A note on collections of graphs with non-surjective lambda labelings

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Abstract

The $\lambda$-number of a graph $G$, denoted $\lambda(G)$, is the smallest integer $k$ such that there exists a function from $V(G)$ into \{0, 1, 2, \ldots, k\} under which adjacent vertices receive integers which differ by at least 2 and vertices at distance two receive integers which differ by at least 1. We establish the infinitude of the collection of connected graphs $G$ with fixed maximum degree $\Delta \geq 4$ and fixed $\lambda$-number $\lambda + t$, $1 \leq t \leq \Delta - 1$ such that no $\lambda$-labeling of $G$ into \{0, 1, 2, \ldots, \lambda(G)\} is surjective. Also, from among graphs with no surjective $\lambda$-labelings, we construct connected graphs with maximum degree 3, $\lambda$-number 5 and arbitrarily large order.

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1. Introduction

Introduced by Griggs and Yeh [12] as an extension of $T$-colorings, an $L(2, 1)$-labeling of a graph $G$ is a function from $V(G)$ into \{0, 1, 2, \ldots, $\lambda(G)$\} such that

- vertices which are adjacent receive integers that differ by at least 2,
- vertices which are distance two apart receive integers that differ by at least 1, and
- at least one vertex is labeled 0.

The $\lambda$-number of a graph $G$, denoted $\lambda(G)$, is the smallest $k$ for which there exists an $L(2, 1)$-labeling of $G$ into \{0, 1, 2, \ldots, $\lambda(G)$\}. Each $L(2, 1)$-labeling of $G$ into \{0, 1, 2, \ldots, $\lambda(G)$\} is called a $\lambda$-labeling of $G$.

If $L$ is a $\lambda$-labeling of $G$, then each integer in \{1, 2, \ldots, $\lambda(G) - 1$\} which is not assigned by $L$ to any vertex is called a hole of $L$. Clearly, $L$ is surjective if and only if $L$ has no holes. Any graph $G$ which has at least one surjective $\lambda$-labeling will be called full-colorable; otherwise, it will be called non-full colorable.

The area of $L(2, 1)$-labelings has inspired much literature, devoted to such issues as the $\lambda$-numbers of graphs in specific classes (see [2,8,10,13–17]), the relationship between $\lambda(G)$ and other invariants of $G$ (see [7,11]) and the generalization of $L(2, 1)$-labelings (see [1,3,6]). Recently, Fishburn and Roberts [4,5] and Georges and Mauro [9] have considered the graph

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invariant $\rho(G)$, the minimum number of holes taken over all $\lambda$-labelings of $G$. In particular, the former authors have shown that $\rho(G) = 0$ if $|V(G)| = \lambda(G) + 1$, and that $\rho(G) = 0$ if $G$ is any tree except a claw. They have analyzed extensively the colorability of graphs with fixed orders $n \leq 10$, maximum degree $\Delta = 3, 4$ and $\lambda$-number 5, and have proven the following theorems.

**Theorem 1.1** (Fishburn and Roberts [4]). For any fixed positive integer $m$, there exists a non-full colorable graph with $\lambda(G) = 5$, maximum degree $\Delta = 3$ and order $n = 10m - 1$.

**Theorem 1.2** (Fishburn and Roberts [4]). For fixed $\Delta \geq 3$, there exists a non-full colorable graph $G$ with maximum degree $\Delta$, order $n = 2\Delta + 2$ and $\lambda$-number $\Delta + 2$.

**Theorem 1.3** (Fishburn and Roberts [4]). For fixed $\Delta \geq 3$ and fixed $\lambda$, $\Delta + 1 \leq \lambda \leq 2\Delta - 1$, there exists a non-full colorable graph $G$ with maximum degree $\Delta$, order $n \geq \lambda + 1$ and $\lambda$-number $\lambda$.

In this note, we continue the exploration of Fishburn and Roberts, and settle one of their open questions, by establishing the infinitude of the collection of non-full colorable, connected graphs with maximum degree $\Delta$ and $\lambda$-number $\Delta + t$ for fixed $\Delta \geq 4$ and fixed $t$, $1 \leq t \leq \Delta - 1$. Letting $\mathcal{F}_\Delta(\lambda)$ be the collection of connected, non-tree, non-full colorable graphs with maximum degree $\Delta$ and $\lambda$-number $\lambda$, we prove in particular the following two theorems:

**Theorem 2.1.** For fixed $\Delta \geq 4$, $\mathcal{F}_\Delta(\Delta+1)$ is an infinite set.

**Theorem 3.1.** For fixed $\Delta \geq 4$ and fixed $t$, $2 \leq t \leq \Delta - 1$, $\mathcal{F}_\Delta(\Delta+t)$ is an infinite set.

We also construct graphs in $\mathcal{F}_3(5)$ with arbitrarily large orders $n$ for $n = 0 \mod 5, n = 4 \mod 5$ and $n = 3 \mod 5$.

2. **Proof of Theorem 2.1**

In order to establish that $\mathcal{F}_\Delta(\Delta+1)$ is infinite, we construct graphs $\mathcal{F}_\Delta(m)$, $m \geq 1$, having the properties that $\mathcal{F}_\Delta(m)$ has maximum degree $\Delta$, $\lambda(\mathcal{F}_\Delta(m)) = \Delta + 1$ and every $\lambda$-labeling of $\mathcal{F}_\Delta(m)$ is non-surjective. Each graph $\mathcal{F}_\Delta(m)$ will be formed by stitching together the pairwise isomorphic graphs $A_\Delta(1), A_\Delta(2), \ldots, A_\Delta(m)$, defined below, which have maximum degree $\Delta$ and $\lambda$-number $\Delta + 1$.

We begin with the definition of $A_\Delta(i)$ for the three cases $\Delta = 2j$ for $j \geq 3$, $\Delta = 2j$ for $j = 2$, and $\Delta = 2j + 1$ for $j \geq 2$.

1. For $\Delta = 2j$, $j \geq 3$, let $A_\Delta(i)$ be the graph with vertex set $R_i \cup U_i \cup W_i$, where

$$R_i = \{r_i\};$$
$$U_i = \{u_{i,1}, u_{i,2}, \ldots, u_{i,j+1}\};$$
$$W_i = \{w_{i,1}, w_{i,2}, \ldots, w_{i,j-1}\},$$

and where $E(A_2(j)(i))$ is the smallest set such that the subgraph induced by $U_i \cup R_i$ is isomorphic to $(\text{the complete graph minus an edge}) \ K_{j+2} - \{u_{i,j}, u_{i,j+1}\}$ and the subgraph induced by $R_i \cup W_i$ is isomorphic to $K_{1,j-1}$ where $d(r_i) = j - 1$. In Fig. 1, we illustrate $A_6(i)$.

Noting that the $\lambda$-number of the subgraph induced by $U_i \cup R_i$ is $2j + 1$, we have $\lambda(A_2(j)(i)) \geq 2j + 1$.

2. For $\Delta = 4$, we define $A_4(i)$ as shown in Fig. 2:

It is clear by inspection that $\lambda(A_4(i)) = 5$.

3. For $\Delta = 2j + 1$, $j \geq 2$, let $A_{2j+1}(i)$ be the graph with vertex set $R_i \cup U_i \cup W_i$, where

$$R_i = \{r_i\};$$
$$U_i = \{u_{i,1}, u_{i,2}, \ldots, u_{i,j+1}\};$$
$$W_i = \{w_{i,1}, w_{i,2}, \ldots, w_{i,j}\},$$

and where $E(A_{2j+1}(i))$ is the smallest set such that the subgraph induced by $U_i \cup R_i$ is isomorphic to $K_{j+2}$ and the subgraph induced by $R_i \cup W_i$ is isomorphic to $K_{1,j}$ where $d(r_i) = j$. In Fig. 3, we illustrate $A_5(i)$. Noting that the $\lambda$-number of the subgraph induced by $U_i \cup R_i$ is $2j + 2$, we have $\lambda(A_{2j+1}(i)) \geq 2j + 2$. 


We now turn to the construction of $F_{a2}(m)$ for fixed $m \geq 1$, analyzing separately the five exhaustive cases $A = 2j$ for $j > 3$, $A = 6$, $A = 2j + 1$ for $j \geq 3$ and $A = 5$. Since in each case it will be trivially true that $F_{a2}(1)$ is an element of $Z_{a2}(A + 1)$, we argue that $F_{a2}(m) \in Z_{a2}(A + 1)$ only for $m \geq 2$.

Case 1: For $A = 2j$, $j > 3$, let $V(F_{2j}(m)) = \bigcup_{i=1}^{m} V(A_{2j}^{i}(m))$ and let $E(F_{2j}(m)) = E \cup E^{*}$, where $E = \bigcup_{i=1}^{m} E(A_{2j}^{i}(m))$ and $E^{*} = \bigcup_{t=1}^{j-1} \{\{w_{i,t}, u_{i+1,t}\} | 1 \leq t \leq j - 1\} \cup \{\{w_{i,j-1}, u_{i+1,j}\}, \{w_{i,j-1}, u_{i+1,j+1}\}\}$. To show that $F_{2j}(m) \in Z_{2j}(2j + 1)$, we establish

(1) $\lambda(F_{2j}(m)) = 2j + 1$, and
(2) every $\lambda$-labeling of $F_{2j}(m)$ has a hole.

Since $A_{2j}(i)$ is a subgraph of $F_{2j}(m)$ with $\lambda$-number at least $2j + 1$, we establish that $\lambda(F_{2j}(m)) = 2j + 1$ by demonstrating an $L(2, 1)$-labeling $L$ of $F_{2j}(m)$ with span $2j + 1$, thus:

$$L(v) = \begin{cases} 0 & \text{if } v = r_{i}, 1 \leq i \leq m, \\ 2t & \text{if } v = u_{i,t}, 1 \leq t \leq j, 1 \leq i \leq m, \\ 2j + 1 & \text{if } v = u_{i,j+1}, 1 \leq i \leq m, \\ 2t + 3 & \text{if } v = w_{i,t}, 1 \leq t \leq j - 2, 1 \leq i \leq m, \\ 3 & \text{if } v = w_{i,j-1}, 1 \leq i \leq m. \end{cases}$$

Now let $R = \{r_{1}, r_{2}, \ldots, r_{m}\}$ and let $L$ be an arbitrary $\lambda$-labeling of $F_{2j}(m)$. We first argue as follows that $L(r_{1}) = L(r_{2}) = \cdots = L(r_{m})$: select vertices $r_{i}$ and $r_{i+1}$ in $R$. Since $d(r_{i}) = d(r_{i+1}) = 2j = A$, the labels of $r_{i}$ and $r_{i+1}$ are necessarily in $[0, 2j + 1]$. With no loss of generality, suppose that $L(r_{i}) = 0$ and $L(r_{i+1}) = 2j + 1$. Since the subgraph $H$ induced by $U_{i+1} \cup \{r_{i+1}\}$ is isomorphic to the complete graph on $j + 1$ vertices minus an edge, then $\lambda(H) = 2j + 1$, and hence the restriction of $L$ on $H$
must assign 0 to some vertex in \(U_{i+1}\). But each vertex in \(U_{i+1}\) is at distance two from \(r_i\), implying a violation of the distance two condition. Therefore \(L(r_1) = L(r_2) = \cdots = L(r_m) = 0\) (resp. \(2j + 1\)). Since \(R\) is a dominating set, then 1 (resp. \(2j\)) must be a hole of \(L\). Thus \(\mathcal{F}_j(2) \subseteq \mathcal{F}_j(2j + 1)\) for \(m \geq 1\), implying that \(\mathcal{F}_j(2j + 1)\) is an infinite set.

Case 2: For \(d = 6\), let \(V(\mathcal{F}_6(m)) = \bigcup_{i=1}^{m} V(A_6(i))\) and let \(E(\mathcal{F}_6(m)) = E \cup E^*\), where \(E = \bigcup_{i=1}^{m} E(A_6(i))\) and \(E^* = \bigcup_{i=1}^{m} \{[w_{i,1}, u_{i,1+1}], [w_{i,1}, u_{i,1+3}], [w_{i,2}, u_{i,1+2}], [w_{i,2}, u_{i,1+4}].\) Since \(A_6(i)\) has \(\lambda\)-number at least 7, it is easily checked that \(\lambda(\mathcal{F}_6(m)) = 7\) through the following \(L(2, 1)\)-labeling \(L\):

\[
L(v) = \begin{cases} 
0 & \text{if } v = r_j, 1 \leq i \leq m, \\
2 & \text{if } v = u_{i,1}, 1 \leq i \leq m, \\
7 & \text{if } v = u_{i,2}, 1 \leq i \leq m, \\
4 & \text{if } v = u_{i,3}, 1 \leq i \leq m, \\
5 & \text{if } v = u_{i,4}, 1 \leq i \leq m, \\
6 & \text{if } v = w_{i,1,1}, 1 \leq i \leq m, \\
3 & \text{if } v = w_{i,2,1}, 1 \leq i \leq m.
\end{cases}
\]

The argument that \(\mathcal{F}_6(m) \subseteq \mathcal{F}_6(7)\) for \(m \geq 1\) is now identical to that given above in Case 1.

Case 3: For \(d = 4\), let \(V(\mathcal{F}_4(m)) = \bigcup_{i=1}^{m} V(A_4(i))\) and let \(E(\mathcal{F}_4(m)) = E \cup E^*\) where \(E = \bigcup_{i=1}^{m} E(A_4(i))\) and \(E^* = \bigcup_{i=1}^{m-1} \{[w_{i,1}, u_{i,1+1}], [w_{i,2}, u_{i,1+2}]\}.\) Since \(A_4(i)\) has \(\lambda\)-number 5, it is easily checked that \(\lambda(\mathcal{F}_4(m)) = 5\) through the following \(L(2, 1)\)-labeling \(L\):

\[
L(v) = \begin{cases} 
0 & \text{if } v = r_j, 1 \leq i \leq m, \\
5 & \text{if } v = u_{i,1,1}, 1 \leq i \leq m, \\
4 & \text{if } v = u_{i,2}, 1 \leq i \leq m, \\
3 & \text{if } v = u_{i,3}, 1 \leq i \leq m, \\
6 & \text{if } v = w_{i,1,1}, 1 \leq i \leq m, \\
3 & \text{if } v = w_{i,2,1}, 1 \leq i \leq m.
\end{cases}
\]

Now let \(R = \{r_1, r_2, \ldots, r_m\}\) and let \(L\) be an arbitrary \(\lambda\)-labeling of \(\mathcal{F}_4(m)\). We first argue as follows that \(L(r_1) = L(r_2) = \cdots = L(r_m)\): since \(d(r_1) = d(r_{j+1}) = 4\), we assume with no loss of generality that \(L(r_1) = 0\) and \(L(r_{j+1}) = 5\). Adjacent vertices \(u_{i+1,1}\) and \(u_{i+1,2}\) must therefore receive distinct labels from \([1, 3]\), so it follows from the distance conditions that either \(w_{i,1}\) or \(w_{i,2}\) must be assigned the label 5, a violation of the distance two condition. Therefore \(L(r_1) = L(r_2) = \cdots = L(r_m) = 0\) (resp. 5). Since \(R\) is a dominating set, then 1 (resp. 4) must be a hole of \(L\). Thus \(\mathcal{F}_4(m) \subseteq \mathcal{F}_4(5)\) for \(m \geq 1\), implying that \(\mathcal{F}_4(5)\) is an infinite set.

Case 4: For \(d = 2j + 1, j \geq 3\), let \(V(\mathcal{F}_{2j+1}(m)) = \bigcup_{i=1}^{m} V(A_{2j+1}(i))\) and let \(E(\mathcal{F}_{2j+1}(m)) = E \cup E^*\) where \(E = \bigcup_{i=1}^{m} E(A_{2j+1}(i))\) and \(E^* = \bigcup_{i=1}^{m-1} \{[w_{i,1,1}, u_{i,1+1}], [w_{i,1, j}, u_{i, j+1}, 1] \cup [w_{i,1, j}, u_{i, j+1, j+1}]\}.\) Since \(A_{2j+1}(i)\) has \(\lambda\)-number at least \(2j + 2\), it is easily checked that \(\lambda(\mathcal{F}_{2j+1}(m)) = 2j + 2\) through the following \(L(2, 1)\)-labeling \(L\):

\[
L(v) = \begin{cases} 
0 & \text{if } v = r_j, 1 \leq i \leq m, \\
2r & \text{if } v = u_{i,1}, 1 \leq i \leq j + 1, 1 \leq i \leq m, \\
2r + 3 & \text{if } v = w_{i,1,1}, 1 \leq i \leq j - 1, 1 \leq i \leq m, \\
3 & \text{if } v = w_{i,2,1}, 1 \leq i \leq m.
\end{cases}
\]

The argument that \(\mathcal{F}_{2j+1}(m) \subseteq \mathcal{F}_{2j+1}(2j + 2)\) for \(m \geq 1\) is now similar to that given in Case 1.

Case 5: For \(d = 5\), let \(V(\mathcal{F}_5(m)) = \bigcup_{i=1}^{m} V(A_5(i))\) and let \(E(\mathcal{F}_5(m)) = E \cup E^*\) where \(E = \bigcup_{i=1}^{m} E(A_5(i))\) and \(E^* = \bigcup_{i=1}^{m-1} \{[w_{i,1,1}, u_{i,1+1}], [w_{i,2,1}, u_{i,1+2}]\}.\) Since \(A_5(i)\) has \(\lambda\)-number at least 6, it is easily checked that \(\lambda(\mathcal{F}_5(m)) = 6\) through the following \(L(2, 1)\)-labeling \(L\):

\[
L(v) = \begin{cases} 
0 & \text{if } v = r_j, 1 \leq i \leq m, \\
2 & \text{if } v = u_{i,1,1}, 1 \leq i \leq m, \\
6 & \text{if } v = u_{i,2}, 1 \leq i \leq m, \\
4 & \text{if } v = u_{i,3}, 1 \leq i \leq m, \\
5 & \text{if } v = w_{i,1,1}, 1 \leq i \leq m, \\
3 & \text{if } v = w_{i,2,1}, 1 \leq i \leq m.
\end{cases}
\]

Now let \(R = \{r_1, r_2, \ldots, r_m\}\) and let \(L\) be an arbitrary \(\lambda\)-labeling of \(\mathcal{F}_5(m)\). We first argue as follows that \(L(r_1) = L(r_2) = \cdots = L(r_m)\): since \(d(r_1) = d(r_{j+1}) = 5\), we assume with no loss of generality that \(L(r_1) = 0\) and \(L(r_{j+1}) = 6\). Then vertices \(u_{i+1,1}\) and \(u_{i+1,2}\) are distance two away from \(r_i\), so the label 0 is necessarily assigned to \(u_{i+1,3}\). Therefore, the vertices \(u_{i+1,1}\) and \(u_{i+1,2}\) are assigned distinct labels in \([2, 4]\). Suppose with no loss of generality that \(L(u_{i+1,1}) = 2\) and \(L(u_{i+1,2}) = 4\). Then the distance conditions require that \(L(u_{i,2}) \geq 7\), a contradiction. Therefore \(L(r_1) = L(r_2) = \cdots = L(r_m) = 0\) (resp. 6). Since \(R\) is a dominating set, then 1 (resp. 5) must be a hole of \(L\). Thus \(\mathcal{F}_5(m) \subseteq \mathcal{F}_5(6)\) for \(m \geq 1\), implying that \(\mathcal{F}_5(6)\) is an infinite set.
3. Proof of Theorem 3.1

In order to establish that $\mathcal{F}_3(A + t)$ is infinite for fixed $t$, $2 \leq t \leq A - 1$, we construct graphs $\mathcal{K}_A, m \geq 1$, having the properties that $\mathcal{K}_A(m)$ has maximum degree $A$, and every $\lambda$-labeling of $\mathcal{K}_A(m)$ has a hole. The desired construction will be accomplished by a particular stitching of the graphs $B_{A, i}(t)$, $1 \leq i \leq m$, each of which is isomorphic to the complete bipartite graph $K_{A, i}$. Throughout, we will denote the set of vertices in the smaller part of $B_{A, i}(t)$ by $X_i = \{x_{i,0}, x_{i,1}, x_{i,2}, \ldots, x_{i,t-1}\}$ and the set of vertices in the larger part by $Y_i = \{y_{i,0}, y_{i,1}, y_{i,2}, \ldots, y_{i,A-1}\}$. We note that the $\lambda$-number of $B_{A, i}(t)$ is well known to be $A + t$, and that under each $\lambda$-labeling $L$ of $B_{A, i}(t)$ the vertices of each part are assigned consecutive integers. Therefore, the labels assigned by $L$ to the smaller (resp. larger) part of $B_{A, i}(t)$ are either 0, 1, 2, $\ldots$, $t - 1$ (resp. $t + 1, t + 2, \ldots, A + t$ or $A + 1, A + 2, \ldots, A + t$ (resp. 0, 1, 2, $\ldots$, $A - 1$). It follows that each $\lambda$-labeling of $B_{A, i}(t)$ has exactly one hole at either $t$ or $A$. We also note that if $H$ is a graph with fixed subgraph $H'$ isomorphic to $B_{A, i}(t)$ and if $\lambda(H) = \lambda(H')$, then for every $\lambda$-labeling $L$ of $H$, either $L$ or $\lambda(H) - L$ assigns the labels 0, 1, $\ldots$, $t - 1$ to the vertices in the smaller part of $H'$. With no loss of generality, then, we may restrict our attention to only those $\lambda$-labelings of $H$ which assign the labels 0, 1, $\ldots$, $t - 1$ to the vertices in the smaller part of $H'$.

Suppose that $H$ is a graph with vertex set $V(H) = V(B_{A, i}(t)) \cup V(B_{A, j}(t))$ and edge set $E(H) = E(B_{A, i}(t)) \cup E(B_{A, j}(t)) \cup E^*$, where $E^*$ is any set containing only independent edges of the form $(y_{i,\alpha}, y_{j,\beta})$, $0 \leq \alpha, \beta \leq A - 1$ (i.e. $E^*$ is some matching, not necessarily complete, between $Y_i$ and $Y_j$). Suppose also that $G$ is a graph with subgraph $H$ and $\lambda(G) = A + t$, and let $L$ be a $\lambda$-labeling of $G$ which assigns precisely the elements of $\{0, 1, \ldots, t - 1\}$ to the vertices in $X_i$. Then no vertex in $Y_j$ incident to an edge in $E^*$ can receive a label in $\{0, 1, 2, \ldots, t - 1\}$ and no vertex in $Y_j$ receives labels in $\{t + 1, t + 2, \ldots, A + t\}$ and hence the vertices in $X_j$ receive labels in $\{0, 1, 2, \ldots, t - 1\}$.

We now move on to the proof of Theorem 3.1, considering the three exhaustive cases $A \geq 5$ and $2 \leq t \leq A - 2$; $A = 4$ and $t = 2$; $t = A - 1$. Since in each case it will be trivially true that $\mathcal{K}_A(m)$ is an element of $\mathcal{F}_3(A + t)$, we argue each case only for $m \geq 1$.

Case 1: $A \geq 5$ and $2 \leq t \leq A - 2$. Let $\mathcal{K}_A(m)$ be the graph with vertex set $V(\mathcal{K}_A(m)) = \bigcup_{i=1}^m V(\mathcal{B}_{A, i}(t))$ and let $E(\mathcal{K}_A(m)) = E \cup E^*$ where $E = \bigcup_{i=1}^m E(\mathcal{B}_{A, i}(t))$ and $E^* = \bigcup_{i=1}^m \{(y_{i,\alpha}, y_{i+1,\gamma}) \mid 0 \leq \alpha, \gamma \leq A - t, \gamma = (\beta + 2) \bmod A\}$. Since $\mathcal{B}_{A, i}(1)$ is a subgraph of $\mathcal{K}_A(m)$, then $\lambda(\mathcal{K}_A(m)) \geq A + t$. But it is easily seen that $\lambda(\mathcal{K}_A(m)) = A + t$ via the $L(2, 1)$-labeling which assigns $t + 1 + \beta$ to $y_{i,\beta}$ and $\alpha$ to $x_{i,\alpha}$ for $0 \leq \beta \leq A - 1, 0 \leq \alpha \leq t - 1, 1 \leq i \leq m$.

Now let $L$ be any $\lambda$-labeling of $\mathcal{K}_A(m)$ where, with no loss of generality, the vertices in $X_i$ are assigned labels from $\{0, 1, 2, \ldots, t - 1\}$. Since $E^*$ is a union of matchings each with cardinality $A - t + 1$, then by Observation 3.2 and induction, the vertices in $X_i$ are assigned labels from $\{0, 1, 2, \ldots, t - 1\}$ for each $i, 1 \leq i \leq m$. It thus follows that $L$ has a hole at $t$ and that $\mathcal{K}_A(m) \in \mathcal{F}_3(A + t)$ for $m \geq 1$.

Case 2: $A = 4$ and $t = 2$. Let $\mathcal{K}_A(m)$ be the graph with vertex set $V(\mathcal{K}_A(m)) = \bigcup_{i=1}^m V(\mathcal{B}_{A, i}(t))$ and let $E(\mathcal{K}_A(m)) = E \cup E^*$ where $E = \bigcup_{i=1}^m E(\mathcal{B}_{A, i}(t))$ and $E^* = \bigcup_{i=1}^m \{(y_{i,0}, y_{i+1,3}), (y_{i,2}, y_{i+1,0}), (y_{i,3}, y_{i+1,1})\}$. As above, it can be argued that $\lambda(\mathcal{K}_A(m)) = 6$ via the labeling which assigns $3 + \beta$ to $y_{i,\beta}$ and $\alpha$ to $x_{i,\alpha}$ for $0 \leq \beta \leq 3, 0 \leq \alpha \leq t - 1, 1 \leq i \leq m$. Since $E^*$ is a union of matchings of order $3 > A - t = 2$, it follows by Observation 3.2 that every $\lambda$-labeling of $\mathcal{K}_A(m)$ has a hole at $t$ and that therefore $\mathcal{K}_A(m) \in \mathcal{F}_3(A + t)$ for $m \geq 1$.

Case 3: $t = A - 1$. Let $\mathcal{K}_A(m)$ be the graph with vertex set $V(\mathcal{K}_A(m)) = \bigcup_{i=1}^m V(\mathcal{B}_{A, i}(t))$ and let $E(\mathcal{K}_A(m)) = E \cup E^*$ where $E = \bigcup_{i=1}^m E(\mathcal{B}_{A, i}(t))$ and $E^* = \bigcup_{i=1}^m \{(y_{i,0}, y_{i+1,2}), (y_{i,1}, y_{i+1,3})\}$. As above, it can be argued that $\lambda(\mathcal{K}_A(m)) = 2A - 1$ via the labeling which assigns $\beta + 1$ to $y_{i,\beta}$ and $\alpha$ to $x_{i,\alpha}$ for $0 \leq \beta \leq A - 1, 0 \leq \alpha \leq A - 2, 1 \leq i \leq m$. By Observation 3.2, it again follows that every $\lambda$-labeling of $\mathcal{K}_A(m)$ has a hole at $t = A - 1$ and that $\mathcal{K}_A(m) \in \mathcal{F}_3(A + t)$ for $m \geq 1$.

4. On the infinitude of $\mathcal{F}_3(5)$

In [4], Fishburn and Roberts considered the number of elements of $\mathcal{F}_3(5)$ with fixed order $n$. Among other results, they proved that $\mathcal{F}_3(5)$ contains precisely four elements with order less than 6, no elements with order 6 or 7, at least one element with order 8, at least three with order 9 and at least two with order 10. They also showed that $\mathcal{F}_3(5)$ contains at least one element with order $n = 10m - 1, m \geq 1$. In this section, we extend certain constructions of Fishburn and Roberts to show that $\mathcal{F}_3(5)$...
contains at least one graph of order \( n \) for each sufficiently large \( n \) with \( n \equiv 0 \pmod{5} \), \( 4 \equiv 0 \pmod{5} \) and \( 3 \equiv 0 \pmod{5} \). Throughout, \( B_{3,2}(i) \) and \( B_{2,2}(i) \) shall be defined as in Section 3.

We first argue the case \( n \equiv 0 \pmod{5} \) for \( n \geq 5 \). As already noted, \( B_{3,2}(1) \) is a non-full colorable graph of order 5. For \( k \geq 2 \), let \( H(k) \) be the graph of order \( 5k \) with vertex set \( \bigcup_{i=1}^{k} V(B_{3,2}(i)) \) and edge set \( E \cup E^{**} \) where \( E = \bigcup_{i=1}^{k} E(B_{3,2}(i)) \) and \( E^{**} = (\bigcup_{i=1}^{k-1} \{ \{ y_{i,2}, y_{i+1,0} \} \}) \cup \{ \{ y_{k,2}, y_{1,0} \} \} \). Then this graph, illustrated in Fishburn and Roberts for the case \( k = 2 \), has \( \lambda \)-number at least \( \lambda(B_{3,2}(i)) = 5 \). However, by assigning 0 to \( x_{1,0} \), 1 to \( x_{1,1} \), 3 to \( y_{1,0} \), 4 to \( y_{1,1} \), and 5 to \( y_{2,2} \) for \( 1 \leq i \leq k \), we construct an \( (2, 1) \)-labeling of \( H(k) \) with span 5. Thus, \( \lambda(H(k)) = 5 \). To see that \( H(k) \) is not full colorable, let \( L \) denote an arbitrary \( \lambda \)-labeling of \( H(k) \) where, with no loss of generality, the vertices \( x_{1,0} \) and \( x_{1,1} \) receive labels in \( \{ 0, 1 \} \) and the vertices \( y_{1,0}, y_{1,1} \) and \( y_{2,2} \) receive labels in \( \{ 3, 5 \} \). By symmetry, we assume with no loss of generality that \( L(x_{1,0}) = 0 \) and \( L(x_{1,1}) = 1 \). If \( L(y_{1,2}) = 4 \), then the distance conditions require \( L(y_{2,2}) = 0 \), implying that \( L(y_{2,1}) = 0 \) for \( i = 1, 2 \). Hence, either \( x_{2,0} \) or \( x_{2,1} \) is assigned 4 under \( L \), a violation of the distance two condition. A similar argument in consideration of the vertices of \( B_{3,2}(1) \) and \( B_{2,2}(2) \) shows that \( L(y_{1,0}) \neq 4 \). Thus, \( L(y_{1,1}) = 4 \) and by symmetry, we may assume \( L(y_{1,1}) = 3 \) and \( L(y_{2,1}) = 5 \). It is now an easy matter to argue by induction that under \( L \), the vertices \( x_{i,0} \) and \( x_{i,1} \) receive labels in \( \{ 0, 1 \} \), the vertices \( y_{i,0} \) and \( y_{i,1} \) receive labels in \( \{ 3, 5 \} \) and the vertex \( y_{1,1} \) receives label 4. Since 2 cannot be assigned by \( L \), we are done.

For the case \( n = 4 \equiv 0 \pmod{5} \) where \( n \geq 9 \), we let \( H'(k) \) be the graph of order \( 5k - 1 \) with vertex set \( \bigcup_{i=1}^{k-1} V(B_{3,2}(i)) \cup V(B_{2,2}(k)) \) and edge set \( E \cup E^{**} \) where \( E = (\bigcup_{i=1}^{k-1} E(B_{3,2}(i))) \cup E(B_{2,2}(k)) \) and \( E^{**} = (\bigcup_{i=1}^{k-1} \{ \{ y_{i,2}, y_{i+1,0} \} \}) \cup \{ \{ y_{k,2}, y_{1,0} \} \} \). Then loosely speaking, this graph consists of \( k - 1 \) copies of \( K_{3,2} \) and one copy of \( K_{2,2} \), stitched together in a way analogous to the stitching in the graph \( H(k) \). By assigning labels in \( \{ 0, 1 \} \) to vertices \( x_{i,0} \) and \( x_{i,1} \), 1 \( \leq i \leq k \), and assigning 4 to vertices \( y_{i,1} \), 1 \( \leq i \leq k - 1 \), and assigning labels in \( \{ 3, 5 \} \) to the remaining vertices, it is easily seen that \( \lambda(H'(k)) = 5 \). Assuming with no loss of generality that \( L \) is a \( \lambda \)-labeling of \( H'(k) \) which assigns 0 and 1 to the vertices \( x_{1,0} \) and \( x_{1,1} \), it is then argued as above that \( L \) cannot assign 2 to any vertex.

For the case \( n = 3 \equiv 0 \pmod{5} \) where \( n \geq 13 \), we let \( H''(k) \) be the graph of order \( 5k - 2 \) with vertex set \( \bigcup_{i=1}^{k-3} V(B_{3,2}(i)) \cup V(B_{2,2}(k - 2)) \cup V(B_{2,2}(k - 1)) \cup V(B_{2,2}(k)) \) and edge set \( E \cup E^{**} \) where \( E = (\bigcup_{i=1}^{k-3} E(B_{3,2}(i))) \cup E(B_{2,2}(k - 2)) \cup E(B_{2,2}(k - 1)) \cup E(B_{2,2}(k)) \) and \( E^{**} = (\bigcup_{i=1}^{k-3} \{ \{ y_{i,2}, y_{i+1,0} \} \}) \cup \{ \{ y_{k-2,1}, y_{k-1,0} \}, \{ y_{k-1,2}, y_{k,0} \}, \{ y_{k,1}, y_{1,0} \} \} \). Then loosely speaking, this graph consists of \( k - 2 \) copies of \( K_{3,2} \) and two copies of \( K_{2,2} \), stitched together in a way which places exactly one copy of \( K_{3,2} \) \((B_{3,2}(k - 1)) \) between the two copies of \( K_{2,2} \) \((B_{2,2}(k - 2)) \) and \( B_{2,2}(k)) \). By assigning labels in \( \{ 0, 1 \} \) to vertices in the smaller parts of the copies of \( K_{3,2} \) and 4 to the vertex \( y_{1,1} \) in each copy of \( K_{3,2} \), it is easy to construct an \( (2, 1) \)-labeling with span 5, thus demonstrating that \( \lambda(H''(k)) = 5 \). The argument that every \( \lambda \)-labeling of \( H''(k) \) has a hole is tedious, but similar in nature to those already given.

5. Closing remarks

In [9], Georges and Mauro establish an element of \( \mathcal{A}_3(2A) \) with order \( A^2 + A \), for each \( A \geq 1 \). Thus, \( \mathcal{A}_3(2A) \) is not empty, yet the infinitude of \( \mathcal{A}_3(2A) \) remains an open question. We conjecture that \( \mathcal{A}_3(2A) \) is finite, and that for \( t > A \), \( \mathcal{A}_3(A + t) \) is empty.

For general \( n \), we have been unable to establish graphs in \( \mathcal{A}_3(5) \) with order \( n = 1 \mod{5} \) and \( 2 \mod{5} \), so these too are open questions.

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