# On the Structure of Counterexamples to Symmetric Orderings for BDD's

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**Abstract**

Binary Decision Diagrams (BDDs) are used to represent boolean functions in a variety of applications. The size of a reduced ordered BDD depends on the ordering of variables. Several researchers have suggested grouping symmetric variables as a promising heuristic for finding good orderings. In this paper we study the conjecture which states that symmetric variables gather in at least one of the optimum variable orders. First, we prove some useful properties of partially symmetric functions. Next, we develop a faster procedure for finding counterexamples to this conjecture that exploits the partitioning of boolean functions into nn-equivalence classes. Third, we study the structure of counterexamples and devise a new and simple method to generate new counterexamples from given counterexamples. Finally, we present different kinds of counterexamples, which show that boolean functions are very diverse with respect to where symmetric orders can fall in the range from optimal orders to worst-case orders.

## 1 Introduction

Binary Decision Diagrams (BDDs) are used to represent boolean functions in a variety of applications. The size of a BDD is known to be sensitive to a chosen variable order. In search of good variable orders, a variant of sifting-based reordering algorithms heuristically applies the aggregate criterion of symmetry and keeps symmetric variables together in groups. Groups of symmetric variables are then moved at a time during the sifting procedure. An instance of this variant is *symmetric sifting* which is described in [PS94][PS95].

Heuristic strategies of this sort normally produce good results as reported in [PS95]. Nevertheless, failures to result in optimums occur in some cases. In [PS95] the function $f(x, y, z) = xy + yz' + y'z$ was given as a counterexample

[1] Research supported in part by NSF grant CCR 9732186

for which *"no symmetric order is optimal"* and it was later referred to as such by [S98] and [LM00]. Unfortunately we must point out that it is wrong to regard function $f$ as a counterexample. The function $f$ is partially symmetric in variables $y$ and $z$. Yet two of the symmetric orders of $f$ are in fact optimum orders. Namely, symmetric orders $< x\ y\ z >$ and $< x\ z\ y >$ are both optimum orders. Thus the function $f$ is actually a positive example and should not be mistaken as a counterexample. The function $f$ is repeated in Example 2.1 and all the six variable orders are listed in Table 1. The reader may look up the table and verify that two of the symmetric orders are actually optimum orders.

Albeit a mistake, it gives rise to the question: how well do symmetric variables contribute to optimum variable orders? Thus we have formed the conjecture: *"symmetric variables gather in at least one optimum variable order."* In this paper we study this conjecture in detail. First, we prove some useful properties of partially symmetric functions. Next, we develop a faster procedure for finding counterexamples to this conjecture by partitioning boolean functions into nn-equivalence classes. Third, we study the structure of counterexamples and devise a new and simple method to generate new counterexamples from given counterexamples. Finally, we present different kinds of counterexamples, which show that boolean functions are very diverse with respect to where symmetric orders can fall in the range from optimal orders to worst-case orders. Recently, we discovered that a counterexample has been reported in [MMD94]. However, they do not present any details on the structures and properties of counterexamples.

This paper is organized as follows: We review some basic ideas on BDDs in section 2. Section 3 explores the aggregation criterion symmetric variables. Sub-section 3.1 reviews related concepts on symmetry and sub-section 3.2 elaborates on how to form symmetric groups. Section 4 presents theorems about the properties of partially symmetric functions. In section 5, we describe the search method for counterexamples. Functions are partitioned into equivalence classes to improve upon the search method. Section 6 describes the structure of counterexamples and a simple method to generate new counterexamples from existing counterexamples. Counterexamples are reported in section 7, followed by positive examples in section 8. Section 9 briefs our conclusions. In Appendix 1 more counterexamples are listed, and Appendix 2 describes additional results on symmetric orders.

## 2 Basics on BDDs

Let $B$ denote the boolean set $\{0, 1\}$. Boolean variables can assume values from $B = \{0, 1\}$. A literal is either a variable $x$ or its negation $\bar{x}$. A product is a set of literals that does not contain a literal and its negation. A minterm over a set of variables $A = \{x_1, \ldots, x_n\}$ is a product that contains either positively or negatively all variables of $A$. A boolean function of $n$ variables is a mapping $f$ :

$B^n \to B$. We express boolean functions in sum of minterm list form and omit sum of product form whenever space saving is an important consideration. The minterms can be denoted in the following way. If $(i_1, \ldots, i_n)$ is an $n$-tuple of zeroes and ones, then $x_1^{i_1} \ldots x_n^{i_n}$ is a typical minterm, where $x_j^{i_j} = x_j'$ if $i_j = 0$ and $x_j$ otherwise. When $n$ is clear from the context, we write such an $n$-tuple as a decimal number.

A BDD representing a boolean function $f$ is a single rooted directed acyclic graph where the Shannon decomposition

$$f = \bar{x}_i f_{x_i=0} + x_i f_{x_i=1} \ (1 \leq i \leq n)$$

is carried out in each node [A78][L59].

The graph has one root and has two terminal nodes labeled by a boolean constant 0 or 1. Each internal node is labeled by a boolean variable $x_i$ and has two outgoing edges labeled 0 and 1 corresponding to the cases where the variable evaluates to 0 or 1. The two successors of an internal node each represents the function with the variable set to one of its two values 0 or 1. For any value assignment to the variables, the function value is determined by tracing a path from the root to a terminal node, following the 1 branch if the variable of the internal node takes on the value 1, or else following the 0 branch if the variable of the internal node takes on the value 0, terminating at a terminal node, whose value (0 or 1) is the value of the function.

A BDD is called *ordered* if each variable is encountered at most once on each path from the root to a terminal node and if the variables are encountered in the same order on all such paths. An ordered BDD is called reduced (ROBDD) if it contains vertices neither with isomorphic sub-graphs nor with both edges pointing to the same node. Hence every node of a ROBDD represents a distinct boolean function. For each boolean function the ROBDD corresponding to a given variable order is unique up to isomorphism [B86].

The size of a ROBDD is defined as the number of internal nodes. A level of a ROBDD is the set of nodes labeled with the same variable. The top level refers to the root. The bottom level refers to the set of nodes sitting immediately one level above the terminal nodes. An order of the variables formed from bottom level to the top level is called a variable order, denoted by $\pi$ and written as $< x_{i_1} \ldots x_{i_n} >$, where $x_{i_1} \ldots x_{i_n}$ is a permutation of the variables of $f(x_1, \ldots, x_n)$. In the following, only ROBDDs are considered. For brevity these graphs are called BDDs.

## 3 Symmetric Variables

Symmetry is one of the properties which boolean functions possess. Symmetric variables are the aggregation criterion that we explore. Hence, we review some related concepts on symmetry before describing the methods applied for testing symmetric variables.

**Definition 1.** A boolean function $f(x_1, \ldots, x_n)$ is called *totally symmetric* iff its value is unchanged by any permutation of its variables. For example, the function $f(x, y) = x \oplus y$ is totally symmetric.

Obviously, a function $f$ is symmetric if and only if its function value only depends on the the number of 1's in the input vector and not on their positions. Hence, the BDD size of a totally symmetric function is independent of the variable orders.

**Definition 2.** A boolean function $f(x_1, \ldots, x_n)$ is said to be *partially symmetric* iff there exists at least one subset of at least two variables in which $f$ is totally symmetric. In particular, let $\lambda = \{x_{i_1}, \ldots, x_{i_k}\}$ be a subset of input variables, where $k \geq 2$. $f$ is *partially symmetric* in the variables $x_{i_1}, \ldots, x_{i_k}$, iff the value of $f$ remains invariant for all the permutations of the variables $x_{i_1}, \ldots, x_{i_k}$. The variables $x_{i_1}, \ldots, x_{i_k}$ are said to be *symmetric variables*. The set of variables $x_{i_1}, \ldots, x_{i_k}$ thus formed is called a *symmetric group*, written as $(x_{i_1} \ldots, x_{i_k})$. The *symmetric partition* $\rho$ on the set of input variables $X = \{x_1, \ldots, x_n\}$, written as $\rho = \{(\lambda_1), \ldots, (\lambda_k)\}$, consists of disjoint groups of variables where each $(\lambda_i)$ is a maximal symmetric group.

[Example 2.1] $f(x, y, z) = xy + yz' + y'z$. $f$ is partially symmetric in the variables $y$ and $z$ even though the given expression is not "syntactically" symmetric in $y$ and $z$, and the symmetric variables $y$ and $z$ form one symmetric group $(y\ z)$. The symmetric partition of variables is $\rho = \{(x)(y\ z)\}$.

[Definition 3.] Let $f$ be a function of $n$ variables $X = \{x_1, \ldots, x_n\}$ and let $\rho = \{(\lambda_1), \ldots, (\lambda_k)\}$ be the symmetric partition of $X$. A variable order $\pi$ is said to be a *symmetric order* iff $\pi$ consists of symmetric groups and symmetric variables are located side by side in any order. Formally, $\pi =< \lambda_{i_1}, \ldots, \lambda_{i_k} >$ $, where\ \lambda_{i_j} \in \rho\ for\ j = 1, \ldots, k.$

[Example 3.1] In Example 2.1, $f$ is partially symmetric in $(y,\ z)$. Thus, the variable order $< x\ y\ z >$ is a symmetric order, since $y$ and $z$ are located side by side; whereas the variable order $< y\ x\ z >$ is not, since the symmetric variables $y$ and $z$ are separated by $x$.

In a similar vein, besides the non-symmetric variables , the value of a partially symmetric function $f$ depends on the number of 1's in the input vectors of symmetric variables and not on their individual positions. Correspondingly, the size of BDD of a partially symmetric function $f$ is independent of the permutations of symmetric variables, other variables being fixed at their respective positions. To make this point clear, consider the function $f$ in Example 2.1. The sizes of BDDs associated with each variable order are listed as follows:

| $\pi$ # | < *variable order* > | $\lvert BDD_\pi \rvert$ |
|---|---|---|
| 1 | $< x\ y\ z\ >$ | 4 |
| 2 | $< x\ z\ y\ >$ | 4 |
| 3 | $< y\ x\ z\ >$ | 4 |
| 4 | $< y\ z\ x\ >$ | 5 |
| 5 | $< z\ x\ y\ >$ | 4 |
| 6 | $< z\ y\ x\ >$ | 5 |

Table 1. All the variable orders for $f = xy + y'z + yz'$

The first column labeled $\pi\#$ indicates the order number. The second column labeled < *variable order* > contains variable orders read from bottom level to top level. The third column labeled $|BDD_\pi|$ contains the sizes of BDDs built for each associated variable order.

As it was revealed in the above table, permutation of the symmetric variables $y$ and $z$ in variable orders $\pi_1$ and $\pi_2$ does not have an impact on the size of resulting BDDs, when $x$ is fixed at the bottom level. Neither does the size of BDD when $x$ is fixed at the top level and only $y$, $z$ are swapped in variable orders $\pi_4$ and $\pi_6$. Likewise, the size of BDD remains invariant for the variable orders $\pi_3$ and $\pi_5$ where $x$ is fixed at the middle level and $y$, $z$ are swapped.

### *3.2 Forming Symmetric Groups*

To group symmetric variables into symmetric groups, an efficient method for testing symmetric variables is apparently desirable. To this end, we transpose the corresponding columns of the associated function $f$, instead of computing the function value for each permutation of variables. The form of the transposed function $f'$ is compared with the form of the original function $f$ to determine whether the function $f$ remains invariant or not.

**Example**. Let $f(x_1, x_2, x_3) = x_1'x_2x_3 + x_1x_2'x_3$. Writing $f$ as a matrix gives

$$\begin{bmatrix} 0\ 1\ 1 \\ 1\ 0\ 1 \end{bmatrix}$$

Testing for the symmetry of the variables $x_1$ and $x_2$ involves transposing the $x_1$ column and the $x_2$ column to obtain $f'$ as follows:

$$\begin{bmatrix} 1\ 0\ 1 \\ 0\ 1\ 1 \end{bmatrix}$$

Comparing $f$ and $f'$, we found that $f'$ differs from $f$ only in the order of the rows. Hence we can decide that the value of $f$ remains invariant for the permutation of $x_1$ and $x_2$. Therefore we may group variables $x_1$ and $x_2$ to form one symmetric group $(x_1x_2)$. Now, applying $(x_1x_3)$ to the matrix for $f$ gives

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

which is not the same as $f$. Neither is the resulting matrix given by the permutation of $(x_2\ x_3)$. Thus, $f$ is partially symmetric in the only one symmetric group $(x_1\ x_2)$.

The number of transpositions for a function of $n$ variables is $n(n-1)/2$. Thus in general, it would require $n(n-1)/2$ tests. However, since the symmetric relation on the set of $n$ variables $A = \{x_1 \dots x_n\}$ is an equivalence relation, the number of tests may be reduced to $(n-1)$ transpositions when $f$ is totally symmetric. In particular, if one generates the pair-wise transposition in such a manner that (1,2), (1,3), ..., (1,n) are tested first, then all the symmetric groups would be generated if $f$ is totally symmetric [H65].

Based on these ideas, we design a procedure *Form_Symmetric_Groups()* for grouping symmetric variables into symmetric groups whose pseudo-code is omitted. It takes as an argument the input function $f$ of $n$ variables and returns as value the symmetric partition $G$.

## 4 Properties of Partially Symmetric Functions

In this section, we develop fundamental theorems about the symmetric sets of partially symmetric functions. Let $X = \{x_1, \dots, x_n\}$ and let $\rho = \{\rho_1, \dots, \rho_k\}$ be any partition of $X$. Assume that the number of variables in $\rho_i$ is $r_i$ for $i = 1, \dots, k$. The system $(P_X, \vee, \wedge, 0, 1)$ is a lattice, where $P_X$ is the family of all partitions of X. $\rho \vee \tau$ and $\rho \wedge \tau$ denote respectively the partitions induced by the join and intersection of the two equivalence relations $\rho$ and $\tau$. Let $0 = \{(x_1), \dots, (x_n)\}$ and $1 = \{(x_1, \dots, x_n)\}$. For example, let $X = \{x_1\ x_2\ x_3\ x_4\}$ and, $\rho = \{(x_1), (x_2), (x_3x_4)\}$, and $\tau = \{(x_1x_2), (x_3x_4)\}$. Then $\rho$ and $\tau$ are two partitions on $X$. And $\rho$ is smaller than $\tau$, i.e. $\rho < \tau$.

**Theorem 1.** Let $f_1$ and $f_2$ be two partially symmetric functions with the same symmetric partition $\rho$ on the same set of variables. Then,

**1.** $f_1 + f_2$ is also partially symmetric with a partition $\rho' \geq \rho$.

**Proof.** Straightforward. □

**Example.** Let $f_1(x_0, x_1, x_2, x_3) = \Sigma m(7\ 12)$ and $f_2(x_0, x_1, x_2, x_3) = \Sigma m(11\ 12)$. Both $f_1$ and $f_2$ are partially symmetric in the variable pair $(x_2\ x_3)$; i.e., $\rho_{f_1} = \rho_{f_2} = \{(x_0), (x_1), (x_2\ x_3)\}$. Then, $f(x_0, x_1, x_2, x_3) = f_1 + f_2 = \Sigma\ m(7\ 11\ 12)$. And $f$ is partially symmetric in the variable groups $(x_0\ x_1)$ and $(x_2\ x_3)$; i.e., $\rho_f = \{(x_0\ x_1), (x_2\ x_3)\}$ Thus, $\rho_f > \rho_{f_1}$ and $\rho_f > \rho_{f_2}$.

**2.** $f_1 \cdot f_2$ is partially symmetric with a partition $\rho' \geq \rho$.

**Proof.** Let $x_i$, $x_j$ be any variables in symmetric group $c$ of $\rho$, and let $t$ be any term containing $x_i$ and $x_j$. Then, let $t'$ be the term when $x_i$ and $x_j$ are switched in $t$. Since $f_1$ is symmetric in $(x_i\ x_j)$, hence $t' \in f_1$ and similarly $t' \in f_2$. Therefore $t' \in f_1 \cdot f_2$. □

**Example.** Let $f_1(x_0, x_1, x_2) = \Sigma\ m(1\ 2\ 5\ 6\ 7)$ and $f_2(x_0, x_1, x_2) = \Sigma\ m(4\ 7)$. Then both $f_1$ and $f_2$ are partially symmetric in the variable pair $(x_1\ x_2)$. However, taking the intersection of $f_1$ and $f_2$ results in the function: $f\ (x_0, x_1, x_2) = f_1 \cdot f_2 = \Sigma\ m(7)$, which is totally symmetric. All three variables form one symmetry group of $f$, i.e., $(x_0\ x_1\ x_2)$.

**3.** $\bar{f}_1$, $\bar{f}_2$ are also symmetric with the same partition.

**4.** Let $xf$ denote the operation of multiplying $f$ by a new variable $x$ (this may be generalized to multiplying by a literal). Then $xf$ has a symmetric partition $\rho_{xf} \geq \rho_f$.

**Example.** Let $f\ (x_0, x_1, x_2) = \Sigma\ m(4\ 7) = x_0 x_1' x_2' + x_0 x_1 x_2$. $f$ is partially symmetric in $(x_1\ x_2)$. Then multiplying $f$ by a new variable $x_3$, we have $x_3 f(x_0, x_1, x_2, x_3) = \Sigma\ m(9\ 15) = x_0 x_1' x_2' x_3 + x_0 x_1 x_2 x_3$ which is partially symmetric in $(x_1\ x_2)$ and $(x_0\ x_3)$. Thus, the symmetric partition of $x_3 f$, $\rho_{x_3 f} = \{(x_1\ x_2),\ (x_0\ x_3)\}$, is strictly larger than the symmetric partition of $f$, $\rho_f = \{(x_0),\ (x_1\ x_2)\}$.

## 5 The Search Method

Since the conjecture states that symmetric variables gather in at least one of the optimum variable orders; in other words, it conjectures that at least one optimum variable order is a symmetric order. Hence, a counterexample to the conjecture being studied would require a function to have none of its optimum orders a symmetric order.

**Definition 4.** A boolean function $f$ is said to be a *counterexample* iff none of its optimum orders is a symmetric order.

The first method which we developed to find counterexamples to the conjecture being studied was based on exhaustive search. The search method consisted of 3 steps as follows: First, generate all functions of $n$ variables. There are $2^{2^n}$ functions of $n$ variables in all. Second, for each generated function $f$, compute all the optimum variable orders from the total of $n!$ variable orders and the symmetric partition of $f$. Third, if none of the optimum orders is a symmetric order, then $f$ is a counterexample.

**NN-Equivalence Class**

Since $2^{2^n}$ is an enormously large number to search through, we make further improvements upon the exhaustive search method. This is accomplished by partitioning the boolean functions into equivalence classes. ROBDDs for equivalent functions are isomorphic. Thus we build the ROBDDs of representative function from each class only. If the representative of an equivalence class $c$ is a counterexample, then we generate all the equivalent functions in $c$ and check for more counterexamples. The size of search space is hence reduced considerably. Consequently the run time is decreased drastically. Detailed explanations follow in subsequent paragraphs. The following definition provides the criterion for partitioning boolean functions.

**Definition 5.** Let $f$ and $g$ be two boolean functions of $n$ variables. We say that $f$ and $g$ are $nn-equivalent$ iff either one may be obtained from the other by complementing some or all the input variables or one is the complement of the other.

**Example 5.1** Let $f\ (x_1, x_2) = x_1\ x_2$ and $g\ (x_1, x_2) = f(x_1', x_2) = x_1' x_2$. Then $f$ and $g$ are $nn - equivalent$, since $g$ is obtained from $f$ (or vice-versa) by complementing one of the variables, which is $x_1$ in this example.

**Example 5.2** Let $f\ (x_1, x_2) = x_1\ x_2$ and $g\ (x_1, x_2) = f'(x_1, x_2) = x_1' + x_2'$. Then $f$ and $g$ are $nn - equivalent$, since $g$ is the complement of $f$.

The correctness of the improved search method is established by the following theorem.

**Theorem 2.** Boolean functions in the same $nn - equivalent$ class possess essentially the same ROBDD structures for all the variable orders, and differ only in the complemented edges and/or the values which label the terminal nodes.

**Proof.** Let $f$ and $g$ be two boolean functions of $n$ variables.

Case 1. Let $g$ be obtained from $f$ by complementing some or all variables. In this case, the ROBDDs of $f$ and $g$ have exactly the same size on each individual level and so agree on the total size. Their ROBDDs differ only in the labels of edges of the complemented variables.

Case 2. Let $g$ be the complement of $f$. Then ROBDD of $f$ and $g$ differ only in the values which label the terminal nodes. □

The total number of equivalence classes under $nn - equivalence$ is

$$\frac{1}{2^{n+1}}\ (2^{2^n}\ + (2^n - 1)\ 2^{2^{n-1}+1})$$

for boolean function of $n$ variables [Ha65]. For instance, the five $nn-equivalence$ classes for $n = 2$ are as follows:

$[0,\ 1]$ $[x_1'\ x_2',\ x_1'\ x_2,\ x_1\ x_2',\ x_1\ x_2,\ x_1'+x_2',\ x_1'+x_2,\ x_1+x_2',\ x_1+x_2]$
$[x_1, x_1']$ $[x_2, x_2']$ $[x_1\ x_2 + x_1'\ x_2',\ x_1'\ x_2 + x_1\ x_2']$

Table 2 lists the total number of functions $2^{2^n}$ and the total number of $nn - equivalence$ classes for the number of variables from 1 to 5. It can be seen that the ratio of $T_n$ to $2^{2^n}$ decreases dramatically to 0.0156 for $n = 5$.

| $n$ | $2^{2^n}$ | $T_n$ | $T_n/2^{2^n}$ |
|---|---|---|---|
| 1 | 4 | 2 | 0.50 |
| 2 | 16 | 5 | 0.3125 |
| 3 | 256 | 30 | 0.1172 |
| 4 | 65536 | 2,288 | 0.0349 |
| 5 | 4,294,967,296 | 67,172,352 | 0.0156 |

Table 2. Total Number of $nn - equivalence$ classes

Functions are ordered by their associated minterm lists in lexical order. The function ordered first in each class is chosen as the representative. For instance, the last class for $n = 2$ consists of following two functions in terms of minterm lists: $[\Sigma m(0,3),\ \Sigma m(1,2)]$, and $(0,3) < (1,2)$ in order. Since $\Sigma m(0,3)$ is the first in order, it is thus chosen as the representative for the class.

Since ROBDDs of nn-equivalent functions are isomorphic by Theorem 2, to search for counterexamples we begin with the representative $f$ from each class. If $f$ is a counterexample, we may pursue the functions that are nn-equivalent to $f$ for more counterexamples by Theorem 3. Theorem 4 tells us how many counterexamples we may expect to get for each class.

**Theorem 3.** Suppose $f$ is a counterexample and has the symmetric partition $\{(\lambda_1), \ldots, (\lambda_k)\}$. Then $g$ is also a counterexample iff $g = \bar{f}$ or $g$ is obtained from $f$ by complementing some or all the $\lambda_i$.

**Proof:** By Theorem 2, $f$ and $g$ are $nn-equivalent$, hence they have essentially the same BDD structure. Thus $f$ and $g$ have the same collection of optimum orders. Furthermore, it is obvious that $g$ has the same symmetric partition as $f$ does. Therefore, $g$ is also a counterexample if $f$ is. □

**Example.** Let $f(x_0, x_1, x_2, x_3) = \Sigma m(0\ 1\ 2\ 4\ 9\ 10\ 12\ 15)$. Then $f$ has the symmetric partition $\{(x_0)\ (x_1\ x_2\ x_3)\}$. And has a total of six optimum orders giving optimum BDD size 7, namely, $< x_1\ x_0\ x_2\ x_3 >$, $< x_1\ x_0\ x_3\ x_2 >$, $< x_2\ x_0\ x_1\ x_3 >$, $< x_2\ x_0\ x_3\ x_1 >$, $< x_3\ x_0\ x_1\ x_2 >$, $< x_3\ x_0\ x_2\ x_1 >$. As it can be seen that no optimum order is symmetric, hence $f$ is a counterexample. By complementing $(x_0)$, we get $g1 = f(x_0', x_1, x_2, x_3) = \Sigma m(1\ 2\ 4\ 7\ 8\ 9\ 10\ 12)$. By complementing $(x_1\ x_2\ x_3)$, we get $g2 = f(x_0, x_1', x_2', x_3') = \Sigma m(3\ 5\ 6\ 7\ 8\ 11\ 13\ 14)$. By complementing both $(x_0)$ and $(x_1\ x_2\ x_3)$, we get $g3 = f(x_0', x_1', x_2', x_3') = \Sigma m(0\ 3\ 5\ 6\ 7\ 8\ 11\ 13\ 14\ 15)$. Since $g1$, $g2$, and $g3$ all have the same collection of optimum orders and the same symmetric partition, therefore all of them are all counterexamples.

However, when $g$ is obtained from $f$ by complementing a subset of variables of some symmetric group, then $g$ may or may not be a counterexample.

**Example.** Complementing $x_1$ of $f$ in the preceding example to obtain $g = f(x_0, x_1', x_2, x_3) = \Sigma m(0\ 4\ 5\ 6\ 8\ 11\ 13\ 14)$. The function $g$ still has the same collection of optimum orders as $f$ has, but the symmetric partition of $g$ has changed to $\{(x_0)(x_1)(x_2\ x_3)\}$. So $g$ is no longer a counterexample.

**Example.** Nevertheless, now let $f(x_0, x_1, x_2, x_3, x_4) = \Sigma m(1\ 2\ 3\ 4\ 5\ 6\ 7\ 24\ 25\ 30)$. Function $f$ has symmetric partition $\{(x_0\ x_1)(x_2\ x_3)(x_4)\}$ and is a counterexample. Complementing $x_0$ of $f$ to obtain $g = f(x_0', x_1, x_2, x_3, x_4, x_5) =$ $\Sigma m(8\ 9\ 14\ 17\ 18\ 19\ 20\ 21\ 22\ 23)$. The function $g$ has a different symmetric partition $\{(x_0)(x_1)(x_2\ x_3)(x_4)\}$ as expected. But is now also a counterexample. The reason is that the collection of four optimum orders are the same for both $f$ and $g$, namely, $< x_2\ x_4\ x_3\ x_0\ x_1 >$ $< x_2\ x_4\ x_3\ x_1\ x_0 >$, $< x_3\ x_4\ x_2\ x_0\ x_1 >$, and $< x_3\ x_4\ x_2\ x_1\ x_0 >$. As it can be seen that the symmetric variables $x_2$ and $x_3$ in the only symmetric group of $g$ which is $(x_2\ x_3)$ are still kept side by side. Therefore, $g$ is still a counterexample.

**Theorem 4.** Suppose function $f$ is a counterexample and functions in the same nn-equivalent class $c$ that are counterexamples have the maximum size of symmetric partition $k$, $\{(\lambda_1), \ldots, (\lambda_k)\}$. Then we can obtain $2^k$ counterexamples from the nn-equivalence class $c$.

**Proof:** Straightforward from Theorem 3. □

**Example.** Repeating the function given in the preceding example, we have: $f(x_0, x_1, x_2, x_3, x_4) = \Sigma m(1\ 2\ 3\ 4\ 5\ 6\ 7\ 24\ 25\ 30)$. $f$ is a counterexample with symmetric partition $\{(x_0\ x_1)(x_2\ x_3)(x_4)\}$ of size 3. Hence we can expect to generate at least $2^3 = 8$ counterexamples in all. After complementing $x_0$ to obtain $g = f(x_0', x_1, x_2, x_3, x_4, x_5) = \Sigma m(8\ 9\ 14\ 17\ 18\ 19\ 20\ 21\ 22\ 23)$, with symmetric partition $\{(x_0)(x_1)(x_2\ x_3)(x_4)\}$ of size 4. We found that $g$ is also a counterexample. Therefore, we may increase the number of expected counterexamples from $2^3 = 8$ to $2^4 = 16$.

We now present an outline of the improved search algorithm. Procedure *Find_Counterexamples*() takes as input the number of variables $n$ of the boolean functions, and returns as a value a collection of counterexamples $C$.

**Procedure** *Find_Counterexamples (n)*
$C \leftarrow$ empty;
**for** (all the representatives $f$ from each class of functions of $n$ variables) **do**
  **if** ($f$ is a counterexample) **then**
    $C \leftarrow C \cup \{f\}$;
    **for** (all the nn-equivalent functions $g$) **do**
      **if** ($g$ is a counterexample) **then**
      $C \leftarrow C \cup \{g\}$;
**return** ($C$);

## 6 Structure and Generation of Counterexamples

In this section, we study the structure of counterexamples under the operations of join ($\vee$), intersection ($\wedge$) and negation ($\neg$). We also give a simple method for generating more counterexamples from given counterexamples.

**Theorem 5.** The set of counterexamples of $n$ variables is not closed under join ($\vee$), intersection ($\wedge$), but is closed under negation ($\neg$).

**Proof:** We prove that the set of counterexamples of $n$ variables is neither closed under $\vee$ nor closed under $\wedge$ by contradiction. Assume that the set of counterexamples of $n$ variables is closed under $\vee$ and $\wedge$. For the case of join ($\vee$), let $f_1(x_0, x_1, x_2, x_3, x_4) = \Sigma\ m(1\ 2\ 3\ 4\ 5\ 6\ 8\ 16\ 30\ 31)$ and let $f_2(x_0, x_1, x_2, x_3, x_4) = \Sigma\ m(1\ 2\ 3\ 4\ 5\ 6\ 15\ 23\ 24\ 25)$. Both $f_1$ and $f_2$ are counterexamples and partially symmetric in $(x_1\ x_2)$ and $(x_3\ x_4)$. Then, $(f_1 \vee f_2)(x_0, x_1, x_2, x_3, x_4) = \Sigma\ m(1\ 2\ 3\ 4\ 5\ 6\ 8\ 15\ 16\ 23\ 24\ 25\ 30\ 31)$ is not a counterexample although it is also partially symmetric in the same symmetric groups. For the case of intersection ($\wedge$), let $f_1(x_0, x_1, x_2, x_3) = \Sigma\ m(0\ 1\ 7\ 11\ 13\ 14)$, and $f_2(x_0, x_1, x_2, x_3) = \Sigma\ m(1\ 2\ 4\ 8\ 14\ 15)$. Both $f_1$ and $f_2$ are counterexamples and have the same symmetric partition $\rho_{f_1} = \rho_{f_2} = \{(x_0\ x_1 x_2), (x_3)\}$. But $(f_1 \wedge f_2)(x_0, x_1, x_2, x_3) = \Sigma\ m(1\ 14)$ is no longer a counterexample although it still has the same symmetric partition.

Furthermore, since $f$ and $\bar{f}$ are symmetric with the same sets of variables by Theorem 1.3, and have the same BDD structures except for the labels of terminal nodes, hence the set of counterexamples of $n$ variables is closed under $\neg$. □

**Theorem 6.** Suppose $f$ is a counterexample. Then,

**1.** $xf$ is also a counterexample, where $x$ is a new variable.

**Proof:** Note that any counterexample $f$ cannot be a constant function 0 or 1. There is an ROBDD for $xf$ with the structure where the root is labeled with the new variable $x$ and the positive cofactor is represented by the ROBDD for $f$ and the negative cofactor is represented by **0**.

Therefore, the optimum ROBDD size for $xf$, denoted as $W$, is $\leq\ 1 +$ optimum ROBDD size for $f$, denoted as $V$. Because $f$ is a counterexample, optimum symmetric orders for $f$, gives ROBDD size, denoted as $X$, $\geq 1 +$ optimum ROBDD size for $f$.

Now $xf$ depends on $x$ since $f$ is not a constant function. So the ROBDD size for any symmetric order for $xf$ must be at least $1 + X$.

Thus $W \leq 1 + V \leq X < 1\ + X$. Hence $xf$ is a counterexample. □

**2.** $xf\ + x'$ is also a counterexample, where $x$ is a new variable.

The proof is similar to the preceding one. □.

The advantage of $x\ f\ + x'$ is that the symmetry partition of $f$ remain intact. Repeated applications of the same operation only partition the subsequent new variables into one symmetric group.

**Corollary.** For every n $\geq$ 4, there is a counterexample. (See Definition 4. for the definition of counterexample.)

**Counterexample Not Preserved:** Following example shows that while both $f1$ and $f2$ are counterexamples, the operation of $u1f1\ + u2f2$ may not yield a counterexample.

**Example.** Let $f1(x_0, x_1, x_2, x_3) = \Sigma\ m(0\ 1\ 6\ 10\ 12\ 15)$ and $f2(x_0, x_1, x_2, x_3) = \Sigma\ m(0\ 1\ 7\ 11\ 13\ 14)$. Both $f1$ and $f2$ are counterexamples and have the same symmetric partition $\{(x_0\ x_1\ x_2)(x_3)\}$. Applying the operation $u1f1 + u2f2$ yields $f$ as follows: $f(x_0, x_1, x_2, x_3, x_4, x_5) = \Sigma\ m(1\ 2\ 3\ 5\ 6\ 7\ \ 26\ 27\ 29\ 31\ 42\ 43\ 45\ 47\ 50\ 51\ 53\ 55\ 57\ 59\ 62\ 63)$. $f$ has the same symmetric partition but is not a counterexample.

## 7 Counterexamples

In this section we report counterexamples. For functions of 3 variables, there does not exist any counterexample. Counterexamples are found beginning with functions of 4 variables. There are 80 counterexamples in all for functions of 4 variables and 1262800 counterexamples in all for functions of 5 variables[MMD94].

In none of the counterexamples would a symmetric order make an optimum order. Meanwhile symmetric orders may have 2 kinds of impact on the worst orders described as follows: Type 1. None of the symmetric orders is a worst order. Examples are demonstrated by Counterexample 1 for 5 variables and Counterexample 2 for 6 variables. Type 2. Some of the symmetric orders are worst orders. Examples are demonstrated by Counterexample 3 for 5 variables and Counterexample 4 for 6 variables. All the 80 counterexamples for functions of 4 variables belong to the second type.

For each counterexample, the function is expressed in minterm list form. The statistics of the optimum/worst orders obtained from each function are described.

**Counterexample 1.** $f(x_0, x_1, x_2, x_3, x_4) = \Sigma m(1, 2, 3, 4, 5, 6, 7, 24, 25, 30)$

(Optimum/Worst) size : 9 / 13 (#Optimum/#Worst) : 4 / 8

The function $f$ is partially symmetric in the two symmetric groups $(x_0\ x_1)$ and $(x_2\ x_3)$. All the 24 symmetric orders give sub-optimum BDD sizes between 10 and 12, neither the best nor the worst. For example, symmetric order $< x_4\ x_3\ x_2\ x_1\ x_0 >$ gives ROBDD size 10, and symmetric order $< x_2\ x_3\ x_0\ x_1\ x_4 >$ gives ROBDD size 12.

**Counterexample 2.** $f(x_0, x_1, x_2, x_3, x_4, x_5) = \Sigma\, m(2, 3, 4, 5, 6, 7, 10, 12, 54, 62)$.

(Optimum/Worst) size : 10 / 17 (#Optimum/#Worst) : 4 / 8

The function $f$ is symmetric in the variable groups $(x_0\ x_1)$ and $(x_3\ x_4)$. It has 4 optimum orders that yield the optimum ROBDD size 10. In addition, it has 8 worst orders that yield the worst ROBDD size 17.

All the 24 symmetric orders give sub-optimum sizes between 11 and 15, neither the optimum nor the worst. For example, symmetric order $< x_5\ x_2\ x_0\ x_1\ x_4\ x_3 >$ gives ROBDD size 11, and symmetric order $< x_5\ x_0\ x_1\ x_4\ x_3\ x_2 >$ gives ROBDD size 15.

**Counterexample 3.** $f(x_0, x_1, x_2, x_3, x_4) = \Sigma m(0, 1, 6, 30, 31)$

(Optimum/Worst) size : 8 / 11 (#Optimum/#Worst) : 8 / 24

The function $f$ is partially symmetric in the two symmetric groups $(x_0\ x_1)$ and $(x_2\ x_3)$. There are a total of 24 symmetric orders. While none of the symmetric orders give optimum BDD size 8, eight symmetric orders are among the worst orders which give worst BDD size 11. Remaining 16 symmetric orders give sub-optimum BDD size 9, for example, symmetric orders $< x_0\ x_1\ x_4\ x_2\ x_3 >$ and $< x_4\ x_3\ x_2\ x_1\ x_0 >$ are such sub-optimum orders.

As long as variable $x_4$ makes the top level of ROBDD, any permutation of symmetric variables at the lower levels would cause the worst sized BDD to be built, for example, $< x_0\ x_1\ x_2\ x_3\ x_4 >$ and $< x_3\ x_1\ x_2\ x_0\ x_4 >$ are two such symmetric orders among the worst.

**Counterexample 4.** $f(x_0, x_1, x_2, x_3, x_4, x_5) = \Sigma\ m(2, 3, 4, 5, 6, 7, 10, 14, 53, 61)$

(Optimum/Worst) size : 9 / 17 (#Optimum/#Worst) : 2 / 8

The function $f$ is partially symmetric in one symmetric group $(x_0\ x_1)$. There are a total of 240 symmetric orders. While none of the symmetric orders give optimum BDD size 9, all the worst orders are symmetric orders which give worst BDD size 17. Remaining 232 symmetric orders give sub-optimum BDD sizes between 10 and 16. For example, one best symmetric order is $< x_4\ x_5\ x_3\ x_2\ x_1\ x_0 >$ which gives BDD size 10.

## 8 Positive Examples

During the search for counterexamples, we have also found examples such that symmetric orders make contributions to optimum orders in various degrees. We describe these various degrees of contributions made by symmetric orders along with their relations with worst orders with appropriate examples.

- All the symmetric orders are optimum orders as well as worst orders. This situation holds for totally symmetric functions, such as $f(x_0, x_1, x_2) = x_0x_1x_2$,
- All the symmetric orders are optimum orders. And none of the symmetric orders are worst orders.
  **Example 6.1** Let $f(x_0, x_1, x_2) = x_0x_1 + x_2$. $f$ is symmetric in the variable pair $(x_0, x_1)$. There are 4 symmetric orders. All the symmetric orders give optimum BDD size 3. None of the worst orders, which give worst BDD size 4, are symmetric orders.
- Some of the symmetric orders are optimum orders. We may further distinguish 3 types of impact that symmetric orders may have on worst orders.
  [(1)] Symmetric orders are either optimum orders or sub-optimum orders. None of the symmetric orders are worst orders.
  **Example 6.2** $f(x_0, x_1, x_2, x_3, x_4) = \Sigma m(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$

  (Optimum/Worst) size: 7 / 12 (#Optimum/#Worst) : 8 / 8

  $f$ is symmetric in the two variable groups $(x_1, x_2)$ and $(x_3, x_4)$. There are 24 symmetric orders. Symmetric orders give BDD sizes between 7 and 10.

  In this example, all of the eight optimum orders, are also symmetric orders: 1:$< x_0\ x_1\ x_2\ x_3\ x_4 >$ 2:$< x_0\ x_1\ x_2\ x_4\ x_3 >$ 3:$< x_0\ x_2\ x_1\ x_3\ x_4 >$ 4:$< x_0\ x_2\ x_1\ x_4\ x_3 >$ 5:$< x_1\ x_2\ x_3\ x_4\ x_0 >$ 6:$< x_1\ x_2\ x_4\ x_3\ x_0 >$ 7:$< x_2\ x_1\ x_3\ x_4\ x_0 >$ 8:$< x_2\ x_1\ x_4\ x_3\ x_0 >$.

  Listed as below, are the eight worst orders but none of them are symmetric orders: 1:$< x_3\ x_1\ x_0\ x_2\ x_4 >$ 2:$< x_3\ x_1\ x_0\ x_4\ x_2 >$ 3:$< x_3\ x_2\ x_0\ x_1\ x_4 >$ 4:$< x_3\ x_2\ x_0\ x_4\ x_1 >$ 5:$< x_4\ x_1\ x_0\ x_2\ x_3 >$ 6:$< x_4\ x_1\ x_0\ x_3\ x_2 >$ 7:$< x_4\ x_2\ x_0\ x_1\ x_3 >$ 8:$< x_4\ x_2\ x_0\ x_3\ x_1 >$.

  Some of the symmetric orders give sub-optimum BDDs. For instance, the symmetric order $< x_1\ x_2\ x_0\ x_3\ x_4 >$ yields a BDD of size 8, and the

symmetric order $< x_4\ x_3\ x_0\ x_1\ x_2 >$ yields a BDD of size 10.
[(2)] Some of the symmetric orders are worst orders. Some of the symmetric orders are sub-optimum orders.

**Example 6.3** $f(x_0, x_1, x_2, x_3) = \Sigma m(0, 1, 6, 14)$

(Optimum/Worst) size : 5 / 8 (#Optimum/#Worst) : 12 / 4

$f$ is symmetric in the variable pair $(x_1\ x_2)$. It has 12 symmetric orders. Among the 12 symmetric orders, 4 of them are optimum orders, 4 of them are worst orders, and the remaining 4 of them are sub-optimum orders.

In the following list of twelve optimum orders, it can be seen that four of optimum orders are symmetric orders, namely, $\pi_3$, $\pi_4$, $\pi_9$, and $\pi_{10}$: $\pi_1$:$< x_0\ x_1\ x_3\ x_2 >$ $\pi_2$:$< x_0\ x_2\ x_3\ x_1 >$ $\pi_3$:$< x_0\ x_3\ x_1\ x_2 >$ $\pi_4$:$< x_0\ x_3\ x_2\ x_1 >$ $\pi_5$:$< x_1\ x_0\ x_3\ x_2 >$ $\pi_6$:$< x_1\ x_3\ x_0\ x_2 >$ $\pi_7$:$< x_2\ x_0\ x_3\ x_1 >$ $\pi_8$:$< x_2\ x_3\ x_0\ x_1 >$ $\pi_9$:$< x_3\ x_0\ x_1\ x_2 >$ $\pi_{10}$:$< x_3\ x_0\ x_2\ x_1 >$ $\pi_{11}$:$< x_3\ x_1\ x_0\ x_2 >$ $\pi_{12}$:$< x_3\ x_2\ x_0\ x_1 >$.

And all the worst orders are symmetric orders: $\pi_1$:$< x_1\ x_2\ x_0\ x_3 >$ $\pi_2$:$< x_1\ x_2\ x_3\ x_0 >$ $\pi_3$:$< x_2\ x_1\ x_0\ x_3 >$ $\pi_4$:$< x_2\ x_1\ x_3\ x_0 >$.
The remaining four symmetric orders: $< x_0\ x_1\ x_2\ x_3 >$, $< x_0\ x_2\ x_1\ x_3 >$, $< x_3\ x_1\ x_2\ x_0 >$, and $< x_3\ x_2\ x_1\ x_0 >$, all yield sub-optimum BDDs of the same one size 6.
[(3)] Symmetric orders are either optimum orders or worst orders. No symmetric orders are sub-optimum orders. One such example is the function given in Example 2.1.

# 9 Conclusions

We have shown that counterexamples exist for the conjecture being studied for *every* $n \geq 4$. Hence the conjecture stating that symmetric variables gather in at least one of the optimum orders does not hold in general.

Nevertheless, the impact of symmetric variables on the optimality of variable orders is far from conclusive. Symmetric orders of a function $f$ can give corresponding BDDs optimum size, sub-optimum size, and/or the worst size. Symmetric orders may be optimum orders just as well as they may be worst orders or sub-optimum orders.

For more counterexamples, the interested reader may refer to the Appendix 1. We have computed the maximum distance between the the best BDD size and the worst BDD size that symmetric orders may yield for functions of variables 3, 4, and 5. The results are reported in Appendix 2. We hope this study would aid in future research on heuristics for optimum/good variable orders. The alert reader may have noticed that in all the counterexamples listed here, there is a symmetric order with ROBDD size equal to 1 + optimal ROBDD size. It would be interesting to come up with counterexamples where the sizes differ by more than one, or to prove that there is always a symmetric order with ROBDD size equal to $c+$ optimal ROBDD size, where $c$ is independent of $n$, or even that there is always one symmetric order with ROBDD size equal to $c$(optimal ROBDD size).

## Appendix 1.

Each row in the following two tables lists a counterexample. Each counterexample is specified with the number of variables, in the first column labeled $n$, and the list of minterms, in the second column labeled $f(x_0, \ldots, x_n) = \Sigma\ m(\ldots)$. Each counterexample is a partially symmetric function and has its symmetric groups specified in the third column, labeled *groups*. The fourth column, labeled *sym.ord's* $(b/w)$, states the best and worst BDD sizes given by symmetric orders. The fifth, labeled $(o/w)$, states the optimum and worst BDD sizes. The last column, labeled $\#(o/w)$, states the number of optimum and worst orders.

| $n$ | $f(x_0, \ldots, x_n) = \Sigma\ m(\ldots)$ | *groups* | *sym. ord's* $b/w$ | $o/w$ | $\#(o/w)$ |
|---|---|---|---|---|---|
| 5 | (1 2 3 4 5 6 7 24 26 29) | $(x_0\ x_1)(x_2\ x_4)$ | 10 / 12 | 9 / 13 | 4 / 8 |
| 5 | (1 2 3 4 5 6 8 11 13 14) | $(x_2\ x_3\ x_4)$ | 9 / 11 | 8 / 13 | 12 / 12 |
| 5 | (1 2 3 4 5 6 9 10 12 15) | $(x_2\ x_3\ x_4)$ | 9 / 11 | 8 / 13 | 12 / 12 |
| 5 | (1 2 3 4 5 8 16 27 29 31) | $(x_0\ x_1)(x_2\ x_3)$ | 12 / 13 | 11 / 14 | 20 / 26 |
| 6 | (2 3 4 5 6 7 8 48 55 63) | $(x_0\ x_1)(x_3\ x_4)$ | 13 / 18 | 12 / 20 | 4 / 24 |
| 6 | (2 3 4 5 6 7 9 49 54 62) | $(x_0\ x_1)(x_3\ x_4)$ | 13 / 18 | 12 / 20 | 4 / 24 |
| 6 | (2 3 4 5 6 7 10 12 55 63) | $(x_0\ x_1)(x_3\ x_4)$ | 11 / 16 | 10 / 18 | 4 / 16 |
| 6 | (2 3 4 5 6 7 11 13 54 62) | $(x_0\ x_1)(x_3\ x_4)$ | 11 / 16 | 10 / 18 | 4 / 16 |
| 6 | (2 3 4 5 6 7 11 13 55 63) | $(x_0\ x_1)(x_3\ x_4)$ | 11 / 15 | 10 / 17 | 4 / 8 |

Table 3. Counterexamples for which no symmetric orders are worst orders.

| $n$ | $f(x_0,\ldots,x_n) = \Sigma\ m(\ldots)$ | $groups$ | $sym.\ ord's$ $b/w$ | $o/w$ | $\#(o/w)$ |
|---|---|---|---|---|---|
| 5 | (1 2 3 4 5 6 8 16 30 31) | $(x_0\ x_1)(x_2\ x_3)$ | 11 / 13 | 10 / 13 | 4 / 40 |
| 5 | (1 2 3 4 5 6 15 23 24 25) | $(x_0\ x_1)(x_2\ x_3)$ | 11 / 13 | 10 / 13 | 4 / 40 |
| 5 | (1 2 3 4 5 7 14 22 24 29) | $(x_0\ x_1)$ | 12 / 15 | 11 / 15 | 4 / 12 |
| 5 | (1 2 3 4 5 8 9 22 24 25) | $(x_2\ x_3)$ | 10 / 14 | 9 / 14 | 6 / 16 |
| 5 | (1 2 3 4 5 8 16 26 29 30) | $(x_0\ x_1)$ | 12 / 15 | 11 / 15 | 4 / 8 |
| 6 | (2 3 4 5 6 7 10 14 52 60) | $(x_0\ x_1)$ | 10 / 17 | 9 / 17 | 2 / 8 |
| 6 | (2 3 4 5 6 7 10 14 60 62) | $(x_0\ x_1)$ | 10 / 17 | 9 / 17 | 2 / 8 |
| 6 | (2 3 4 5 6 7 11 15 53 61) | $(x_0\ x_1)$ | 10 / 17 | 9 / 17 | 2 / 8 |
| 6 | (2 3 4 5 6 7 12 14 56 60) | $(x_0\ x_1)$ | 10 / 17 | 9 / 17 | 2 / 8 |
| 6 | (2 3 4 5 6 7 13 15 50 58) | $(x_0\ x_1)$ | 10 / 17 | 9 / 17 | 2 / 8 |

Table 4. Counterexamples for which some symmetric orders are worst orders.

## Appendix 2.

We have measured the distances from the worst ROBDD sizes associated with symmetric orders to the optimum ROBDD sizes for all the functions of 3, 4, and 5 variables respectively and have obtained following results.

In the case of functions of 3 variables, the maximum distance is 1. An instance is found in the function $f(x\ y\ z) = xy + y'z + yz'$ of Example 2.1, where one of the worst symmetric orders, such as $< y\ z\ x >$, yields BDD size 5 and the optimum BDD size of the function is 4. The sizes of BDDs associated with each variable order are listed in Table 1 above.

As for functions of 4 variables, the maximum distance is 4. An example is the following partially symmetric function:

$$f(x_0, x_1, x_2, x_3) = \Sigma m(0, 1, 2, 3, 4, 7, 12, 15)$$

$f$ is symmetric in the variables $x_2$ and $x_3$, and is not a counterexample. Its worst symmetric orders, such as $< x_1 x_2 x_3 x_0 >$, yield BDD size 9 and its optimum BDD size is 5. Thus, the maximum distance is 4.

As for functions of 5 variables, our computation has not finished with all the functions due to time limit. Yet the maximum distance found so far is 9. An example is given by the the following partially symmetric function:

$$f(x_0, x_1, x_2, x_3, x_4) = \Sigma m(0, 1, 2, 3, 4, 5, 10, 14, 18, 22, 24, 25, 26, 27).$$

$f$ is symmetric in the $x_0$ and $x_1$. While not being a counterexample either, one of its worst symmetric orders $< x_3\ x_0\ x_1\ x_2\ x_4 >$ gives ROBDD size 16 and its optimum ROBDD size is 7. Therefore, the distance is 9.

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