On transitive Cayley graphs of strong semilattices of right (left) groups

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\textbf{ABSTRACT}

We investigate Cayley graphs of strong semilattices of right (left) groups, of right (left) zero semigroups, and of groups. We show under which conditions they enjoy the property of automorphism vertex transitivity in analogy to Cayley graphs of groups.

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Cayley graphs of semigroups and mainly of groups have been extensively studied and many interesting results have been obtained, for example, in [1,3–5,8,10]. Recent studies investigate transitivity of Cayley graphs of groups and semigroups [9], of right and left groups and of Clifford semigroups (strong semilattices of groups) [12,13]. In this paper we consider conditions for automorphism vertex transitivity of strong semilattices of right (left) groups. The given proofs use extensively results from [9]. Undefined concepts from Graph Theory can be found in [2]; for the algebraic concepts compare, for example, [4,11,14].

1. Basic definitions and results

A semigroup $S$ is said to be a \textit{right (left) zero semigroup} if $xy = y$ ($xy = x$) for all $x, y \in S$. Let $G$ be a group, $R_n$, for $n \in \mathbb{N}$, the $n$-element right zero semigroup, and set $S = G \times R_n$. Define the multiplication on $S$ componentwise by $(g, r)(g', r') = (gg', r')$ for $g, g' \in G, r, r' \in R_n$. In [1] this semigroup was called a \textit{right zero union of groups (RZUG)} over $G$. Correspondingly, if $L_n$ for $n \in \mathbb{N}$ is the $n$-element left zero semigroup, we set $S = G \times L_n$ and define the multiplication on $S$ componentwise by $(g, l)(g', l') = (gg', l)$ for $g, g' \in G, l, l' \in L_n$. In [1] this semigroup was called a \textit{left zero union of the groups (LZUG)} over $G$. Note that RZUG over $G$ and LZUG over $G$ are exactly the right and the left groups over $G$, where a semigroup $S$ is called a \textit{right (left) group}, if it is uniquely right (left) solvable, i.e. for all $r, t \in S$ there exists a unique $s \in S$ such that $rs = t$ ($sr = t$), see [4] or [7]. A semigroup $S$ is said to be \textit{completely simple} if it has no proper ideals and has a minimal idempotent with respect to the partial order $e \leq f: \Leftrightarrow eef = fe$. 

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If \((Y, \leq)\) is a nonempty partially ordered set such that the meet \(\alpha \land \beta\) of \(\alpha\) and \(\beta\) exists for every \(\alpha, \beta \in Y\), we say that \((Y, \leq)\) is a (lower) semilattice. A semigroup \(S\) is said to be a semilattice of (disjoint) semigroups \((S_\alpha, \circ_\alpha), \alpha \in Y\), if

1. \(Y\) is a semilattice,
2. \(S = \bigcup_{\alpha \in Y} S_\alpha\),
3. \(S_\alpha S_\beta \subseteq S_{\alpha \land \beta}\),

and a strong semilattice of semigroups, if in addition for all \(\beta \geq \alpha\) in \(Y\) there exists a semigroup homomorphism \(f_{\beta, \alpha} : S_\beta \rightarrow S_\alpha\), called a defining homomorphism, with

4. for all \(\alpha \in Y\), \(f_{\alpha, \alpha} = \text{id}_{S_\alpha}\), the identity mapping,
5. for all \(\alpha, \beta, v \in Y\) with \(\alpha \leq \beta \leq \gamma\), we have \(f_{\beta, \alpha} \circ f_{\gamma, \beta} = f_{\gamma, \alpha}\),

where the multiplication on \(S = \bigcup_{\alpha \in Y} S_\alpha\) is defined for \(x \in S_\alpha\) and \(y \in S_\beta\) by

\[ xy = f_{\alpha, \alpha \land \beta}(x)f_{\beta, \alpha \land \beta}(y) \]

If \(A\) is a nonempty subset of \(S\), then by \(\langle A \rangle\) we mean the subsemigroup of \(S\) generated by \(A\). The subsemigroup \(\langle A \rangle\) consists of all elements of \(S\) that can be expressed as finite products of elements of \(A\).

Let \(X, Y\) be sets. A constant mapping \(c_Y : X \rightarrow Y\) for \(y \in Y\) is defined by \(c_Y(x) = y\) for all \(x \in X\). The identity mapping \(id_X : X \rightarrow X\) is defined by \(id_X(x) = x\) for all \(x \in X\). Let \(X_1 \times X_2\) be the cartesian product of a set \(X_1\) and a set \(X_2\). The usual projections of this product onto \(X_i\) are denoted by \(p_i\), for every \(i \in \{1, 2\}\).

Let \((V_1, E_1)\) and \((V_2, E_2)\) be digraphs. A mapping \(\varphi : V_1 \rightarrow V_2\) is called a (digraph) homomorphism if \((u, v) \in E_1\) implies \((\varphi(u), \varphi(v)) \in E_2\), i.e. \(\varphi\) preserves arcs. We write \(\varphi : (V_1, E_1) \rightarrow (V_2, E_2)\). A digraph homomorphism \(\varphi : (V, E) \rightarrow (V, E)\) is called a (digraph) endomorphism. If \(\varphi : (V_1, E_1) \rightarrow (V_2, E_2)\) is a bijective digraph homomorphism and \(\varphi^{-1}\) is also a digraph homomorphism, then \(\varphi\) is called a (digraph) isomorphism. A digraph isomorphism \(\varphi : (V, E) \rightarrow (V, E)\) is called a (digraph) automorphism.

The indegree \(\overrightarrow{d}(v)\) of a vertex \(v\) of a digraph \(D\) is the number of vertices of \(D\) that end in \(v\). The outdegree \(\overleftarrow{d}(v)\) of \(v\) is the number of vertices of \(D\) that start from \(v\).

Let \(G\) be a groupoid (semigroup, group, etc.) and \(A \subseteq G\). We define the Cayley graph \(\text{Cay}(G, A)\) as follows: \(G\) is the vertex set and \((u, v), u, v \in G\), is an arc in \(\text{Cay}(G, A)\) if there exists an element \(a \in A\) such that \(v = ua\). The set \(A\) is called the connection set of \(\text{Cay}(G, A)\).

Note that by this way we do not get multiple edges in Cayley graphs. For groupoids this may in some cases imply a loss of information. Note moreover, that there exist several slightly different definitions of Cayley graphs, in particular the definition by left action of the elements of \(A\). Using right action we can write mappings (colour endomorphisms) on the left to get a biact, see [6].

A graph \(D = (V, E)\) is said to be \(\text{Aut}(D)\)-vertex-transitive if, for any two vertices \(x, y \in V\), there exists an automorphism \(\varphi \in \text{Aut}(D)\) such that \(\varphi(x) = y\). More general, a subset \(C \subseteq \text{End}(D)\) is said to act vertex-transitively on \(D\), or \(D\) is \(C\)-vertex-transitive if, for any two vertices \(x, y \in V\), there exists an endomorphism \(\varphi \in C\) such that \(\varphi(x) = y\).

Now let \(S\) be a semigroup and let \(A \subseteq S\). Denote the automorphism group and the endomorphism monoid of \(\text{Cay}(S, A)\) by \(\text{Aut}(S, A)\) and \(\text{End}(S, A)\), respectively. An element \(\varphi \in \text{End}(S, A)\) will be called a colour-preserving endomorphism if \(xa = y\) implies \(\varphi(x)a = \varphi(y)\), for every \(x, y \in S\) and \(a \in A\). Denote by \(\text{ColEnd}(S, A)\) and \(\text{ColAut}(S, A)\) the sets of all colour-preserving endomorphisms and automorphisms of \(\text{Cay}(S, A)\), respectively. Note that this way colour-preserving endomorphisms are right \(\langle A \rangle\)-act homomorphisms [6].

**Theorem 1.1** ([5, Theorem 2.1]). Let \(S\) be a semigroup, and let \(A\) be a subset of \(S\) which generates a subsemigroup \(\langle A \rangle\) such that all principal left ideals of \(\langle A \rangle\) are finite. Then, the Cayley graph \(\text{Cay}(S, A)\) is \(\text{ColAut}(S, A)\)-vertex-transitive if and only if the following conditions hold:

1. \(Sa = S\), for all \(a \in A\);
2. \(\langle A \rangle\) is a left group;
3. \(|s(A)|\) is independent of the choice of \(s \in S\).

**Theorem 1.2** ([5, Theorem 2.2]). Let \(S\) be a semigroup, and let \(A\) be a subset of \(S\) which generates a subsemigroup \(\langle A \rangle\) such that all principal left ideals of \(\langle A \rangle\) are finite. Then, the Cayley graph \(\text{Cay}(S, A)\) is \(\text{Aut}(S, A)\)-vertex-transitive if and only if the following conditions hold:

1. \(SA = S\);
2. \(\langle A \rangle\) is a completely simple semigroup;
3. the Cayley graph \(\text{Cay}(\langle A \rangle, A)\) is \(\text{Aut}(\langle A \rangle, A)\)-vertex-transitive;
4. \(|s(A)|\) is independent of the choice of \(s \in S\).

Clearly, in the situation of **Theorems 1.1 and 1.2** we always get \(\text{Aut}(S, A)\)-vertex-transitivity if \(A = \emptyset\). So, for the rest of this paper we will assume that \(A \neq \emptyset\).
2. General lemmas

Lemma 2.1 ([9, Lemma 1.2]). Let $D = (V, E)$ be finite and $\text{Aut}(D)$-vertex-transitive. Then the indegree $\overleftarrow{d}(v)$ is the same for each vertex $v$, and is equal to the outdegree $\overrightarrow{d}(v)$ of $v$.

Lemma 2.2. Let $Y$ be a finite semilattice, $S = \bigcup_{y \in Y} S_y$ a strong semilattice of semigroups, $A$ a nonempty subset of $S$, and let $x_\beta \in S_\beta$ and $y_\alpha \in S_\alpha$. If $(x_\beta, y_\alpha)$ is an arc in $\text{Cay}(S, A)$, then $\beta \geq \alpha$.

Proof. Let $(x_\beta, y_\alpha)$ be an arc in $\text{Cay}(S, A)$. Then there exists $a \in A$ such that $y_\alpha = x_\beta a$, say $a \in S_\gamma$ for some $\gamma \in Y$. Hence $y_\alpha = \sum_{\beta, \gamma \in Y}(x_\beta)f_{\gamma, \beta, \gamma}(a)$ and thus $\alpha = \beta \land \gamma$. Therefore $\beta \geq \alpha$. \hfill $\Box$

Lemma 2.3. Let $Y$ be a finite semilattice, $S = \bigcup_{y \in Y} S_y$ a strong semilattice of semigroups, $A$ a nonempty subset of $S$, and let $\beta$ be maximal in $Y$. Then, for all $v \in S_\beta$, the indegrees of $v$ in $\text{Cay}(S_\beta, A \cap S_\beta)$ and in $\text{Cay}(S, A)$ are equal.

Proof. Let $\beta$ be a maximal element of $Y$, and let $v \in S_\beta$. Then by Lemma 2.2, there is no $\alpha \neq \beta$ such that $(x_\alpha, v)$ is an arc in $\text{Cay}(S, A)$. Therefore, the indegrees of $v$ in $\text{Cay}(S_\beta, A \cap S_\beta)$ and in $\text{Cay}(S, A)$ are equal. \hfill $\Box$

Lemma 2.4. Let $Y$ be a finite semilattice, $S = \bigcup_{y \in Y} S_y$ a strong semilattice of semigroups, $A$ a nonempty subset of $S$, and let $\beta$ be a maximal element of $Y$. If $\text{Cay}(S, A)$ is $\text{Aut}(S, A)$-vertex-transitive, then $A \cap S_\beta = A$ (i.e. $A \subseteq S_\beta$).

Proof. We will prove the statement by contraposition. Suppose $A \cap S_\beta \neq A$. Consider two cases

Case 1. $A \cap S_\beta = \emptyset$. Then, in $\text{Cay}(S_\beta, A \cap S_\beta)$, $\overleftarrow{d}(v) = 0$ for all $v \in S_\beta$. Since $\beta$ is maximal in $Y$, in $\text{Cay}(S, A)$, we get $\overleftarrow{d}(v) = 0$ for all $v \in S_\beta$ by Lemma 2.3. Because of $A \neq \emptyset$, in $\text{Cay}(S, A)$, we get $\overleftarrow{d}(v) \geq 1$ for all $v \in S_\beta$. Hence $\text{Cay}(S, A)$ is not $\text{Aut}(S, A)$-vertex-transitive by Lemma 2.1.

Case 2. $A \cap S_\beta \neq \emptyset$. Since $S_\beta$ is finite, let $S_\beta = \{v_1, \ldots, v_n\}$. Then, in $\text{Cay}(S_\beta, A \cap S_\beta)$, we have $\sum_{i=1}^{n} \overleftarrow{d}(v_i) = \sum_{i=1}^{n} \overleftarrow{d}(v_i)$. By Lemma 2.3, $\sum_{i=1}^{n} \overleftarrow{d}(v_i)$ in $\text{Cay}(S_\beta, A \cap S_\beta)$ and $\sum_{i=1}^{n} \overleftarrow{d}(v_i)$ in $\text{Cay}(S, A)$ are equal. Since $A \cap S_\beta \neq \emptyset$ and $A \cap S_\beta \neq \emptyset$, there exists $a \in A \setminus (A \cap S_\beta)$, say $a \in S_\gamma$, for some $\gamma \notin Y$ and $\gamma \neq \beta$. Therefore, $(v_i, v_j, a)$ is an arc in $\text{Cay}(S, A)$ where $v_i \in S_\beta$ and $v_j a \in S_{\gamma \land \beta}$, and thus $\sum_{i=1}^{n} \overleftarrow{d}(v_i)$ in $\text{Cay}(S_\beta, A \cap S_\beta)$ is less than $\sum_{i=1}^{n} \overleftarrow{d}(v_i)$ in $\text{Cay}(S, A)$. Hence, in $\text{Cay}(S, A)$, $\sum_{i=1}^{n} \overleftarrow{d}(v_i) < \sum_{i=1}^{n} \overleftarrow{d}(v_i)$.

Lemma 2.5. Let $Y$ be a finite semilattice, $S = \bigcup_{y \in Y} S_y$ a strong semilattice of finite semigroups, $A$ a nonempty subset of $S$, and let $\text{Cay}(S, A)$ be $\text{Aut}(S, A)$-vertex-transitive. Then

(a) $Y$ has the maximum $m$, and
(b) $A \subseteq S_m$.

Proof. We will prove (a) by contraposition.

(a) Suppose $Y$ has no maximum. Since $Y$ is finite, there are maximal elements $\alpha, \beta \in Y$ such that $\alpha \neq \beta$. Consider two cases

Case 1. $S_\alpha \cap A = \emptyset$. Then $\text{Cay}(S_\beta, A \cap S_\beta)$ is not $\text{Aut}(S, A)$-vertex-transitive by Lemma 2.4.

Case 2. $S_\beta \cap A = \emptyset$. Then $S_\beta \cap A \neq \emptyset$ and $\text{Cay}(S, A)$ is not $\text{Aut}(S, A)$-vertex-transitive by Lemma 2.4.

(b) Let $m$ be the maximum of $Y$. By Lemma 2.4 again, we get $A \subseteq S_m$. \hfill $\Box$

Example 5.1. Figs. 2 and 3 show that (a) and (b) are not sufficient for $\text{Cay}(S, A)$ to be $\text{Aut}(S, A)$-vertex-transitive.

Lemma 2.6. Let $S = G \times R_n$ be a finite right group where $G$ is a group, $R_n = \{r_1, r_2, \ldots, r_n\}$ a right zero semigroup, and take $A \subseteq S$ nonempty. Then $(A) = (G') \times R$ is a right group contained in $S$ where $G' \subseteq G$ and $R \subseteq R_n$.

Proof. Let $G' = \{g \in G|(g, r) \in A$ for some $r \in R_n\}$ and $R = \{r \in R_n |(g, r) \in A$ for some $g \in G\}$. We show that $(A) = (G') \times R$. Take $x, y \in (G') \times R$. Then $x = (g_1, g_2, \ldots, g_n, r)$ and $y = (g'_1, g'_2, \ldots, g'_n, r')$ for some $g_i, g'_i \in G'$, $r, r' \in R$. Hence, $xy = (g_1g_2, \ldots, g_ng_n, g'_1g'_2, \ldots, g'_ng'_n, rr') \in (G') \times R$, and thus $(G') \times R$ is a right group contained in $S$. It is clear that $A \subseteq (G') \times R$. Thus $(A) \subseteq (G') \times R$. Take $z \in (G') \times R$. Then $z = (g_1, g_2, \ldots, g_n, r)$ for some $g_i \in G'$, $r \in R$. Choose $g' \in G$ such that $(g', r) \in A$. Since $G$ is a finite group, there exists $q \in \mathbb{N}$ such that $(g')^q = e$ where $e$ is an identity of $G$. Since $g_i \in G$, there exists $b_i \in R$ such that $(g_i, b_i) \in A$. Hence,

$$z = (g_1g_2, \ldots, g_n, r) = (g_1, g_2, \ldots, g_n, e, r) = (g_1, g_2, \ldots, g_n, (g')^q, r) = ((g_1, b_1)(g_2, b_2) \cdots (g_n, b_n)(g')^q, r) = (g_1, b_1)(g_2, b_2) \cdots (g_n, b_n) (g', r)(g', r) \cdots (g', r) \in (A),$$

and thus $(G') \times R \subseteq (A)$. \hfill $\Box$
As a left dual, we have

**Lemma 2.7.** Let \( S = G \times L_0 \) be a finite left group where \( G \) is a group, \( L_0 = \{ l_1, l_2, \ldots, l_n \} \) a left zero semigroup, and let \( A \) be any nonempty subset of \( S \). Then \( \langle A \rangle = \langle G^r \rangle \times L \) is a left group contained in \( S \) where \( G^r \subseteq G \) and \( L \subseteq L_0 \).

**Lemma 2.8.** Let \( Y \) be a finite semilattice, \( S = \bigcup_{\alpha \in Y} S_\alpha \) a strong semilattice of right groups such that \( S_\alpha = G_\alpha \times L_\alpha \) where \( G_\alpha \) are groups, \( L_\alpha = \{ l_1^\alpha, l_2^\alpha, \ldots, l_n^\alpha \} \) right zero semigroups, \( Y \) has the maximum \( m, A \subseteq S_m \), and let \( \alpha \in Y \). Then, for all \( s \in S_\alpha \), \( |s(A)| = |f_{m,\alpha}(\langle A \rangle)| \).

**Proof.** Let \( s = (g, r) \in S_\alpha \). Since \( \langle A \rangle \) is a right group and a subsemigroup of \( S_m \), we have \( f_{m,\alpha}(\langle A \rangle) = \langle G^r \rangle \times L \), where \( G^r \subseteq G, L \subseteq L_0 \) by **Lemma 2.6**. Therefore,

\[
|s(A)| = |(g, r) f_{m,\alpha}(\langle A \rangle)| = |g (G^r) \times L| = |(G^r) \times L| = |f_{m,\alpha}(\langle A \rangle)|. \quad \square
\]

Note that the result of the following lemma is not left dual to **Lemma 2.8** in the direct sense.

**Lemma 2.9.** Let \( Y \) be a finite semilattice, \( S = \bigcup_{\alpha \in Y} S_\alpha \) a strong semilattice of left groups such that \( S_\alpha = G_\alpha \times L_\alpha \) where \( G_\alpha \) are groups, \( L_\alpha = \{ l_1^\alpha, l_2^\alpha, \ldots, l_n^\alpha \} \) left zero semigroups, \( p_1 \) the first projection, \( Y \) has the maximum \( m, A \subseteq S_m \), and let \( \alpha \in Y \). Then, for all \( s \in S_\alpha \), \( |s(A)| = |p_1 f_{m,\alpha}(\langle A \rangle)| \).

**Proof.** Let \( s = (g, l) \in S_\alpha \). Since \( \langle A \rangle \) is a left subgroup of \( S_m \), we have \( f_{m,\alpha}(\langle A \rangle) = \langle G^r \rangle \times L \), where \( G^r \subseteq G, L \subseteq L_0 \) by **Lemma 2.7**. Therefore,

\[
|s(A)| = |(g, l) f_{m,\alpha}(\langle A \rangle)| = |g (G^r) \times L_\alpha| = |(G^r) \times L_\alpha| = |p_1 f_{m,\alpha}(\langle A \rangle)|. \quad \square
\]

### 3. Right groups

In this section we consider strong semilattices of right groups with automorphism vertex transitive Cayley graph.

**Lemma 3.1.** Let \( S \) be a finite right zero semigroup and let \( A \) be a nonempty subset of \( S \). Then \( \text{Cay}(S, A) \) is \( \text{Aut}(S, A) \)-vertex-transitive if and only if \( A = S \).

**Proof.** (\( \Leftarrow \)) Suppose \( S = A \). Since \( S \) is a right zero semigroup, we get \( xy = y \) for all \( x, y \in S \). Hence \( \text{Cay}(S, A) \) is a complete digraph in which every vertex has a loop. Let \( u, v \in S \). We will prove that there exists \( \varphi \in \text{Aut}(\text{Cay}(S, A)) \) such that \( \varphi(u) = v \).

Define

\[
\varphi(x) = \begin{cases} 
  u & \text{if } x = v \\
  v & \text{if } x = u \\
  x & \text{otherwise.}
\end{cases}
\]

It is clear that \( \varphi \) is a bijection and that \( \varphi = \varphi^{-1} \) preserves arcs because \( \text{Cay}(S, A) \) is a complete digraph in which every vertex has a loop. Hence \( \text{Cay}(S, A) \) is \( \text{Aut}(S, A) \)-vertex-transitive.

(\( \Rightarrow \)) We will prove that \( \text{Cay}(S, A) \) is not \( \text{Aut}(S, A) \)-vertex-transitive, if \( A \neq S \). Suppose \( A \neq S \). Then there exists \( s_0 \in S \setminus A \). Since \( S \) is a right zero semigroup, we get \( s_0A = A \) and thus \( (s_0, a) \) is an arc in \( \text{Cay}(S, A) \) for all \( a \in A \). Hence \( d(s_0) = |A| > 1 \).

Since \( s_0 \notin A = s_0A \), we have \( d(s_0) = 0 \). Now \( \text{Cay}(S, A) \) is not \( \text{Aut}(S, A) \)-vertex-transitive by **Lemma 2.1**. \( \square \)

**Theorem 3.2** ([9, Theorem 3.1]). Let \( Y \) be a finite semilattice, \( S = \bigcup_{\alpha \in Y} G_\alpha \) a strong semilattice of groups, and let \( A \) be a nonempty subset of \( S \). Then the \( \text{Cay}(S, A) \) is \( \text{Aut}(S, A) \)-vertex-transitive if and only if

(a) \( Y \) has the maximum \( m \),
(b) \( A \subseteq G_m \), and
(c) the restrictions of \( f_{m,\alpha} \) to \( \langle A \rangle \) are injections for all \( \alpha \in Y \).
Now we consider $\text{ColAut}(S, A)$-vertex-transitive Cayley graphs.

**Theorem 3.3.** Let $Y$ be a finite semilattice, $S = \bigcup_{a \in Y} G_a$ a strong semilattice of groups, and let $A$ be a nonempty subset of $S$. Then the following conditions are equivalent.

(i) $Y$ has the maximum $m$,
(ii) $A \subseteq S_m$, and
(iii) the restrictions of $f_{m, a}$ to $|A|$ are injections for all $a \in Y$; 
(iv) $\text{Cay}(S, A)$ is $\text{ColAut}(S, A)$-vertex-transitive;
(v) $\text{Cay}(S, A)$ is $\text{Aut}(S, A)$-vertex-transitive.

Proof. (i)$\Rightarrow$(ii) We prove (1)–(3) of Theorem 1.1 for $S$, then the result follows.

(1) Let $a \in A$ and $\alpha \in Y$. Since $A \subseteq S_m$, we get $f_{m, a}(a) \in G_a$, and thus $G_a a = G_a f_{m, a}((a)) = G_a$. Therefore $S_a = (\bigcup_{\alpha \in Y} G_a) a = \bigcup_{\alpha \in Y} (G_a a) = \bigcup_{\alpha \in Y} G_a = S$ since the $S_a$ are distinct.

(2) Since $A \subseteq S_m$, we get that $(A)$ is a subgroup of $G_m$ and, in particular, a left group.

(3) Let $s, s' \in S$. Then $s \in G_a$ and $s' \in G_b$ for some $\alpha, \beta \in Y$. By (c), we get $|s(A)| = |s(f_{m, a}((A)))| = |f_{m, a}((A))| = |A|$ and $|s'(A)| = |s'(f_{m, b}((A)))| = |f_{m, b}((A))| = |A|$. Hence $|s(A)| = |s'(A)|$.

(ii)$\Rightarrow$(iii) Obvious.

(iii)$\Rightarrow$(i) By Theorem 3.2. \hfill \Box

The following theorem shows that the Cayley graph of a strong semilattice of right groups is $\text{ColAut}(S, A)$-vertex-transitive only if it is a strong semilattice of groups.

**Theorem 3.4.** Let $Y$ be a finite semilattice, $S = \bigcup_{a \in Y} S_a$ a strong semilattice of right groups such that $S_a = G_a \times R_{n_a}$ where $G_a$ are groups, $R_{n_a} = \{r_1^a, r_2^a, \ldots, r_{n_a}^a\}$ right zero semigroups, and let $A$ be a nonempty subset of $S$.

If the Cayley graph $\text{Cay}(S, A)$ is $\text{ColAut}(S, A)$-vertex-transitive, then $|R_{n_a}| = 1$ for all $a \in Y$.

Proof. By Lemma 2.5, there exists the maximum $m$ of $Y$ and $A \subseteq S_m$. We will prove, by contraposition, that the Cayley graph $\text{Cay}(S, A)$ is not $\text{ColAut}(S, A)$-vertex-transitive, if there exists $\beta \in Y$ such that $|R_{n_\beta}| \neq 1$. Suppose that there exists $\beta \in Y$ such that $|R_{n_\beta}| \neq 1$. Then we have $S_\beta = G_\beta \times R_{n_\beta}$ with $|R_{n_\beta}| > 1$. Choose $a \in A \subseteq S_m$. Then $f_{m, \beta}(a) = (g, r) \in S_\beta$ for some $g \in G_\beta, r \in R_{n_\beta}$. Hence,

\[
S_\beta a = \left(G_\beta \times R_{n_\beta}\right) f_{m, \beta}(a) \\
= \left(G_\beta \times R_{n_\beta}\right) (g, r) \\
= G_\beta g \times R_{n_\beta} r \\
= G_\beta \times \{r\} \\
= G_\beta \times R_{n_\beta}, \quad \text{since } |R_{n_\beta}| > 1, \\
= S_\beta,
\]

and thus $S_a = \left(\bigcup_{\alpha \in Y} S_a\right) a = \bigcup_{\alpha \in Y} (S_a a) \neq \bigcup_{\alpha \in Y} S_a = S$ since the $S_a$ are distinct. By Theorem 1.1, we get $\text{Cay}(S, A)$ is not $\text{ColAut}(S, A)$-vertex-transitive. \hfill \Box

**Example 3.2** will illustrate this result.

**Corollary 3.5.** Let $Y$ be a finite semilattice, $S = \bigcup_{a \in Y} S_a$ a strong semilattice of right zero semigroups, and let $A$ be a nonempty subset of $S$. If the Cayley graph $\text{Cay}(S, A)$ is $\text{ColAut}(S, A)$-vertex-transitive, then $|R_{n_a}| = 1$ for all $a \in Y$.

Proof. It is clear that right zero semigroups are in particular right groups since $S_a \cong G \times S_a$ for all $a \in Y$, where $G$ is the trivial group.

By Theorem 3.4, we obtain that $|R_{n_a}| = 1$ for all $a \in Y$. \hfill \Box

**Corollary 3.6.** Let $S = G \times R_a$ be a finite right group, and let $A$ be a nonempty subset of $S$. If $\text{Cay}(S, A)$ is $\text{ColAut}(S, A)$-vertex-transitive, then $S$ is a group, i.e. $|R_{n_a}| = 1$.

Proof. This is a direct consequence of Theorem 3.4. \hfill \Box

Now we consider $\text{Aut}(S, A)$-vertex-transitive Cayley graphs of a strong semilattice of right groups.

**Theorem 3.7.** Let $Y$ be a finite semilattice, $S = \bigcup_{a \in Y} S_a$ a strong semilattice of right groups such that $S_a = G_a \times R_{n_a}$ where $G_a$ are groups, $R_{n_a} = \{r_1^a, r_2^a, \ldots, r_{n_a}^a\}$ right zero semigroups, $p_2$ the second projection, and let $A$ be a nonempty subset of $S$. Then the Cayley graph $\text{Cay}(S, A)$ is $\text{Aut}(S, A)$-vertex-transitive if and only if the following conditions hold:

(a) $Y$ has the maximum $m$,
(b) $A \subseteq S_m$. 


Lemma 2.8

Theorem 1.2

Theorem 1.2

Lemma 3.1

Let $Y$ be a finite semilattice, $S \subseteq S_m$, we get

$$S_\alpha A = (G_\alpha \times R_{\alpha \beta})_{f_{m,\alpha}}(A)$$

$$= G_\alpha \times R_{\alpha \beta} \text{ by (c)}$$

$$= S_\alpha \text{ for all } \alpha \in Y.$$ 

Therefore, $SA = \bigcup_{\alpha \in Y} S_\alpha A = \bigcup_{\alpha \in Y} (G_\alpha \times R_{\alpha \beta})_{f_{m,\alpha}}(A) = \bigcup_{\alpha \in Y} S_\alpha = S$ since the $S_\alpha$ are distinct.

(2) Since $A \subseteq S_m$, we obtain that $\langle A \rangle$ is a right group and a subsemigroup of $S_m$ and, in particular, a completely simple semigroup.

(3) This is (e).

(4) Let $s, s' \in S$. Then $s \in S_\alpha$ and $s' \in S_\beta$ for some $\alpha, \beta \in Y$. By Lemma 2.8, we get $|s(A)| = |f_{m,\alpha}(\langle A \rangle)|$ and $|s'(A)| = |f_{m,\beta}(\langle A \rangle)|$. Hence $|s(A)| = |s'(A)|$ by (d).

(⇒) From Lemma 2.5 it is clear that (a) and (b) are necessary. Let $m$ be the maximum of $Y$, $A \subseteq S_m$. We will prove, by contraposition, that the Cayley graph $Cay(S, A)$ is not $\text{Aut}(S, A)$-vertex-transitive, if

(a) there exists $\beta \in Y$ such that $|p_2(f_{m,\beta}(A))| < |R_{\alpha \beta}|$, or

(b) there exists $\beta \in Y$ such that the restriction of $f_{m,\beta}$ to $\langle A \rangle$ is not an injection, or

γ) the Cayley graph $Cay(S, A)$ is not $\text{Aut}(\langle A \rangle, A)$-vertex-transitive.

(a) Suppose that there exists $\beta \in Y$ such that $|p_2(f_{m,\beta}(A))| < |R_{\alpha \beta}|$. Then, $f_{m,\beta}(\langle A \rangle) = (G_{\beta} \times R_{\beta \beta})_{f_{m,\beta}}(A)$ where $G_{\beta} \subseteq G_{\alpha \beta}, R_{\beta \beta} \subseteq R_{\alpha \beta}$ by Lemma 2.6. Hence,

$$S_{\beta} A = (G_{\beta} \times R_{\beta \beta})_{f_{m,\beta}}(A)$$

$$= (G_{\beta} \times R_{\beta \beta}), \text{ since } |p_2(f_{m,\beta}(A))| < |R_{\alpha \beta}|.$$ 

and thus $SA = \bigcup_{\alpha \in Y} S_\alpha A = \bigcup_{\alpha \in Y} (G_\alpha \times R_{\alpha \beta})_{f_{m,\alpha}}(A)$ because the $S_\alpha$ are distinct. By Theorem 1.2(1), we obtain that $\text{Cay}(S, A)$ is not $\text{Aut}(S, A)$-vertex-transitive.

(β) Let now $|p_2(f_{m,\alpha}(A))| = |R_{\alpha \beta}|$ for all $\alpha \in Y$, and suppose that there exists $\beta \in Y$ such that the restrictions of $f_{m,\beta}$ to $\langle A \rangle$ is not an injection. Then $|\langle A \rangle| > |f_{m,\beta}(\langle A \rangle)|$. Let $s \in S_m$ and $s' \in S_\beta$. By Lemma 2.8, we have $|s(A)| = |\langle A \rangle|$ and $|s'(A)| = |f_{m,\beta}(\langle A \rangle)|$. Therefore $|s(A)| \neq |s'(A)|$. By Theorem 1.2(4), we obtain that the Cayley graph $Cay(S, A)$ is not $\text{Aut}(S, A)$-vertex-transitive.

(γ) By Theorem 1.2(3). □

Example 5.3 will illustrate the situation.

Parallel to Corollaries 3.5 and 3.6 we specialise the preceding theorem.

Corollary 3.8. Let $Y$ be a finite semilattice, $S = \bigcup_{\alpha \in Y} R_\alpha$ a strong semilattice of right zero semigroups, and let $A$ be a nonempty subset of $S$. Then the Cayley graph $Cay(S, A)$ is $\text{Aut}(S, A)$-vertex-transitive if and only if the following conditions hold:

(a) $Y$ has the maximum $m$,
(b) $A = R_m$, and
(c) $f_{m,\alpha}$ is an isomorphism for all $\alpha \in Y$.

Proof. ($\Leftarrow$) Since $A = S_m$ and $f_{m,\alpha}$ is an isomorphism for all $\alpha \in Y$, we get that the restrictions of $f_{m,\alpha}$ to $\langle A \rangle = R_m$ are injections. By Lemma 3.1, we have that the Cayley graph $\text{Cay}(\langle A \rangle, A)$ is $\text{Aut}(\langle A \rangle, A)$-vertex-transitive. Therefore the Cayley graph $Cay(S, A)$ is $\text{Aut}(S, A)$-vertex-transitive by Theorem 3.7

($\Rightarrow$) Obvious from Theorem 3.7. □

Corollary 3.9. Let $S = G \times R_\alpha$ be a finite right group, $A$ a nonempty subset of $S$, and let $p_2$ be the second projection. Then $\text{Cay}(S, A)$ is $\text{Aut}(S, A)$-vertex-transitive if and only if the following conditions hold:

(a) $p_2(A) = R_\alpha$, and
(b) the Cayley graph $\text{Cay}(\langle A \rangle, A)$ is $\text{Aut}(\langle A \rangle, A)$-vertex-transitive.

Proof. This is a direct consequence of Theorem 3.7. □

From [9, Theorem 2.3], we already known that if $A = (b) \times R_\alpha$, then $\text{Cay}(S, A)$ is $\text{Aut}(S, A)$-vertex-transitive, which is a part of Corollary 3.9.
4. Left groups

Now we consider left groups instead of right groups.

Theorem 4.1. Let $Y$ be a finite semilattice, $S = \bigcup_{\alpha \in Y} S_{\alpha}$ a strong semilattice of left groups such that $S_{\alpha} = G_\alpha \times L_{n_\alpha}$ where $G_\alpha$ are groups, $L_{n_\alpha} = \{ r_1^\alpha, r_2^\alpha, \ldots, r_{n_\alpha}^\alpha \}$ left zero semigroups, $p_1$ the first projection, and let $A$ be a nonempty subset of $S$. Then the following conditions are equivalent.

(i) (a) $Y$ has the maximum $m$,
(b) $A \subseteq S_m$, and
(c) $|p_1((A))| = |p_1(f_{m,\alpha}((A)))|$ for all $\alpha \in Y$;
(ii) $\text{Cay}(S, A)$ is $\text{ColAut}(S, A)$-vertex-transitive;
(iii) $\text{Cay}(S, A)$ is $\text{Aut}(S, A)$-vertex-transitive.

Proof. (i) $\Rightarrow$ (ii) We prove (1)–(3) of Theorem 1.1 for $S$, then the result follows.

1. Let $a \in A$ and $\alpha \in Y$. Since $A \subseteq S_m, f_{m,\alpha}(a) = (g, l) \in S_\alpha$ for some $g \in G_\alpha, l \in L_{n_\alpha}$, and thus

$$S_\alpha x = (G_\alpha \times L_{n_\alpha}) f_{m,\alpha}((a)) = (G_\alpha \times L_{n_\alpha})(g, l) = G_\alpha g \times L_{n_\alpha} l = G_\alpha \times L_{n_\alpha} = S_\alpha.$$

Therefore $S_\alpha = (\bigcup_{\alpha \in Y} S_\alpha) a = \bigcup_{\alpha \in Y} (S_\alpha a) = \bigcup_{\alpha \in Y} S_\alpha = S$ since the $S_\alpha$ are distinct.

2. Since $A \subseteq S_m$, we obtain that $(A)$ is a left group and a subsemigroup of $S_m$ by Lemma 2.7.

3. Let $s, s' \in S$. Then $s \in S_\alpha$ and $s' \in S_\beta$ for some $\alpha, \beta \in Y$. By Lemma 2.9, we get $|s(A)| = |p_1(f_{m,\alpha}((A)))|$ and $|s'(A)| = |p_1(f_{m,\beta}((A)))|$. By (c), we obtain $|s(A)| = |s'(A)|$.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (i) We know from Lemma 2.5 that (a) and (b) are necessary. We will prove, by contraposition, that the Cayley graph $\text{Cay}(S, A)$ is not $\text{Aut}(S, A)$-vertex-transitive, if there exists $\beta \in Y$ such that $|p_1((A))| \neq |p_1(f_{m,\beta}((A)))|$. Let $m$ be the maximum of $Y, A \subseteq S_m$, and suppose that there exists $\beta \in Y$ such that $|p_1((A))| \neq |p_1(f_{m,\beta}((A)))|$. Let $s \in S_m$ and $s' \in S_\beta$. By Lemma 2.9, we get $|s(A)| = |p_1((A))|$ and $|s'(A)| = |p_1(f_{m,\beta}((A)))|$. Therefore $|s(A)| \neq |s'(A)|$. By Theorem 1.2(3), we obtain that the Cayley graph $\text{Cay}(S, A)$ is not $\text{Aut}(S, A)$-vertex-transitive.

Example 5.4 will illustrate the situation.

The preceding result is again specialised now for strong semilattices of left zero semigroups.

Corollary 4.2. Let $Y$ be a finite semilattice, $S = \bigcup_{\alpha \in Y} L_{n_\alpha}$ a strong semilattice of left zero semigroups, and let $A$ be a nonempty subset of $S$. Then the following conditions are equivalent.

(i) (a) $Y$ has the maximum $m$, and
(b) $A \subseteq L_{m}$;
(ii) $\text{Cay}(S, A)$ is $\text{ColAut}(S, A)$-vertex-transitive;
(iii) $\text{Cay}(S, A)$ is $\text{Aut}(S, A)$-vertex-transitive.

Proof. (i) $\Rightarrow$ (ii) Since now $|p_1((A))| = |p_1(f_{m,\alpha}((A)))| = 1$, we get $\text{Cay}(S, A)$ is $\text{ColAut}(S, A)$-vertex-transitive by Theorem 4.1.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (i) This follows from Theorem 4.1. □

Corollary 4.3. Let $S = G \times L_m$ be a finite left group where $G$ is a group, $L_m = \{ l_1, l_2, l_3, \ldots, l_m \}$, and let $A$ be any nonempty subset of $S$. Then the Cayley graph $\text{Cay}(S, A)$ is always $\text{ColAut}(S, A)$-vertex-transitive (and thus $\text{Aut}(S, A)$-vertex-transitive).

Remark 4.4. Let $S_\alpha$ be groups for all $\alpha \in Y$. Then $(A)$ and $f_{m,\alpha}(A)$ are groups, and thus $|p_1((A))| = |(A)|$ and $|p_1(f_{m,\alpha}((A)))| = |f_{m,\alpha}((A))|$. Therefore, we have that Theorem 3.3 is also a corollary of Theorem 4.1.

5. Examples

In all examples we consider the semilattice $Y = \{ \alpha, \beta, \gamma \}$. The index $\alpha$ denotes elements from the semigroup in the bottom, the index $\beta$ denotes elements from the top left semigroup, and $\gamma$ from the top right semigroup, moreover 0 (with the respective index) denotes the identity element of the respective group.

Example 5.1. Let $S_\alpha = R_3 = \{ r_1, r_2, r_3 \}$, $S_\beta = \mathbb{Z}_2 \times R_2 = \{ (0_\beta, r_1), (0_\beta, r_2), (1_\beta, r_1), (1_\beta, r_2) \}$, $S_\gamma = R_2 = \{ r_1^\gamma, r_2^\gamma \}$. Take $p_2 = f_{\beta,\alpha} : S_\beta \to S_\alpha$, and define $f_{\gamma,\alpha} : S_\gamma \to S_\alpha$ by $f_{\gamma,\alpha}(r_1^\gamma) = r_1$, and $f_{\gamma,\alpha}(r_2^\gamma) = r_2$. Then $S = \bigcup_{\alpha \in Y} S_{\alpha}$ is a strong semilattice of right groups (Fig. 1).
In Fig. 2, we don’t have (a) from Lemma 2.5, and we see that Cay(S, A) is not Aut(S, A)-vertex-transitive. If we take Y = {α, β}, i.e. we take off r'₁, r'₂ and the respective two arcs, then we have (a) but not (b) from Lemma 2.5. The resulting figure shows that Cay(S, A) is not Aut(S, A)-vertex-transitive. In Fig. 3, we have (a) and (b) from Lemma 2.5, and we see that Cay(S, A) is not Aut(S_α ∪ S_β, A)-vertex-transitive.

**Example 5.2.** Let S_α = Z_2 = {0_α, 1_α}, S_β = Z_2 × R_2 = {(0_β, r_1), (0_β, r_2), (1_β, r_1), (1_β, r_2)}, S_γ = Z_2 × Z_2 = {(0, 0)_γ, (0, 1)_γ, (1, 0)_γ, (1, 1)_γ}, where (0, 0)_γ is the identity of Z_2 × Z_2. Take p_1 = f_{β, α} : S_β → S_α, and p_2 = f_{γ, α} : S_γ → S_α. Then S = ∪_{α∈Y} S_α a strong semilattice of right groups (Fig. 4).
In Fig. 5, we have $S_β = Z_2 \times R_2$ such that $|R_2| > 1$, and we see that Cay$(S, A)$ is not Aut$(S, A)$-vertex-transitive and thus not ColAut$(S, A)$-vertex-transitive which explains Theorem 3.4. In Fig. 6, we have (a), (b), and (c) from Theorem 3.3, and we see that Cay$(S, A)$ is ColAut$(S, A)$-vertex-transitive.

Example 5.3. Let $S_α = Z_2 \times R_2 = \{(0_α, r_1), (0_α, r_2), (1_α, r_1), (1_α, r_2)\}$, $S_β = Z_2 \times R_2 = \{(0_β, r_1), (0_β, r_2), (1_β, r_1), (1_β, r_2)\}$, $S_γ = Z_2 \times R_2 = \{(0_γ, r_1), (0_γ, r_2), (1_γ, r_1), (1_γ, r_2)\}$.

Define the map $f_{β,α} : S_β \rightarrow S_α$ by

\[
\begin{align*}
f_{β,α}((0_β, r_1)) &= (0_α, r_1) \\
f_{β,α}((0_β, r_2)) &= (0_α, r_2) \quad (f_{β,α} \text{ is not an injection}) \\
f_{β,α}((1_β, r_1)) &= (0_α, r_1) \\
f_{β,α}((1_β, r_2)) &= (0_α, r_2), \quad \text{i.e. } f_{β,α} = c_0 \times id_{R_2},
\end{align*}
\]

and $f_{γ,α} : S_γ \rightarrow S_α$ by

\[
\begin{align*}
f_{γ,α}((0_γ, r_1)) &= (0_α, r_1) \\
f_{γ,α}((0_γ, r_2)) &= (0_α, r_2) \quad (f_{γ,α} \text{ is an isomorphism}) \\
f_{γ,α}((1_γ, r_1)) &= (1_α, r_1) \\
f_{γ,α}((1_γ, r_2)) &= (1_α, r_2), \quad \text{i.e. } f_{γ,α} = id_{Z_2} \times id_{R_2}.
\end{align*}
\]

Then $S = \bigcup_{α \in Y} S_α$ is a strong semilattice of right groups (Fig. 7).

In Fig. 8, we have (a), (b), (c), (e), and not (d) from Theorem 3.7, and we see that Cay$(S, A)$ is not Aut$(S, A)$-vertex-transitive.

In Fig. 9, we have (a), (b), (c) and (d), but not (e) from Theorem 3.7, and we see that Cay$(S, A)$ is not Aut$(S, A)$-vertex-transitive.

In Fig. 10, we have (a), (b), (c), (d) and (e) from Theorem 3.7, and we see that Cay$(S, A)$ is Aut$(S, A)$-vertex-transitive but not ColAut$(S, A)$-vertex-transitive.
Example 5.4. Let \( S_\alpha = \mathbb{Z}_2 \times L_2 = \{(0_\alpha, l_1), (0_\alpha, l_2), (1_\alpha, l_1), (1_\alpha, l_2)\} \), \( S_\beta = \mathbb{Z}_2 \times L_2 = \{(0_\beta, l_1), (0_\beta, l_2), (1_\beta, l_1), (1_\beta, l_2)\} \), \( S_\gamma = \mathbb{Z}_2 \times L_2 = \{(0_\gamma, l_1), (0_\gamma, l_2), (1_\gamma, l_1), (1_\gamma, l_2)\} \).

Define the map \( f_{\beta, \alpha} : S_\beta \to S_\alpha \) by

\[
\begin{align*}
f_{\beta, \alpha}((0_\beta, l_1)) &= (0_\alpha, l_1) \\
f_{\beta, \alpha}((0_\beta, l_2)) &= (0_\alpha, l_2) \quad (f_{\beta, \alpha} \text{ is not an injection}) \\
f_{\beta, \alpha}((1_\beta, l_1)) &= (0_\alpha, l_1) \\
f_{\beta, \alpha}((1_\beta, l_2)) &= (0_\alpha, l_2) \quad \text{i.e. } f_{\beta, \alpha} = c_0 \times id_{L_2},
\end{align*}
\]

and \( f_{\gamma, \alpha} : S_\gamma \to S_\alpha \) by

\[
\begin{align*}
f_{\gamma, \alpha}((0_\gamma, l_1)) &= (0_\alpha, l_1) \\
f_{\gamma, \alpha}((0_\gamma, l_2)) &= (0_\alpha, l_2) \quad (f_{\gamma, \alpha} \text{ is an isomorphism}) \\
f_{\gamma, \alpha}((1_\gamma, l_1)) &= (1_\alpha, l_1) \\
f_{\gamma, \alpha}((1_\gamma, l_2)) &= (1_\alpha, l_2) \quad \text{i.e. } f_{\gamma, \alpha} = id_{\mathbb{Z}_2} \times id_{L_2}.
\end{align*}
\]

Then \( S = \bigcup_{\alpha \in Y} S_\alpha \) is a strong semilattice of left groups (Fig. 11).

In Fig. 12, we have (a), (b), and not (c) from Theorem 4.1, and we see that \( \text{Cay}(S_\alpha \cup S_\beta, A) \) is not \( \text{Aut}(S_\alpha \cup S_\beta, A) \)-vertex-transitive. In Fig. 13, we have (a), (b), and (c) from Theorem 4.1, and we see that \( \text{Cay}(S_\alpha \cup S_\gamma, A) \) is \( \text{ColAut}(S_\alpha \cup S_\gamma, A) \)-vertex-transitive and thus also \( \text{Aut}(S_\alpha \cup S_\gamma, A) \)-vertex-transitive.
Fig. 12. $\text{Cay}(S_\alpha \cup S_\beta, \{(0_\beta, l_1), (1_\beta, l_2)\})$.

Fig. 13. $\text{Cay}(S_\alpha \cup S_\gamma, \{(0_\gamma, l_1), (1_\gamma, l_1)\})$: thickline, $\text{Cay}(S_\gamma, \{(0_\gamma, l_1), (1_\gamma, l_2)\})$: thinline.

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