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The method of simplified Tikhonov regularization for dealing with the inverse time-dependent heat source problem ☆

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ABSTRACT

This paper investigates the inverse problem of determining a heat source using a parabolic equation where data are given at some fixed location. The problem is ill-posed, i.e., the solution (if it exists) does not depend continuously on the data. A simplified Tikhonov regularization method is given and an order optimal stability estimate is obtained. A numerical example shows that the regularization method is effective and stable.

1. Introduction

Consider the following inverse problem: to find a pair of functions $(u(x,t), f(t))$ which satisfy the heat equation in a quarter infinite domain as follows:

$$\begin{cases} u_t - u_{xx} = f(t), & x > 0, t > 0, \\ u(x,0) = 0, & x \geq 0, \\ u(0,t) = 0, & t \geq 0, \\ u(x,t)|_{x\to\infty} \text{bounded}, & t \geq 0, \\ u(1,t) = g(t), & t \geq 0, \end{cases} \tag{1.1}$$

where $f(t)$ denotes the source (sink) term. Our purpose is to identify $f(t)$ from the additional data $u(1,t) = g(t)$. Since the data $g(t)$ are based on (physical) observation, there must be measurement errors, and we assume the measured data function $g_\delta(t) \in L^2(\mathbb{R})$, satisfying

$$\|g - g_\delta\| \leq \delta, \tag{1.2}$$

where $\|\cdot\|$ denotes the L^2-norm, and the constant $\delta > 0$ represents a noise level.

This problem can be seen as a problem of source identification from measured data for a parabolic equation which is important in many branches of engineering sciences. A typical example is groundwater pollutant source estimation in cities with large populations. In this application, it is crucial to accurately identify which companies are responsible for the contamination [1]. For a heat source of the form $f = f(u)$, the inverse source problem was studied by [2]. In [3], the authors considered the heat source as a function of both space and time, but additive or separable. But many researchers viewed the source as a function of space or time only. In [4,5], the authors determined the heat source depending on one variable in

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a bounded domain by using a boundary-element method and an iterative algorithm. In [6], the authors identified the heat source as space dependent only, by a fundamental solution method. In [7], the authors identified the heat source as time dependent only, by a fundamental solution method. In [8], the author identified the heat source as time dependent only, by the Lie-group shooting method (LGSM).

As we know, there has been lots of research on identification of heat source adopting numerical algorithms [9–16]. But to the author's knowledge there are few papers, using the regularization method, with strict theoretical analysis, on identifying the heat source. Recently, in [17,18], the authors identified the heat source depending only on a spatial variable by the wavelet dual least squares method and the Fourier regularization method, respectively.

The problem of identifying the heat source is ill-posed in Hadamard's sense. That is, any small change in the input data can result in a dramatic change to the solution. The ill-posedness can be seen by solving the problem in the frequency domain. In order to analyze the problem (1.1) in $L^2(\mathbb{R})$, we define all functions to be zero for $t < 0$. The notation $\|\cdot\|$ denotes the L^2-norm, and

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi t} f(t) dt \tag{1.3}$$

is the Fourier transform of the function $f(t)$.
The problem (1.1) can now be formulated in frequency space as follows:

$$\begin{cases} i\xi\hat{u}(x,\xi) - \hat{u}_{xx}(x,\xi) = \hat{f}(\xi), & x > 0, \xi \in \mathbb{R}, \\ \hat{u}(0,\xi) = 0, & \xi \in \mathbb{R}, \\ \hat{u}(x,\xi)|_{x\to\infty} \text{bounded}, & \xi \in \mathbb{R}, \\ \hat{u}(1,\xi) = \hat{g}(\xi), & \xi \in \mathbb{R}. \end{cases} \tag{1.4}$$

By elementary calculations, we get

$$\hat{u}(x,\xi) = \frac{1 - e^{-\sqrt{i\xi}x}}{i\xi} \hat{f}(\xi). \tag{1.5}$$

Substituting $\hat{u}(1,\xi) = \hat{g}(\xi)$ into (1.5) leads to

$$\hat{f}(\xi) = \frac{i\xi}{1 - e^{-\sqrt{i\xi}}} \hat{g}(\xi). \tag{1.6}$$

So

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi t} \frac{i\xi}{1 - e^{-\sqrt{i\xi}}} \hat{g}(\xi) d\xi. \tag{1.7}$$

From the right hand side of (1.6) or (1.7), we know that

$$\left| \frac{i\xi}{1 - e^{-\sqrt{i\xi}}} \right| = \frac{|\xi|}{\sqrt{1 - 2e^{-\sqrt{\frac{|\xi|}{2}}} \cos\sqrt{\frac{|\xi|}{2}} + e^{-2\sqrt{\frac{|\xi|}{2}}}}} \to \infty, \quad \text{as } \xi \to \infty. \tag{1.8}$$

Therefore when we consider our problem in $L^2(\mathbb{R})$, the exact data function $\hat{g}(\xi)$ must decay. However, the measured data function $g_\delta(t)$, which is merely in $L^2(\mathbb{R})$, does not possess such a decay property in general. Thus if we try to obtain the heat source $f(t)$, the high frequency components in the error are magnified and can destroy the solution. So the problem (1.1) is mildly ill-posed and the degree of the ill-posedness is equivalent to that of the first-order numerical differentiation. It is impossible to solve the problem (1.1) by using classical methods. In the following section, we will use a simplified Tikhonov regularization method to deal with the ill-posed problem. Before doing that, we impose an a priori bound on the input data, i.e.,

$$\|f(\cdot)\|_{H^p} \le E, \quad p > 0, \tag{1.9}$$

where $E > 0$ is a constant, and $\|\cdot\|_{H^p}$ denotes the norm in the Sobolev space $H^p(\mathbb{R})$ defined by

$$\|f(\cdot)\|_{H^p} := \left(\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 (1+\xi^2)^p d\xi \right)^{\frac{1}{2}}. \tag{1.10}$$

The major object of this paper is to provide a simplified Tikhonov regularization method. Meanwhile, the Hölder type estimate of the stability between the regularization solution and the exact solution is obtained. In particular, the error estimate which we obtained is order optimal, according to the general theory of regularization [19].

The simplified Tikhonov regularization method is based on the Tikhonov regularization method. Skillfully simplifying the filter obtained by the Tikhonov regularization, a better regularization approximation solution of the inverse problem was obtained. This idea initially came from Carasso, the author who modified the filter obtained by the Tikhonov regularization method and obtained the order optimal error estimate in [20]. By this method, Fu [21] considered the inverse heat conduction problem with a general sideways parabolic equation, and Cheng et al. [22,23] considered the spherically symmetric inverse problem.

2. Some auxiliary results

In this section, we will give three important lemmas.

Lemma 2.1. *For $\xi \in \mathbb{R}$, the inequality*

$$\left|\frac{i\xi}{1-e^{-\sqrt{i\xi}}}\right| \le \frac{|\xi|}{1-e^{-\sqrt{\frac{|\xi|}{2}}}} \tag{2.1}$$

holds.

Lemma 2.2. *If $x > 1$, the following inequality:*

$$\frac{1}{1-e^{-\sqrt{\frac{x}{2}}}} < 2 \tag{2.2}$$

holds.

Lemma 2.3. *For $0 < \alpha < 1$, the following inequalities:*

$$\sup_{\xi\in\mathbb{R}}\left|\left(1-\frac{1}{1+\alpha^2\xi^2}\right)(1+\xi^2)^{-\frac{p}{2}}\right| \le \max\{\alpha^p, \alpha^2\} \tag{2.3}$$

$$\sup_{\xi\in\mathbb{R}}\left|\frac{|\xi|}{(1-e^{-\sqrt{\frac{|\xi|}{2}}})(1+\alpha^2\xi^2)}\right| \le \frac{2}{\alpha} \tag{2.4}$$

hold.

Proof. Let

$$A(\xi) := \left(1-\frac{1}{1+\alpha^2\xi^2}\right)(1+\xi^2)^{-\frac{p}{2}}. \tag{2.5}$$

The proof of (2.3) will be separated into three cases:
Case 1. $|\xi| \ge \xi_0 := \frac{1}{\alpha}$; we get

$$A(\xi) \le (1+\xi^2)^{-\frac{p}{2}} \le |\xi|^{-p} \le \xi_0^{-p} = \alpha^p. \tag{2.6}$$

Case 2. $1 < |\xi| < \xi_0$; we obtain

$$A(\xi) = \frac{\alpha^2\xi^2}{1+\alpha^2\xi^2}(1+\xi^2)^{-\frac{p}{2}} \le \alpha^2\xi^2(1+\xi^2)^{-\frac{p}{2}} \le \alpha^2|\xi|^{2-p}. \tag{2.7}$$

If $0 < p \le 2$, the above inequality becomes

$$A(\xi) \le \alpha^2\xi_0^{2-p} = \alpha^p. \tag{2.8}$$

If $p > 2$, we get

$$A(\xi) \le \alpha^2|\xi|^{2-p} \le \alpha^2. \tag{2.9}$$

Case 3. $|\xi| \le 1$; we get

$$A(\xi) \le \alpha^2\xi^2(1+\xi^2)^{-\frac{p}{2}} \le \alpha^2. \tag{2.10}$$

Combining (2.6) with (2.8)–(2.10), the first inequality holds.
Let

$$B(\xi) := \frac{|\xi|}{(1-e^{-\sqrt{\frac{|\xi|}{2}}})(1+\alpha^2\xi^2)}, \qquad D(\xi) := \frac{|\xi|}{1-e^{-\sqrt{\frac{|\xi|}{2}}}}. \tag{2.11}$$

Like in the above proof, we divide the second inequality's proof into two cases:

Case 1. $0 \le |\xi| \le \xi_0 := \frac{1}{\alpha}$; we have

$$D(\xi) \le D\left(\frac{1}{\alpha}\right) \le \frac{2}{\alpha}, \quad \text{if } 0 < \alpha < 1. \tag{2.12}$$

So

$$B(\xi) \le \frac{2}{\alpha}. \tag{2.13}$$

Case 2. $|\xi| > \xi_0$; we get

$$D(\xi) \le 2|\xi| \tag{2.14}$$

and

$$B(\xi) \le \frac{2|\xi|}{1+\alpha^2\xi^2}. \tag{2.15}$$

Let

$$L(\xi) := \frac{2\xi}{1+\alpha^2\xi^2}. \tag{2.16}$$

Then

$$L'(\xi) = \frac{2(1-\alpha^2\xi^2)}{(1+\alpha^2\xi^2)^2}. \tag{2.17}$$

Setting $L'(\xi) = 0$, we have $\xi_1 = \frac{1}{\alpha}$. It is easy to see that $\xi_1 = \frac{1}{\alpha}$ is a maximal value point of $L(\xi)$.
So

$$|L(\xi)| \le \left|\frac{2\xi_1}{1+\alpha^2\xi_1^2}\right| \le 2\xi_1 = \frac{2}{\alpha}. \tag{2.18}$$

Combining (2.13) with (2.18), we have that (2.4) holds. □

3. A simplified Tikhonov regularization method

Let us formulate the problem of identifying $f(t)$ from the (exact) data $u(1,t) = g(t)$ as an operator equation:

$$Af = g, \tag{3.1}$$

where $A \in L(L^2(\mathbb{R}), L^2(\mathbb{R}))$.
From (1.6), we know that

$$\frac{1-\mathrm{e}^{-\sqrt{\mathrm{i}\xi}}}{\mathrm{i}\xi}\hat{f}(\xi) = \hat{g}(\xi). \tag{3.2}$$

Obviously, (3.1) is equivalent to the following operator equation:

$$\widehat{Af} = \hat{g}(\xi), \qquad \hat{A} = FSF^{-1}, \tag{3.3}$$

where $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the Fourier operator that maps any $L^2(\mathbb{R})$ function $f(t)$ into its Fourier transform $\hat{f}(\xi)$. From (3.2), we obtain

$$\widehat{Af} = \frac{1-\mathrm{e}^{-\sqrt{\mathrm{i}\xi}}}{\mathrm{i}\xi}\hat{f}(\xi). \tag{3.4}$$

We shall use a Fourier transform to obtain a representation of the approximation solution of equation (3.1). From (3.2), we know that function $g(t)$ lies in the domain of operator A^{-1} provided

$$\|f\|^2 = \int_{-\infty}^{\infty}\left|\frac{\mathrm{i}\xi}{1-\mathrm{e}^{-\sqrt{\mathrm{i}\xi}}}\right|^2 |\hat{g}(\xi)|^2 \mathrm{d}\xi < \infty. \tag{3.5}$$

Note that $|\frac{\mathrm{i}\xi}{1-\mathrm{e}^{-\sqrt{\mathrm{i}\xi}}}| = O(\xi)$, as $|\xi| \to \infty$; therefore (3.5) means that $\hat{g}(\xi)$ has a decay at high frequencies. Such a decay is not likely to occur in the Fourier transform of the measured noisy temperature history $g_\delta(t) \in L^2(\mathbb{R})$. Hence we give an approximate solution of $f(t)$ by means of a Tikhonov regularization method which minimizes the quantity

$$\|Af - g_\delta\|^2 + \alpha^2\|f\|^2 \tag{3.6}$$

over all $f \in L^2(\mathbb{R})$. α will be considered as a regularization parameter. The following lemma will give a regularization approximate of $f(t)$.

Lemma 3.1. *There exists a unique solution to the above minimization problem. It is given by*

$$f_\delta(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi t} \frac{\frac{i\xi}{1-e^{-\sqrt{i\xi}}} \hat{g}_\delta(\xi)}{1 + \alpha^2 \left| \frac{i\xi}{1-e^{-\sqrt{i\xi}}} \right|^2} d\xi, \tag{3.7}$$

where α is a regularization parameter.

Proof. Let I denote the identity operator in $L^2(\mathbb{R})$ and A^* be the adjoint of A. Then by Theorem 2.12 in [24], the unique solution of the minimization problem (3.6) is equal to the solution of the following normal equation:

$$A^*Af_\delta(t) + \alpha^2 f_\delta(t) = A^* g_\delta(t). \tag{3.8}$$

That is,

$$f_\delta(t) = [A^*A + \alpha^2 I]^{-1} A^* g_\delta(t). \tag{3.9}$$

In order to obtain the explicit formula (3.7) from (3.9), we need to apply some properties of $L^2(\mathbb{R})$. By the Parseval formula, we have

$$(\widehat{Au}, \hat{v}) = (Au, v) = (u, A^*v) = (\hat{u}, \widehat{A^*v}). \tag{3.10}$$

From (3.4), we obtain

$$\left(\frac{1-e^{-\sqrt{i\xi}}}{i\xi} \hat{u}, \hat{v} \right) = \left(\hat{u}, \overline{\frac{1-e^{-\sqrt{i\xi}}}{i\xi}} \hat{v} \right), \tag{3.11}$$

so

$$\widehat{A^*v} = \overline{\frac{1-e^{-\sqrt{i\xi}}}{i\xi}} \hat{v}, \tag{3.12}$$

i.e.,

$$\hat{A}^* = \overline{\frac{1-e^{-\sqrt{i\xi}}}{i\xi}}. \tag{3.13}$$

Noting (3.4) and (3.13), we obtain

$$(A^*Au)^\wedge = \overline{\frac{1-e^{-\sqrt{i\xi}}}{i\xi}} \widehat{Au} = \left| \frac{1-e^{-\sqrt{i\xi}}}{i\xi} \right|^2 \hat{u}. \tag{3.14}$$

Due to (3.8), we know that

$$(A^*Af_\delta)^\wedge + \alpha^2 \hat{f}_\delta = \widehat{A^*g_\delta}. \tag{3.15}$$

Noting (3.12) and (3.14), we obtain

$$\left(\left| \frac{1-e^{-\sqrt{i\xi}}}{i\xi} \right|^2 + \alpha^2 \right) \hat{f}_\delta = \overline{\frac{1-e^{-\sqrt{i\xi}}}{i\xi}} \hat{g}_\delta. \tag{3.16}$$

Therefore

$$\hat{f}_\delta(\xi) = \frac{\overline{\frac{1-e^{-\sqrt{i\xi}}}{i\xi}} \hat{g}_\delta(\xi)}{\left|\frac{1-e^{-\sqrt{i\xi}}}{i\xi}\right|^2 + \alpha^2} = \frac{\frac{i\xi}{1-e^{-\sqrt{i\xi}}} \hat{g}_\delta(\xi)}{1 + \alpha^2 \left|\frac{i\xi}{1-e^{-\sqrt{i\xi}}}\right|^2}. \tag{3.17}$$

Finally, (3.7) holds by the inverse Fourier transform. □

Comparing formula (3.17) with formula (1.6), we find that the Tikhonov regularization procedure consists in replacing the unknown $\hat{g}(\xi)$ with an appropriately filtered Fourier transform of the noisy data $\hat{g}_\delta(\xi)$. The filter in (3.17) attenuates the high frequencies in $\hat{g}_\delta(\xi)$ in a manner consistent with the goal of minimizing the quantity (3.6). By means of this, we can use a much better filter, $1/(1+\alpha^2\xi^2)$, to replace the filter $1/(1+\alpha^2|\mathrm{i}\xi/(1-\mathrm{e}^{-\sqrt{\mathrm{i}\xi}})|^2)$ and give another approximation, $f_{\delta,\alpha}(t)$, of solution $f(t)$.

We define a regularized approximate solution of problem (1.1) or (3.1) for noisy data $g_\delta(t)$, which is called the simplified Tikhonov regularized solution, as follows:

$$f_{\delta,\alpha}(t) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi t}\frac{\mathrm{i}\xi}{(1-\mathrm{e}^{-\sqrt{\mathrm{i}\xi}})(1+\alpha^2\xi^2)}\hat{g}_\delta(\xi)\mathrm{d}\xi. \tag{3.18}$$

The main conclusion of this section is:

Theorem 3.2. *Let $f(t)$ given by (1.7) be the exact solution of (1.1) and $f_{\delta,\alpha}(t)$ given by (3.18) be the simplified Tikhonov regularized approximation to $f(t)$. Let $g_\delta(t)$ be the measured data at $x = 1$ satisfying (1.2) and assume that the a priori condition (1.9) holds. If we select*

$$\alpha = \left(\frac{\delta}{E}\right)^{\frac{1}{p+1}}, \tag{3.19}$$

then we obtain the following error estimate:

$$\|f(\cdot)-f_{\delta,\alpha}(\cdot)\| \le 2\delta^{\frac{p}{p+1}}E^{\frac{1}{p+1}}\left(1+\frac{1}{2}\max\left\{1,\left(\frac{\delta}{E}\right)^{\frac{2-p}{p+1}}\right\}\right). \tag{3.20}$$

Proof. By the Parseval formula and (1.6), (3.18), (2.1), (2.2), (2.3), (2.4), (3.19), we have

$$\begin{aligned}
\|f(\cdot)-f_{\delta,\alpha}(\cdot)\| &= \|\hat{f}(\cdot)-\hat{f}_{\delta,\alpha}(\cdot)\| \\
&= \left\|\frac{\mathrm{i}\xi}{1-\mathrm{e}^{-\sqrt{\mathrm{i}\xi}}}\hat{g}(\xi) - \frac{\mathrm{i}\xi}{(1-\mathrm{e}^{-\sqrt{\mathrm{i}\xi}})(1+\alpha^2\xi^2)}\hat{g}_\delta(\xi)\right\| \\
&\le \left\|\frac{\mathrm{i}\xi}{1-\mathrm{e}^{-\sqrt{\mathrm{i}\xi}}}\hat{g}(\xi) - \frac{\mathrm{i}\xi}{(1-\mathrm{e}^{-\sqrt{\mathrm{i}\xi}})(1+\alpha^2\xi^2)}\hat{g}(\xi)\right\| \\
&\quad + \left\|\frac{\mathrm{i}\xi}{(1-\mathrm{e}^{-\sqrt{\mathrm{i}\xi}})(1+\alpha^2\xi^2)}\hat{g}(\xi) - \frac{\mathrm{i}\xi}{(1-\mathrm{e}^{-\sqrt{\mathrm{i}\xi}})(1+\alpha^2\xi^2)}\hat{g}_\delta(\xi)\right\| \\
&= \left\|\frac{\mathrm{i}\xi}{1-\mathrm{e}^{-\sqrt{\mathrm{i}\xi}}}\hat{g}(\xi)\left(1-\frac{1}{1+\alpha^2\xi^2}\right)\right\| + \left\|\frac{\mathrm{i}\xi}{(1-\mathrm{e}^{-\sqrt{\mathrm{i}\xi}})(1+\alpha^2\xi^2)}(\hat{g}(\xi)-\hat{g}_\delta(\xi))\right\| \\
&\le \left\|\hat{f}(\xi)(1+\xi^2)^{\frac{p}{2}}(1+\xi^2)^{-\frac{p}{2}}\left(1-\frac{1}{1+\alpha^2\xi^2}\right)\right\| + \sup_{\xi\in\mathbb{R}}\left|\frac{\mathrm{i}\xi}{(1-\mathrm{e}^{-\sqrt{\mathrm{i}\xi}})(1+\alpha^2\xi^2)}\right| \|\hat{g}(\xi)-\hat{g}_\delta(\xi)\| \\
&\le \sup_{\xi\in\mathbb{R}}\left|(1+\xi^2)^{-\frac{p}{2}}\left(1-\frac{1}{1+\alpha^2\xi^2}\right)\right| \left\|\hat{f}(\xi)(1+\xi^2)^{\frac{p}{2}}\right\| + \sup_{\xi\in\mathbb{R}}\left|\frac{|\xi|}{(1-\mathrm{e}^{-\sqrt{\frac{|\xi|}{2}}})(1+\alpha^2\xi^2)}\right|\delta \\
&\le \max\{\alpha^p,\alpha^2\}E + \frac{2}{\alpha}\delta \\
&= \max\left\{\left(\frac{\delta}{E}\right)^{\frac{p}{p+1}},\left(\frac{\delta}{E}\right)^{\frac{2}{p+1}}\right\}E + 2\left(\frac{\delta}{E}\right)^{\frac{-1}{p+1}}\delta \\
&= 2\delta^{\frac{p}{p+1}}E^{\frac{1}{p+1}}\left(1+\frac{1}{2}\max\left\{1,\left(\frac{\delta}{E}\right)^{\frac{2-p}{p+1}}\right\}\right). \quad \square
\end{aligned}$$

Remark 3.3. If $0 < p \le 2$, $\|f(\cdot)-f_{\delta,\alpha}(\cdot)\| \le 3\delta^{\frac{p}{p+1}}E^{\frac{1}{p+1}} \to 0$ as $\delta \to 0$. If $p > 2$, $\|f(\cdot)-f_{\delta,\alpha}(\cdot)\| \le 2\delta^{\frac{p}{p+1}}E^{\frac{1}{p+1}} + \delta^{\frac{2}{p+1}}E^{\frac{p-1}{p+1}} \to 0$ as $\delta \to 0$. Hence $f_{\delta,\alpha}(t)$ can be viewed as an approximation of the exact solution $f(t)$.

Remark 3.4. In practice, $\|f\|_p$ is usually not known; an exact a priori bound E cannot be obtained. But if we choose $\alpha = \delta^{\frac{1}{p+1}}$, we can also obtain

$$\|f(\cdot)-f_{\delta,\alpha}(\cdot)\| \le 3\delta^{\frac{p}{p+1}} \to 0, \quad \text{as } \delta \to 0. \tag{3.21}$$

This choice is helpful in our realistic computation.

Remark 3.5. If we chose $p = 0$, i.e., the a priori assumption was replaced by $\|f\|_{L^2(\mathbb{R})} \leq E$, then there would only be boundedness of the terms in (3.20) rather than convergence to zero [19].

4. A numerical example

In this section, numerical results are presented, which verify the validity of the theoretical results of this method. It is easy to see that the function

$$u(x,t) = \begin{cases} \dfrac{x+1}{t^{\frac{3}{2}}} \exp\left\{-\dfrac{(x+1)^2}{4t}\right\} - \dfrac{1}{t^{\frac{3}{2}}} \exp\left\{-\dfrac{1}{4t}\right\}, & x > 0, t > 0, \\ 0, & t \leq 0 \end{cases} \tag{4.1}$$

and the function

$$f(t) = \begin{cases} \left(\dfrac{3}{2} t^{-\frac{5}{2}} - \dfrac{1}{4} t^{-\frac{7}{2}}\right) \exp\left\{-\dfrac{1}{4t}\right\}, & t > 0, \\ 0, & t \leq 0 \end{cases} \tag{4.2}$$

are satisfied for the problem (1.1) with exact data

$$g(t) = \begin{cases} \dfrac{2}{t^{\frac{3}{2}}} \exp\left\{-\dfrac{1}{t}\right\} - \dfrac{1}{t^{\frac{3}{2}}} \exp\left\{-\dfrac{1}{4t}\right\}, & t > 0, \\ 0, & t \leq 0. \end{cases} \tag{4.3}$$

Now we will focus on our numerical experiment in order to verify the theoretical results. The range of variable t in the numerical experiment is $[0, 1]$.

Suppose that the sequence $\{g_k\}_{k=0}^n$ represents samples from the function $g(t)$ on an equidistant grid, and n is even; then we add a random uniform perturbation to each data value forming the vector g_δ, i.e.,

$$g_\delta = g + \varepsilon\, \mathrm{rand}n(\mathrm{size}(g)), \tag{4.4}$$

where

$$g = (g(t_1), \ldots, g(t_n))^T, \qquad t_i = (i-1)\Delta t, \qquad \Delta t = \frac{1}{n-1}, \quad i = 1, 2, \ldots, n. \tag{4.5}$$

The function "rand$n(\cdot)$" generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2 = 1$, and standard deviation $\sigma = 1$. "Rand$n$(size(g))" returns an array of random entries that is of the same size as g. The total noise level δ can be measured in the sense of the root mean square error (RMSE) according to

$$\delta = \|g_\delta - g\|_{l^2} = \left(\frac{1}{n}\sum_{i=1}^{n}(g_i - g_{i,\delta})^2\right)^{\frac{1}{2}}. \tag{4.6}$$

We give a simple description of numerical implementation as follows:

Step 1: Take the fast Fourier transform (FFT) for the vector g_δ.

Step 2: Compute the vector (see (3.18))

$$\left\{\frac{\mathrm{i}\xi_k}{(1 - \mathrm{e}^{-\sqrt{\mathrm{i}\xi_k}})(1 + \alpha^2 \xi_k^2)} \hat{g}_\delta(\xi_k)\right\}_{k=-\frac{n}{2}-1}^{\frac{n}{2}}, \tag{4.7}$$

where $\mathrm{i} = \sqrt{-1}$, $\xi_k = 2\pi k$. The regularization parameter α is selected according to Remark 3.4, i.e., $\alpha = \delta^{\frac{1}{p+1}}$.

Step 3: Take the inverse FFT for the vector in (4.7) and obtain $f_{\delta,\alpha}(t)$.

When using the FFT algorithm we implicitly assume that the vector g_δ represents a periodic function. This is not realistic in our application, and thus we need to modify the algorithm. A discussion about the algorithm can be found in [25].

Figs. 1–3 indicate the comparisons between the exact solution $f(t)$ and the simplified Tikhonov regularization solution $f_{\delta,\alpha}(t)$ for $p = \frac{1}{2}$, $p = \frac{3}{5}$ and $p = 1$, respectively, with the perturbations $\varepsilon = 0.001$ and $\varepsilon = 0.0001$. From Figs. 1–3, we conclude that the regularized solution approximates the exact solution as the amount of noise ε decreases. Also, it can be seen from Figs. 1–3 that some inaccuracies appear in these regularization solutions as the parameter p increases, but these results are acceptable.

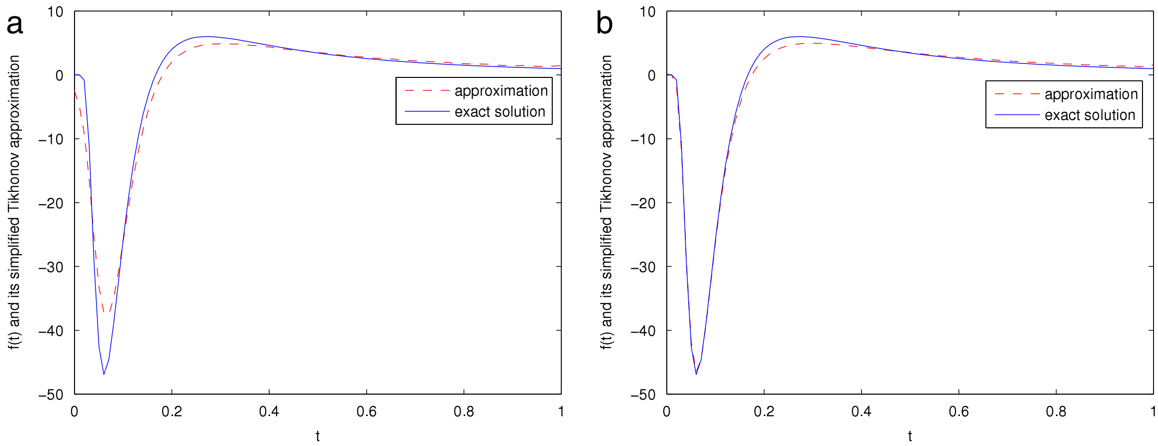

Fig. 1. $p = \frac{1}{2}$: (a) $\varepsilon = 0.001, \alpha = 0.0016$; (b) $\varepsilon = 0.0001, \alpha = 0.0034$.

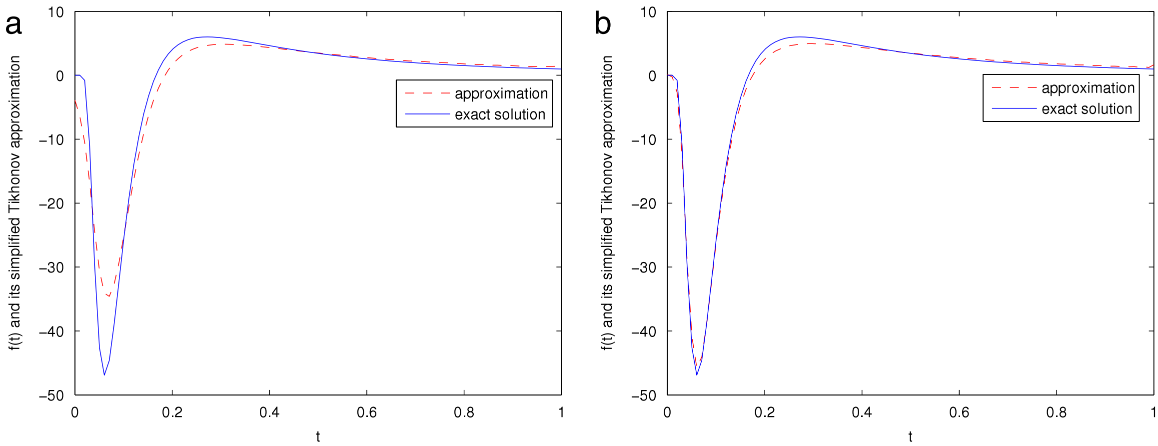

Fig. 2. $p = \frac{3}{5}$: (a) $\varepsilon = 0.001, \alpha = 0.0206$; (b) $\varepsilon = 0.0001, \alpha = 0.0049$.

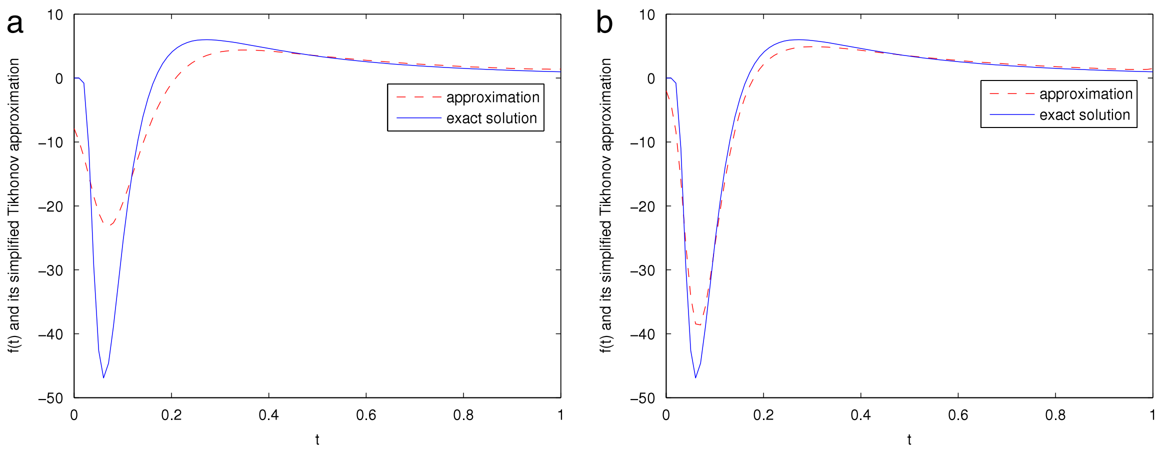

Fig. 3. $p = 1$: (a) $\varepsilon = 0.001, \alpha = 0.0455$; (b) $\varepsilon = 0.0001, \alpha = 0.0144$.

5. Conclusions

In this paper, we considered the identification of an unknown heat source term depending only on time variable in a parabolic equation. This problem is ill-posed, i.e., the solution (if it exists) does not depend on the input data. We obtained the regularization solution by the simplified Tikhonov regularization method and gave a Hölder type error estimate. Moreover the error estimation is order optimal. Meanwhile, a numerical example verified the efficiency and accuracy of the method.

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