symmetrized version of the real (Jordan) canonical form of $A^{-1}H$. $(H,\ A)$ is diagonalizable if, and only if, the real Jordan form is diagonal.

4. Congruence

Any real quadratic form ϕ in n real variables $x(1),\ x(2),\dots,x(n)$ may be written compactly as

$$\phi(x) := x^{\mathrm{t}}Ax = \langle Ax,\ x\rangle,$$

where A is a real symmetric $n\times n$ matrix. Any reversible linear change of variables $x\to y=F^{-1}x$ induces a corresponding change in the matrix representation

$$\phi(x)=(Fy)^{\mathrm{t}}A(Fy)=y^{\mathrm{t}}(F^{\mathrm{t}}AF)y.$$

The transformation

$$A \ \to\ F^{\mathrm{t}}AF$$

with real invertible F is called a *congruence transformation*. A matrix pencil $(H,\ A)$ may be regarded as a single matrix whose entries $h_{ij}-\lambda a_{ij}$ are linear polynomials in the parameter λ. Consequently there is some possibility for confusion in discussing congruence transformations of matrix pencils. Are the entries in F allowed to be functions of λ? In order to be clear the term strict congruence is used to emphasize that the entries of F must belong to the basic scalar field, usually $\mathbb{R}$ or $\mathbb{C}$. However we shall be concerned exclusively with strict congruence and strict equivalence and so will sometimes suppress the adjective strict.

If a real symmetric pencil $(H,\ A)$ is (strictly) congruent to a diagonal pencil $(\Lambda_1,\ \Lambda_2)$ via F, then the columns of F are eigenvectors of (1) and then, a fortiori, $(H,\ A)$ is also (strictly) equivalent to $(\Lambda_1,\ \Lambda_2)$.

The canonical form under congruence is quite complicated to state and we do not need all the details, see [6]. It is based on the Jordan form of $A^{-1}H$ and in all respects except one is the symmetrized version of the canonical form under equivalence. For complex eigenvalues the structure was given in the section on equivalence. The part associated with real eigenvalues λ consists of direct sums of pairs analogous to

$$\delta\begin{bmatrix}0&1&\lambda\\1&\lambda&0\\\lambda&0&0\end{bmatrix},\qquad \begin{bmatrix}0&0&\delta\\0&\delta&0\\\delta&0&0\end{bmatrix},$$

where $\delta=\pm1$ is the sign associated with λ. There is one sign per Jordan block but, in general, the blocks belonging to an eigenvalue need not all have the same sign. In fact the sign characteristic property belongs to the eigenvector x and is only attributed to λ:

$$\delta=\operatorname{sgn}[x^{\mathrm{t}}Ax].$$

No real congruence transformation can change δ, no Hermitian congruence can change δ, but a complex congruence by $\sqrt{-1}$ can, of course, remove it. Of more significance is the fact that a one-sided equivalence transformation by an involutary diagonal matrix can remove δ *without spoiling the symmetry*. If $EH = HE^*$, $EA = AE^*$, then (EH, EA) is still symmetric.

To summarize: every symmetric pencil (H, A) with A invertible is strictly congruent to the symmetrized version of the real (Jordan) canonical form of $A^{-1}H$ with the extra sign structure for real eigenvalues described above. The sign structure may be removed by an equivalence transformation that preserves symmetry, the eigenvalues and the eigenvectors.

Example 6. The symmetric regular pencils

$$[\operatorname{diag}(3, -2, 1), \operatorname{diag}(1, -1, 1)] \quad \text{and} \quad [\operatorname{diag}(3, 2, 1), \operatorname{diag}(1, 1, 1)]$$

are strictly equivalent but not strictly congruent, although they have the same eigenvalues and eigenvectors.

5. Definite pencils

Let us review those special cases when a symmetric pencil (H, A) has a full set of independent real eigenvectors or, in other words, when the quadratic forms for H and A may be diagonalized simultaneously.

Case 1. $A = I$ (the standard form).

Case 2. A is symmetric positive definite (spd) (the classical pencils).

Case 3. (H, A) is definite and $n > 2$. Definitions are given below.

Cases 1 and 2 are the classical cases and we assume that the reader is familiar with them. A central role is played by the Rayleigh quotient ρ defined in Section 1. Being homogeneous of degree 0, ρ acts on the 1-dimensional subspaces of $\mathbb{R}^n$ and one may verify that the only critical points of ρ, where the gradient vanishes, are the eigenspaces of (H, A).

In order to use ρ in the study of pencils with singular A it is necessary to allow ρ to have poles. This provokes no difficulties but the danger, when both H and A are indefinite, is that ρ will not be well defined if both numerator and denominator vanish simultaneously. This suggests the following terminology.

Definition 7. A real symmetric pencil (H, A) is *definite* if its Rayleigh quotient $\rho(x) := x^*Hx/x^*Ax$ is well defined in $\mathbb{R} \cup \infty$, for all nonzero $x \in \mathbb{R}^n$.

We hope that a few readers will share our preference for a definition that is conceptual rather than merely technical. The usual definitions follow.

Definition 8. A real symmetric pencil (H, A) is *definite* if

$$(x^*Hx)^2 + (x^*Ax)^2 > 0,$$

for all nonzero $x \in \mathbb{R}^n$.

Definition 9. A real symmetric pencil (H, A) is *definite* if there exists $\eta \in \mathbb{R}$, $\alpha \in \mathbb{R}$ such that $\eta H + \alpha A$ is spd.

Clearly Definitions 7 and 8 are equivalent since Definition 8 states that numerator^2 + denominator$^2 > 0$ for all the Rayleigh quotients. The connection between Definitions 8 and 9 is more subtle.

Example 10.

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It is a good exercise to verify that for this example $\rho(x)$ alternates between $-\infty$ and $+\infty$ (which are identified in $\mathbb{R} \cup \infty$) with no stationary points. This pencil is definite according to Definition 8 since $(x^{t}Hx)^2 + (x^{t}Ax)^2 = (x(1)^2 + x(2)^2)^2$ but $\operatorname{trace}(\eta H + \alpha A) = 0$ for all η and α and so the pencil cannot be definite according to Definition 9. Moreover, having no real eigenvalues, the pencil cannot be diagonalized.

Nevertheless this pencil is the exception and not the rule as the theorem below reveals. It is only in $\mathbb{R}^2$ that the mismatch between Definitions 8 and 9 can occur. One way to avoid continual reference to the case $n = 2$ is to change Definitions 7 and 8 to 7* and 8* by allowing x to range over $\mathbb{C}^n$ rather than $\mathbb{R}^n$ even when H and A are real.

The choice $x = (1, \sqrt{-1})^*$ shows that the pencil in Example 10 is not definite when regarded as an Hermitian pencil.

Returning to the real case and the connection between Definitions 8 and 9 we cite the next theorem.

Theorem 11. *If the $n \times n$ real symmetric pencil (H, A) is definite by Definition* 8 *and $n > 2$, then the pencil may be diagonalized by a real congruence transformation.*

A difficult proof, due to John Milnor, appears in an early edition of Greub's Linear Algebra in the 1950s. Easier proofs are given in [2]. A corollary of Theorem 11 is that, for $n \neq 2$, Definitions 8 and 9 are equivalent. However a definite Hermitian pencil may be diagonalized by a Hermitian congruence whatever the dimension.

6. Congruence versus equivalence

Now it is time to return to the problem mentioned in the introduction. There it is stated that a symmetric pencil with invertible A is definite if, and only if, it is diagonalizable *and* the eigenvalues of positive type are separated from the eigenvalues of negative type, as indicated in Fig. 1.

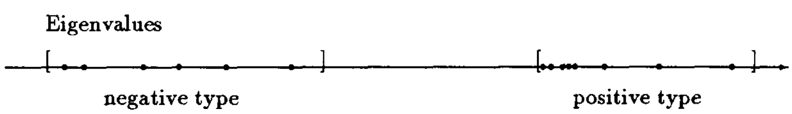

Fig. 1.

The critical observation is that for some regular Hermitian pencils there are equivalence transformations that preserve symmetry but are not congruences. It is legitimate to use such transformations on eigenvalue problems.

Furthermore a symmetry-preserving equivalence can change the inertia of each matrix and hence change the sign types of the eigenvalues. In particular all finite eigenvalues can be given positive type and condition (c) of Theorem 1 is automatically fulfilled.

If the pencil arises in an eigenvalue problem, then the sign characteristic is an artefact of applying too small a transformation class. On the other hand, if the quadratic forms associated with H and A are of primary interest, then only congruence transformations are legitimate.

So far the pencils we have met that are not definite have all arisen in eigenvalue problems.

Notice that the first pencil in Example 6 is diagonal but not definite since, for $z^{t} = (1, \sqrt{2}, 1)$,

$$z^{t}Hz = 0 = z^{t}Az.$$

The situation may be summarized formally.

Definition 12. Two regular Hermitian pencils are *indistinguishable for the eigenproblem* if they have the same elementary divisors and the same associated reducing subspaces.

Reducing subspaces for pencils are the analogues of invariant subspaces for linear operators; $H\mathscr{S} \subset \mathscr{A}\mathscr{S}$.

Example 13. If E is invertible and $EH = HE^*$ and $EA = AE^*$, then (EH, EA) is indistinguishable from a regular Hermitian pencil (H, A).

Theorem 14. *If (H, A) is regular, Hermitian and diagonalizable by strict congruences, then it is indistinguishable from a definite pencil.*

Proof. Let (H, A) be diagonalized by Z:

$$Z^*HZ = \Lambda, \qquad Z^*AZ = \Delta.$$

Λ and Δ are diagonal and Hermitian and therefore real. If (H, A) is regular, then there are real σ and τ such that

$$\det(\sigma H - \tau A) = |\det(Z)|^2 \prod_{i=1}^{n} (\sigma\lambda_i - \tau\delta_i) \neq 0.$$

Let

$$\Psi = \operatorname{diag}(\psi_1, \ldots, \psi_i), \quad \text{where } \psi_i = \operatorname{sign}(\sigma\lambda_i - \tau\delta_i),$$

so that $\sigma\Psi\Lambda - \tau\Psi\Delta$ is positive definite. Now choose invertible $E := Z^{-*}\Psi Z^*$ and verify that (i) (EH, EA) is Hermitian, (ii) $\sigma(EH) - \tau(EA)$ is congruent to the spd matrix $\sigma\Psi\Lambda - \tau\Psi\Delta$ and thus spd, and (iii) $Hz\delta = Az\lambda$ if, and only if, $EHz\delta = EAz\lambda$. By Definition 9 of definiteness, (EA, EH) is definite and, by (iii), it is indistinguishable from (H, A). $\square$

Thus, for eigenvalue problems, by choosing an appropriate representative for each class of indistinguishable regular Hermitian pencils the definite pencils do mark the limit of diagonalizability. In the next section we see to what extent the definite pencils mark the limit of the ability of the Rayleigh quotient to give a full account of spectrum.

7. Rotating a pencil

An easy modification of a single symmetric matrix is translation (or shifting): $A \to A - \sigma I$. The eigenvectors are preserved and the eigenvalues are just translated by σ. It is also possible to shift a matrix pair: $H \to H - \sigma A$, with the same effect. However, there is no intrinsic reason to preserve the second matrix A. So, given a pencil (H, A) there is no difficulty, in principle, in working with any linear fractional transformation (LFT) of it: $H \to H\alpha + A\beta$, $A \to H\gamma + A\delta$ with $\alpha\delta \neq \beta\gamma$. The eigenvectors are preserved and the eigenvalues are transformed by the same LFT.

It is appropriate for pencils to define an eigenvalue as an ordered pair (η, α) or, more precisely, as the ray $\{\tau(\eta, \alpha);\ \tau \in \mathbb{R}\}$ where

$$H\alpha z = A\eta z, \quad z \in \mathbb{C}^n, \quad \alpha \geqslant 0.$$

This acknowledges the projective nature of the generalized eigenvalue problem. However, when the eigenvalues are real, a ray in $\mathbb{R}^2$ through (0, 0) may be characterized by its slope. Instead of designating $(\lambda, 1)$ as the representative of the ray, it is more convenient to use intersection with the unit circle $(\sin\phi, \cos\phi)$.

This representation has been used to good effect in [3,11,12].

It is necessary to identify both $\phi = \frac{1}{2}\pi$, $\phi = -\frac{1}{2}\pi$ with infinite eigenvalues and, more generally, to state the next convention.

Convention.

$$\tfrac{1}{2}\pi + \omega \text{ is identified with } -\tfrac{1}{2}\pi + \omega, \quad 0 \leqslant \omega < \tfrac{1}{2}\pi. \tag{7}$$

The normalization $\alpha \geqslant 0$ in the Cartesian representation (η, α) corresponds to $-\frac{1}{2}\pi < \phi \leqslant \frac{1}{2}\pi$ in the polar representation. If the eigenvalues of a definite pencil are $\{(\sin\theta_i, \cos\theta_i),\ i = 1, \ldots, n\}$, then we refer to the $\{\theta_i\}$ as the polar or angular eigenvalues.

A simple instance of an LFT is the counter-clockwise rotation:

$$(H, A) \overset{\phi}{\to} (H_\phi, A_\phi) = (H, A)\begin{bmatrix} I\cos\phi & -I\sin\phi \\ I\sin\phi & I\cos\phi \end{bmatrix}.$$

The eigenvectors are preserved and the polar eigenvalues are translated by ϕ,

$$\theta_i \to \begin{cases} \theta_i + \phi, & \text{if } \theta_i + \phi \leqslant \frac{1}{2}\pi, \\ \theta_i + \phi - \pi, & \text{otherwise.} \end{cases}$$

This simple transformation suggests that rotations of definite pencils are the natural analogues of translations of symmetric matrices.

Given any definite Hermitian pencil we may consider the class of all its rotations. In each such class there is a distinguished pair (H_ω, A_ω) for which A_ϕ is spd and

$$\operatorname{cond}(A_\omega) := \frac{\lambda_{\max}[A_\omega]}{\lambda_{\min}[A_\omega]} = \min \operatorname{cond}[A_\phi],$$

where the minimum is over all ϕ in $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ for which A_ϕ is spd. We call $\operatorname{cond}(A_\omega)$ the *condition number* of the whole rotation class for the eigenvalue problem. Note that the eigenvectors are invariant and hence orthogonal with respect to each inner product A_ϕ for which A_ϕ is spd; $z_i^* A_\phi z_j = 0$, $i \neq j$. The unit ball for A_ω is the most nearly spherical of all the candidates and gives the best indication of the (Euclidean) angles between the eigenvectors. Indeed if a given pair (H, A) happens to be a rotation of a standard pair (M, I), then this fact will be revealed by (H_ω, A_ω), i.e., $H_\omega = M$, $A_\omega = I$.

The quantity $\mathrm{cond}(A_\omega)$ also arises in perturbation theory for eigenvalues of definite pairs (H, A). Suppose that A is spd and that $H = H(\tau)$, $A = A(\tau)$ are smooth functions of τ in some interval of τ values. Differentiate the eigenvalue equation $Hz \cos\theta = Az \sin\theta$ and let $\dot{z} = \mathrm{d}z/\mathrm{d}\tau$, etc., to find

$$(\dot{H}z + H\dot{z})\cos\theta - Hz\sin\theta\cdot\dot{\theta} = (\dot{A}z + A\dot{z})\sin\theta + Az\cos\theta\cdot\dot{\theta}.$$

Premultiply by z^* and cancel common terms:

$$z^*Az(\sin\theta\tan\theta + \cos\theta)\dot{\theta} = z^*(\dot{H}\cos\theta - \dot{A}\sin\theta)z,$$

so

$$\dot{\theta} = \cos\theta\cdot\frac{z^*(\dot{H}\cos\theta - \dot{A}\sin\theta)z}{z^*Az}.$$

This result may be applied to any rotation with A_ϕ spd:

$$\dot{\theta} = (\theta+\phi)^{\cdot} = \cos(\theta+\phi)\cdot\frac{z^*\left[\dot{H}_\phi\cos(\theta+\phi) - \dot{A}_\phi\sin(\theta+\phi)\right]z}{z^*A_\phi z}$$

$$= \cos(\theta+\phi)\cdot\frac{z^*(\dot{H}\cos\theta - \dot{A}\sin\theta)z}{z^*A_\phi z},$$

since the numerator is invariant under rotation. However neither it nor the denominator is properly scaled for separating the perturbation term. It is tempting to use $(\|H\|^2 + \|A\|^2)^{1/2}$ to get

$$\dot{\theta} = \cos(\theta+\phi)\cdot\frac{(\|H\|^2+\|A\|^2)^{1/2}}{z^*A_\phi z}\cdot\frac{z^*(\dot{H}\cos\theta - \dot{A}\sin\theta)z}{(\|H\|^2+\|A\|^2)^{1/2}},$$

$$|\dot{\theta}| \leqslant \frac{(\|H\|^2+\|A\|^2)^{1/2}}{\lambda_{\min}\left[A_\phi\right]}\cdot\left(\frac{\|\dot{H}\|^2+\|\dot{A}\|^2}{\|H\|^2+\|A\|^2}\right)^{1/2}.$$

The first term could be called the condition number for all the eigenvalues and leads to the problem of maximizing $\lambda_{\min}[A_\phi]$. Stewart [11] shows that

$$\max_\phi \lambda_{\min}\left[A_\phi\right] = c(H, A) := \min_{\|x\|=1}\left[(x^*Hx)^2 + (x^*Ax)^2\right]^{1/2},$$

where $c(H, A)$ is called the Crawford number of the pencil. However $(\|H\|^2 + \|A\|^2)^{1/2} \geqslant \|A_\phi\|$ and if we look again at the expression for $\dot{\theta}$ it seems preferable to rewrite it as

$$\dot{\theta} = \cos(\theta+\phi)\cdot\frac{\|A_\phi\|}{z^*A_\phi z}\cdot\frac{z^*(\dot{H}\cos\theta - \dot{A}\sin\theta)z}{\|A_\phi\|},$$

to obtain

$$|\dot{\theta}| \leqslant \operatorname{cond}(A_\omega) \cdot \frac{\left(\|\dot{H}\|^2 + \|\dot{A}\|^2\right)^{1/2}}{\|A_\omega\|}.$$

The case $A_\omega = I$ reproduces the well-known results for the standard symmetric eigenvalue problem, $|(\tan\theta)^{\cdot}| \leqslant \|\dot{H}\|$, and encourages us to suggest that $\operatorname{cond}(A_\omega)$ is a good choice for condition number. Neither $c(H, A)$ nor $\operatorname{cond}(A_\omega)$ are easy to compute.

If an eigenvalue of a pencil is to be defined as a set of pairs, then the definition of the Rayleigh quotient must be modified in the same way:

$$\rho(x;\ H,\ A) := \{\tau(x^*Hx,\ x^*Ax),\ \tau \neq 0\}.$$

Hence an angle in $(-\frac{1}{2}\pi, \frac{1}{2}\pi]$ may be associated with ρ via arctan (x^*Hx/x^*Ax). In fact it is possible to economize on notation and let ρ stand for either an angle or a ray in $\mathbb{R}^2$ depending on the context.

The minmax characterization of eigenvalues applies to classical pencils (A is spd) with no change. If the angular eigenvalues of (H, A) are ordered by

$$-\tfrac{1}{2}\pi < \theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_n < \tfrac{1}{2}\pi,$$

then

$$\theta_i = \min_{\mathscr{S}^i} \max_{x \in \mathscr{S}^i} \rho(x;\ H,\ A) = \max_{\mathscr{R}^{i-1}} \min_{y \perp \mathscr{R}^{i-1}} \rho(y;\ H,\ A).$$

However, there is a difficulty in applying this to definite pencils because of an indexing problem. The minmax values of ρ are eigenvalues but it is not obvious where, on the unit circle, to start counting. One way to describe the ordering is to invoke the optimal rotation angle ω. Relabel the θ_i so that $\theta_1 + \omega$, when represented in $(-\frac{1}{2}\pi, \frac{1}{2}\pi]$, is minimal among all the $\theta_i + \omega$. Then continue labelling clockwise.

Since each definite pencil is the rotation of a classical pencil, several properties of the latter may be reinterpreted for definite pencils.

8. Indefinite symmetric pencils

Such pencils cannot be diagonalized by equivalences that preserve symmetry. No stable algorithm is available for computing the eigenpairs but, in principle, an extension of the Lanczos algorithm can maintain backward stability by working with groups of vectors instead of a single vector.

Here is a summary of the algorithm presented in [10] for a real symmetric H and A; the Lanczos algorithm with indefinite inner product. Input consists of a starting vector q_1, the matrix A and a real linear operator Op that depends on H, A and a shift σ. It is the eigenvalues closest to σ that are wanted. In the simplest instance $\mathrm{Op} = H$ and $\sigma = \infty$, next come cases when σ is real and $\mathrm{Op} = (H - \sigma A)^{-1}A$, finally when σ is complex, the reader is referred to [10] for definition of Op. In all cases Op is formally self-adjoint with respect to the indefinite inner product associated with A:

$$\langle \mathrm{Op} \cdot x,\ y\rangle_A = \langle x,\ \mathrm{Op} \cdot y\rangle_A,$$

for all $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ and $\langle u, v\rangle_A = v^t A u$.

The output, in the simplest form, consists of an integer j, a matrix $Q = [q_1, \ldots, q_j] \in \mathbb{R}^{n \times j}$, and two symmetric $j \times j$ matrices

$$\Omega = \operatorname{diag}(\omega_1, \ldots, \omega_j), \quad \omega_i \neq 0, \ i = 1, \ldots, j, \tag{8}$$

$$T = \text{tridiagonal}, \tag{9}$$

$$\beta_{j+1} > 0. \tag{10}$$

The columns of Q have Euclidean length 1 and in exact arithmetic $\Omega = Q^*AQ$.

The reduced eigenvalue problem

$$Ts_i = \Omega s_i \theta_i, \quad i = 1, \ldots, j,$$

is used to generate approximate eigenpairs for the pencil (Op, A), namely (θ_i, Qs_i), $i = 1, \ldots, j$. In general some of the approximations are good, others are bad. Fortunately there are computable error estimates for each pair. Part of the estimate may be formed without computing Qs_i, namely $\beta_{j+1} | s_i(j) | / s_i^t \Omega s_i$. If this is not small enough, then Qs_i is not formed.

There is an easily monitored upper bound on the condition number of Q:

$$\operatorname{cond}(Q) := \frac{\sigma_{\max}(Q)}{\sigma_{\min}(Q)} \leqslant \frac{j \| A \|}{\min_i | \omega_i |}.$$

The algorithm is not guaranteed to terminate with $j = n$. The bound on cond(Q) shows that the vulnerability of the algorithm lies in the occurrence of a small value of ω_{j+1} before the wanted eigenvalues have been approximated well enough. In principle, this weakness may be overcome by working with several columns of Q simultaneously. Instead of a 1×1 matrix ω_i with tiny $|\omega_i|$, the algorithm uses a small Hankel matrix Ω_i of the form

$$\Omega_i = \begin{bmatrix} \omega_i & \omega_i^{(2)} \\ \omega_i^{(2)} & \omega_i^{(3)} \end{bmatrix} \quad \text{or} \quad \Omega_i = \begin{bmatrix} \omega_i & \omega_i^{(2)} & \omega_i^{(3)} \\ \omega_i^{(2)} & \omega_i^{(3)} & \omega_i^{(4)} \\ \omega_i^{(3)} & \omega_i^{(4)} & \omega_i^{(5)} \end{bmatrix},$$

such that $\sigma_{\min}(\Omega_i)$ is acceptable.

Another aspect of the algorithm that needs further work is the efficient solution of the eigenproblem for (T, Ω).

Details and numerical examples are given in [10].

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