Two Remarks on Copositive Matrices*

EMILIE HAYNSWORTH
Auburn University
Auburn, Alabama

AND

A. J. HOFFMAN
IBM Research Center
Yorktown Heights, New York

I. INTRODUCTION

Let $M = (m_{ij})$ be a real symmetric matrix of order $n$. $M$ is said to be copositive if $x \geq 0$ implies $(Mx, x) \geq 0$. This concept was first introduced by Motzkin [7]. (A survey of some of this work is contained in [5], and references to uses of the concept in mathematical programming are given in [3]).

In this note, we establish two new properties of copositive matrices:

(a) The set of copositive matrices is a closed convex cone in the space of symmetric matrices, and we offer a contribution to the unsolved problem of describing all extreme rays of this cone (see [1], [2], [4], [6] for background). Let $M$ be a symmetric matrix in which $m_{ii} = 1, m_{ij} = \pm 1$ (we tacitly assume at least one $m_{ij} = -1$). Define $G(M)$ to be the graph on $n$ vertices in which $i$ and $j$ are adjacent vertices if and only if $m_{ij} = -1$. Using a theorem of [8], we shall be able to say when $M$ is respectively copositive, positive semidefinite, copositive, and on an extreme copositive ray in terms of properties of $G(M)$ (Theorem 3.1). For $n \leq 7$, these results were given in [2].

(b) Let $A$ be a symmetric matrix. We say that $A$ has the Perron property if its spectral radius $\rho(A)$ is an eigenvalue, and we shall prove

* Dedicated to Prof. A. M. Ostrowski on his 75th birthday.
that every copositive matrix has the Perron property. Actually, we prove a slightly more general result (Theorem 4.1). Define the polar \( C^* \) of a cone \( C \) to be the set of all \( x \) such that \( (x, y) \geq 0 \) for all \( y \in C \). \( C \) is said to be self-polar if \( C = C^* \). A is said to be copositive with respect to a cone \( C \) if \( (Ax, x) \geq 0 \) for all \( x \in C \). We shall show that \( A \) has the Perron property if and only if it is copositive with respect to some self-polar cone \( C \).

2. Preliminaries

If \( G \) is an undirected graph, without multiple edges or loops, on \( n \) vertices, we define \( A(G) \), the adjacency matrix of \( G \), to be the real symmetric matrix of 0's and 1's in which \( a_{ij} = 1 \) if and only if \( i \) and \( j \) are adjacent vertices. Thus, if \( M \) is a matrix of 1's and \( -1 \)'s as in Section 1,

\[
M = J - 2A(G(M)),
\]

where \( J \) is the matrix every entry of which is 1. If \( G \) and \( H \) are graphs, \( G \subset H \) means every vertex of \( G \) is a vertex of \( H \), every edge of \( G \) is an edge of \( H \), and every edge of \( H \) joining vertices of \( G \) is an edge of \( G \). The symbol \( K_n \) (the complete graph of order \( n \)) denotes a graph on \( n \) vertices, every pair of which are adjacent. The symbol \( K_{m,n} \) (the complete bipartite graph on \( (m,n) \) vertices) denotes a graph on \( m + n \) vertices in which the vertices are partitioned into subsets of \( m \) and \( n \) vertices, no two vertices in the same subset are adjacent, and each pair of vertices of different sets are adjacent. A graph is said to be of diameter 2 if there are two nonadjacent vertices, and whenever \( i \) and \( j \) are nonadjacent vertices there is a vertex \( h \) adjacent to both.

A vector \( x \) is said to be on an extreme ray of a cone \( C \) if \( x = y + z, x, y, z \in C \) implies \( y \) is a nonnegative multiple of \( x \). If \( A \) is a real symmetric matrix and \( A \) is positive semidefinite, then

\[
(Ax, x) = 0 \quad \text{implies} \quad Ax = 0.
\]

From [8], we have the result:

\[
\max_{x \geq 0, 2x_j \geq \frac{1}{2} \chi_j} (A(G)x, x) = \frac{t - 1}{t},
\]

where \( K_t \subset G, K_{t+1} \not\subset G \).

*Linear Algebra and Its Applications* 2(1969), 387–392
3. THEOREM 3.1

Let $M = (m_{ij})$ be a real symmetric matrix for which each $m_{ij} = \pm 1$, each $m_{ii} = 1$. Then

$$M \text{ is copositive if and only if } G(M) \text{ contains no triangles;}$$

$$M \text{ is copositive and on an extreme copositive ray if and only if } G(M) \text{ has no triangles and is of diameter 2.}$$

$$M \text{ is positive semidefinite if and only if } G(M) \text{ is } K_{p,n-p} \text{ for some } 0 < p < n. \text{ In this case, }$$

$$M \text{ is then also copositive and on an extreme ray.}$$

Proof of (3.1). Suppose $G(M)$ contains a triangle $(i_1, i_2, i_3)$. Let $x_{i_1} = x_{i_2} = x_{i_3} = 1$, all other $x_j = 0$. Then $x = (x_1, \ldots, x_n)$ satisfies $(Mx, x) = -3$, so $M$ is not copositive. Suppose $G(M)$ contains no triangle. From (2.1), we have

$$(Mx, x) = (Jx, x) - 2(A(G(M))x, x).$$

(3.4)

Clearly, it is sufficient to prove $(Mx, x) \geq 0$ for $x \geq 0$, $\sum x_j = 1$. But, for such an $x$, $(Jx, x) = 1$. From (3.4), it follows that all we need show is $(A(G(M))x, x) \leq \frac{1}{2}$. But, since $G(M)$ has no triangles, this follows from (2.3).

Proof of (3.2). We assume $G(M)$ has no triangles. Suppose $G(M)$ is not of diameter 2; then there exist vertices $i_1$ and $i_2$, not adjacent in $G(M)$, such that, if the edge joining them were added to $G(M)$, the new graph would still have no triangles. Let $B$ be the matrix in which $b_{i_1i_2} = b_{i_2i_1} = 1$, all other $b_{ij} = 0$. Then

$$M = (M - 2B) + 2B.$$ 

Since $G(M - 2B)$ has no triangles, it is copositive by (3.1). Since $2B$ has nonnegative entries, it is certainly copositive. Thus $M$ is not on an extreme ray.
Conversely, suppose $G(M)$ is of diameter 2,
\begin{equation}
M = N + Q, \tag{3.5}
\end{equation}

$N$ and $Q$ copositive. Let $n_{11} = x > 0$. We first show that, if $n_{ii} = x$ and $i$ and $j$ are adjacent, then $n_{jj} = x$, $n_{ij} = n_{ji} = -x$. This follows at once from the fact that the $2 \times 2$ principal submatrix
\[
\begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}
\]
of $M$ is on an extreme copositive ray in case $n = 2$. Hence, from (3.5), if $n_{ii} = x$, the rest follows. Since we have assumed $n_{11} = x$ and the graph is connected, we deduce that $n_{ii} = x$ for all $i$ and $n_{ij} = -x$ for all $i, j$ such that $m_{ij} = -1$. Suppose now $m_{ij} = +1$. Since $G(M)$ has diameter 2, there exists a vertex $k$ adjacent to both $i$ and $j$ in $G(M)$. Since the $3 \times 3$ submatrix
\[
\begin{pmatrix}
1 & -1 & 1 \\
-1 & 1 & -1 \\
1 & -1 & 1
\end{pmatrix}
\]
of $M$ is on an extreme copositive ray in case $n = 3$ [2], it follows from (3.5) and from the fact that $n_{ii} = x$ that the corresponding entries in $N$ are
\[
\begin{pmatrix}
x & -x & x \\
-x & x & -x \\
x & -x & x
\end{pmatrix}.
\]
Thus $N = xM$, which was to be proven.

**Proof of (3.3).** Suppose $M$ is positive semidefinite. Let $i_0$ and $i_1$ be adjacent vertices of $G(M)$, and let $x = (x_1, \ldots, x_n)$ be defined by $x_{i_0} = x_{i_1} = 1$, all other $x_j = 0$. Then $(MX, x) = 0$. From (2.2), $MX = 0$; hence every other vertex of $G(M)$ is adjacent to exactly one of the two vertices $i_0, i_1$. Let $S_j = \{x|x$ is adjacent to $i_j\}$, $j = 0, 1$. If $x \in S_1$, $y \in S_0$, then, since $x$ is not adjacent to $i_0$, and $y$ and $i_0$ are adjacent, it follows that $x$ is adjacent to $y$. Thus $G(M)$ is the complete bipartite graph cor-
responding to the partition $i_0 \cup S_1, i_1 \cup S_0$ of the vertices of $G(M)$. Conversely, if $G(M) = K_{p,n-p}$, it is clear that $M$ is positive semidefinite of rank 1, and on an extreme copositive ray according to [6].

4. THEOREM 4.1

A symmetric matrix $A$ is co;hositive with respect to a self-polar cone $C$ if, and only if, $A$ has the Perron property.

LEMMA. A closed convex cone $C$ contains its polar $C^*$ if and only if for each vector $x \in \mathbb{R}^n$ there exist vectors $y$ and $z$ such that

$$x = y - z, \quad y, z \in C, \quad (y, z) = 0.$$  \quad (4.1)

Proof. We do not know of any specific reference for this lemma, and we are indebted to Philip Wolfe both for the proof and for the information that it is probably part of the mathematical programming folklore.

Let $y$ be the vector in $C \cup -C$ closest (in the Euclidean norm) to $x$. If $y = x$ or 0, we are finished, so assume otherwise. Assume $y \in C$ (if $y \in -C$, replace $C$ by $-C$, since $A$ is copositive with respect to $-C$). Let $w \in C$, $0 < t < 1$. Then $\|x - y\|^2 \leq \|x - ((1 - t)y + tw)\|^2 = \|(x - y) + t(y - w)\|^2 = \|x - y\|^2 + t^2\|y - w\|^2 + 2t(x - y, y - w)$. It follows that $(x - y, y - w) \geq 0$ for all $w \in C$. Let $w = x\gamma, \gamma > 0$. Then $(x - y, y - w) = (1 - \alpha)(x - y, y) \geq 0$. Since this holds for $x < 1$ and $\alpha > 1$, it follows that $(x - y, y) = 0$. And it follows from $(x - y, y - w) \geq 0$ that $(x - y, w) \leq 0$. Therefore, $z = y - x \in C^* \subset C$, so (4.1) holds.

Conversely, assume that $C^* \subset C$. Then there exists a vector $x \in C^*$ such that $x \notin C$. Write $x = y - z$ with the stipulations of (4.1). Since $x \notin C, z \neq 0$. Then

$$0 \leq (z, x) = (z, y - z) = -(z, z) < 0,$$

a contradiction.

To prove Theorem 4.1, assume $\lambda < 0, -\lambda = \rho(A)$. $Ax = \lambda x, \|x\| = 1$. By the lemma, (4.1) holds. Let $\hat{x} = y + z$. Then $\|\hat{x}\| = 1$. Now,

$$(Ax, x) + (A\hat{x}, \hat{x}) = 2(Ay, y) + 2(Az, z) \geq 0,$$  \quad (4.2)

since $A$ is copositive with respect to $C$. Let $\rho$ be the maximum eigenvalue of $A$. As is well known, $(Aw, w) \leq \rho$ if $\|w\| = 1$. But, from (4.2), we
have $\rho \geq (Ax, x) \geq - (Ax, x) = - \lambda = \rho(A) \geq \rho$. Hence, $\rho = \rho(A)$; i.e., $A$ has the Perron property.

Conversely, assume $A$ has the Perron property, $\rho = \rho(A)$. Let $x_1, \ldots, x_n$ be orthonomal eigenvectors of $A$, with $Ax_1 = \rho x_1$, let $X$ be the square matrix of order $n$ whose $j$th row is $x^j$, let $C$ be the cone of all vectors $y = (y_1, \ldots, y_n)$ such that $(Xy)_1 \geq (\sum_{j=2}^n (Xy)_j)^{1/2}$. Clearly, $C = C^*$ (note $X$ is orthogonal), and $A$ is copositive with respect to $C$.

REFERENCES


Received December 16, 1968

*Linear Algebra and Its Applications* 2 (1969), 387-392