Evaluation of a certain $q$-determinant

Michitomo Nishizawa

Department of Mathematics, School of Science and Engineering, Waseda University, 3-4-1 Ohkubo Shinjuku-ku, Tokyo 169-8555, Japan

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Abstract

A $q$-analogue of Mehta–Wang’s determinant is introduced and evaluated. In the case when $q = 1$, this determinant contains a factor represented by using a special case of the Meixner–Pollaczek polynomial. A recurrence relation for the factor is derived in the case when $0 < |q| < 1$. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction and main result

Determinant expressions of orthogonal function are one of the most important problems in the special function theory. It is well known that the orthogonal polynomial for a given measure can be expressed as determinants. Motivated by Mehta and his collaborators’ works [4,5], we consider an other type of determinant expression. In order to calculate the Mellin transformations of the provability densities of certain random matrix models, they derived the following identities:

$$\det \left[ \Gamma(b + j + i) \right]_{0 \leq i, j \leq n-1} = \prod_{k=0}^{n-1} k! \Gamma(b + k),$$  \hspace{1cm} (1)

$$\det \left[ (a + j - 1) \Gamma(b + j + i) \right]_{0 \leq i, j \leq n-1} = D_n \prod_{k=0}^{n-1} k! \Gamma(b + k).$$  \hspace{1cm} (2)

E-mail address: mnishi@mn.waseda.ac.jp (M. Nishizawa).

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PII: S 0 0 2 4 - 3 7 9 5 ( 0 1 ) 0 0 4 3 8 - 4
where $\Gamma(z)$ is Euler’s gamma function. $D_n$ is represented as
\[ D_n := \det \left[ a\delta_{i,j} - \delta_{i,j+1} + j(b + i)\delta_{i,j-1} \right], \]
where $\delta_{k,l}$ is Kronecker’s delta
\[ \delta_{k,l} = \begin{cases} 1, & k = l, \\ 0, & \text{otherwise}. \end{cases} \]
From this representation, we can see that $D_n$ satisfies the recurrence relation
\[ D_0 = 1, \quad D_1 = a, \quad (n + 1)D_{n+1} = aD_n + n(b + n - 1)D_{n-1}. \tag{3} \]
This relation can be considered as a recurrence relation for a special case of the Meixner–Pollaczek polynomial $P_n^{(\lambda)}(x; \phi)$ (see, for example, [3]). Three-term relation for $P_n^{(\lambda)}(x; \phi)$ is known as
\[ (n + 1)P_n^{(\lambda)}(x; \phi) - 2\left[x \sin \phi + (n + \lambda \cos \phi)\right]P_n^{(\lambda)}(x; \phi) + (n + 2\lambda - 1)P_{n-1}^{(\lambda)}(x; \phi) = 0. \]
Thus, the recurrence relation (3) for $D_n$ coincides with the three-term relation for $n!P_n^{(b/2)}(a\sqrt{-1}/2; \pi/2)/(\sqrt{-1})^a$. Thus, $D_n$ can be represented as
\[ D_n = \frac{n!}{(\sqrt{-1})^n} P_n^{(b/2)} \left( \frac{a\sqrt{-1}}{2}; \frac{\pi}{2} \right) = \frac{\prod_{j=0}^{n-1}(b - j)}{(\sqrt{-1})^n} \, {}_2F_1 \left( \frac{-n, a+b}{2} \right) \left( \frac{1}{2} \right), \]
where
\[ {}_2F_1 \left( \frac{a, b}{c} \right) \]
is Gauss’ hypergeometric function. In this case, the weight function of the polynomial is symmetric and the orthogonality is not with respect to positive measures because coefficients of the recurrence relation have changed their sign. We also note that a generating function $D(z) := \sum_{n=0}^{\infty} D_n z^n / n!$ can be represented as
\[ D(z) = (1 - z)^{-\frac{(b-a)}{2}} (1 + z)^{-\frac{(b-a)}{2}} \]
and satisfies a differential equation
\[ (1 - z^2) \frac{d}{dz} D(z) = (a + bz) D(z). \tag{4} \]
As an extension of these identities, we consider their $q$-analogues. A $q$-analogue of (1) corresponds to a special case of Gram determinant representations derived by Wilson [7]. We construct a $q$-analogue of (2) in this paper. In order to describe it, we introduce some notations. We fix a complex parameter $q$ such that $0 < |q| < 1$. For $a \in \mathbb{C}$ and $m \in \mathbb{Z}_{>0}$, we introduce the following notations:
\[ [a] := \frac{1 - q^a}{1 - q}, \quad [a]_m := \prod_{k=0}^{m-1} [a + k], \quad [m]! := \prod_{k=1}^{m} [k], \]
\[
\begin{align*}
[a]_m & := \frac{[a + m - 1]_m}{[m]!}, \\
(a)_\infty & := \prod_{k=0}^{\infty} (1 - aq^k), \\
(a)_m & := \frac{(a)_\infty}{(aq^m)_\infty}.
\end{align*}
\]

For convenience, we define
\[
[a]_0 = [0]! = (a)_0 = 1, \\
[a]_m = [m]! = (a)_m = 0 \text{ for } m < 0.
\]

q-Analogue of the gamma function is known as
\[
\Gamma(z; q) := (1 - q)^{1-z} \frac{(q)_\infty}{(q^z)_\infty}.
\]

Our main result is stated as follows:

**Theorem 1.** For \(a, b \in \mathbb{C}\), we have
\[
\det[[a + j - i] \Gamma(b + i + j; q)]_{0 \leq i, j \leq n - 1} = q^{na - n(n - 1)(b^2/2 + n(n - 1)(2n - 7)/6) D_{n,q} D_{n-1,q}}
\]
\[
\prod_{k=0}^{n-1} [k] \Gamma(b + k; q). 
\]

\(D_{n,q}\) is the determinant of a matrix \((d_{i,j})_{0 \leq i, j \leq n - 1}\) defined as
\[
d_{i,j} = \begin{cases} 
(1 - q)^{i-j-1} \gamma_j, & i > j, \\
\alpha_j, & i = j, \\
\beta_j, & i = j - 1, \\
0, & i \leq j - 1,
\end{cases}
\]

where
\[
\begin{align*}
\alpha_j & := q^{-a+j} [a - j] + q^{-b-j+1} [j], \\
\beta_j & := q^{-b+j-1} [b + j - 1] [j], \\
\gamma_j & := q^{-b-j+1} - q^{-b+1} - 1.
\end{align*}
\]

\(D_{n,q}\) satisfies the following recurrence relation:
\[
D_{0,q} = 1, \quad D_{1,q} = \alpha_q = q^{-a}[a], \\
D_{n+1,q} = q^{-a+n}[a] D_{n,q} + q^{-a-b}[b + n - 1][n] D_{n-1,q} \quad \text{for } n \geq 1.
\]

We also note that the generating function
\[
D(z; q) := \sum_{n=0}^{\infty} \frac{D_{n,q}}{[n]!} z^n
\]
satisfies a \(q\)-difference equation
\[
(q^a - z^2)(\partial_q D)(z; q) = \left\{([a] \tau_q + q^{-b}[b]z) D\right\}(z; q),
\]
where \(\tau_q\) and \(\partial_q\) are \(q\)-difference operators acting on complex function \(f(z)\) as follows:
\[(\tau_q f)(z)(z) = f(qz), \quad (\partial_q f)(z) := \frac{f(z) - f(qz)}{(1 - q)z}.\]

This equation tends to the differential equation (4) as \(q \to 1\).

It is not known yet whether some models of mathematical physics correspond to these identities. However, they would be interesting from a viewpoint from \(q\)-analysis. It is expected that algebraic structures and symmetries of such \(q\)-determinants have some relations to important works on \(q\)-determinants, for example, [1,2,6].

2. Proof of theorem

In this section, we give a proof of Theorem 1. First, we prepare the following lemma:

**Lemma 2.** For \(A, B \in \mathbb{C}\),

\[
\sum_{k=0}^{j} (-1)^k q^{k(k-1)/2} \left[ \begin{array}{c} j \\ k \end{array} \right] (q^{A+j-k})_k (q^B)_{j-k} = q^{j(A+j-1)} (q^{B-A-j+1})_j.
\]

**Proof.** We prove this lemma by using induction on \(j\). In the case when \(j = 0\) and \(j = 1\), it can be easily shown. If we assume that this lemma holds for \(j\), then we obtain

\[
\begin{aligned}
&\sum_{k=0}^{j+1} (-1)^k q^{k(k-1)/2} \left[ \begin{array}{c} j+1 \\ k \end{array} \right] (q^{A+j+1-k})_k (q^B)_{j+1-k} \\
= &\sum_{k=0}^{j+1} (-1)^k q^{k(k-1)/2} \left[ \begin{array}{c} j \\ k \end{array} \right] + q^{j-k} \left[ \begin{array}{c} j+k \\ j-1 \end{array} \right] (q^{A+j+1-k})_k (q^B)_{j+1-k} \\
= &\sum_{k=0}^{j} (-1)^k q^{k(k-1)/2} \left[ \begin{array}{c} j \\ k \end{array} \right] (q^{A+j-k})_k (q^B)_{j-k} \\
- &q^{A+j} \sum_{k=1}^{j+1} (-1)^{k-1} q^{(k-1)(k-2)/2} \left[ \begin{array}{c} j \\ k-1 \end{array} \right] \times (q^{A+j-k-1})_{k-1} (q^B)_{j-(k-1)} \\
= &q^{j(A+j)} (1 - q^B) (q^{B-A-j+1})_j - q^{j+1} (q^{B-A-j+1})_j \\
= &q^{(j+1)(A+j)} (q^{B-A-j})_{j+1}
\end{aligned}
\]

by using the assumption of induction. \(\square\)
Corollary 3. For $A, B \in \mathbb{C}$ and $j \geq 1$,

$$\sum_{k=0}^{j} (-1)^{k} q^{k(k-1)/2} \binom{j}{k} (1 - q^{j-k})(q^{A+j-k})_{k}(q^{B})_{j-k}$$

$$= q^{(j-1)(A+j-1)}(1 - q^{j})(1 - q^{B})(q^{B-A+j+2})_{j-1}.$$ 

By using the above lemma, we rewrite the matrix $([a + j - i] \Gamma(b + i + j; q))_{i,j}$. Using column and row vectors, we represent the matrices as follows:

$$([a + j - i] \Gamma(b + i + j; q))_{0 \leq i,j \leq n-1} = (C_0, C_1, \ldots, C_{n-1}),$$

where

$$C_j := \begin{pmatrix} [a + j - 0] \Gamma(b + j + 0; q) \\ [a + j - 1] \Gamma(b + j + 1; q) \\ \vdots \\ [a + j - n + 1] \Gamma(b + j + n + 1; q) \end{pmatrix}.$$ 

We transform the $j$th column $C_j$ to

$$\tilde{C}_j := \sum_{k=0}^{j} (-1)^{k} q^{k(k-1)/2} \binom{j}{k} [b + j - k]_{k} C_{j-k}.$$ 

By using Lemma 2 and Corollary 3, $i$th element of $\tilde{C}_j$ is expressed as

$$\sum_{k=0}^{j} (-1)^{k} q^{k(k-1)/2} \binom{j}{k} [b + j - k]_{k} [a + j - k - i] \Gamma(b + i + j - k; q)$$

$$= \frac{\Gamma(b + i; q)}{(1 - q)^{j+1}} \left\{ (1 - q^{a-i}) \sum_{k=0}^{j} (-1)^{k} q^{k(k-1)/2} \binom{j}{k} (q^{b+j-k})_{k}(q^{b+i})_{j-k} 

+ q^{a-i} \sum_{k=0}^{j-1} (-1)^{k} q^{k(k-1)/2} (1 - q^{j-k}) \binom{j}{k} \right\}$$

$$\times (q^{b+j-k})_{k}(q^{b+i})_{j-k} \left\{ q^{j(b+j-1)[a - i][i - j + 1]} + q^{a-i+(j-1)(b+j-1)} 

\times [b + i][j][i - j + 2] \right\}. $$

Therefore, we can factor the determinant

$$\det \left\{ [a + j - i] \Gamma(b + i + j; q) \right\}_{0 \leq i,j \leq n-1}$$
\[
\prod_{k=0}^{n-1} \Gamma(b + k; q) \det \left[ q^{j(b+j-1)}[a - i][i + j + 1]_j + q^{a-i+j-1(b+j-1)} \times [b + i][j][i - j + 2]_{j-1} \right]_{0 \leq i, j \leq n-1}.
\]

We represent the matrix in the right-hand side by using row vectors
\[
\begin{pmatrix}
q^{j(b+j-1)}[a - i][i - j + 1]_j + q^{a-i+j-1(b+j-1)} \\
\times [b + i][j][i - j + 2]_{j-1}
\end{pmatrix}_{i,j}
\]

and replace \( R_i \) by
\[
\tilde{R}_i := \sum_{k=0}^{i} (-1)^k q^{k(k-1)/2} \begin{pmatrix} i \\ k \end{pmatrix} R_{i-k}.
\]

When \( i < j - 1 \), it is obvious that \( \tilde{R}_i = 0 \). In the case when \( i \geq j - 1 \), from the \( q \)-binomial theorem
\[
(x)_m = \sum_{k=0}^{m} (-1)^k q^{k(k-1)/2} \begin{pmatrix} m \\ k \end{pmatrix} x^k,
\]

it follows that the \( j \)th element of \( \tilde{R}_i \) is equal to
\[
[i]! \sum_{k=0}^{i} (-1)^k q^{k(k-1)/2} \left\{ q^{j(b+j-1)}[a - i] + q^{a-i}[k] \right\} \frac{[i - k - j + 1]_j}{[k]![i - k]!} \times \frac{[i - k - j + 1]_{j-1}}{[k]![i - k]!} - q^{a-i+j-1(b+j-1)} \sum_{k=1}^{i} (-1)^k q^{k(k-1)/2} \begin{pmatrix} i - j - 1 \\ k \end{pmatrix} q^k
\]
\[
+ q^{a-i+j-1(b+j-1)} \sum_{k=0}^{i} (-1)^k q^{k(k-1)/2} \begin{pmatrix} i - j + 1 \\ k \end{pmatrix} q^k
\]
Therefore, we obtain

\[
\det \left[ a + j - i \right] \Gamma (b + j + i; q) \right]_{0 \leq i, j \leq n-1} \\
= q^{\sum_{i=0}^{n-1} (a-i) + \sum_{j=0}^{n-1} j (b+j-1)} \det \left[ d_{i,j} \right]_{0 \leq i, j \leq n-1} \prod_{k=0}^{n-1} [k]! \Gamma (b + k; q) \\
= q^{na + n(n-1)b/2 + n(n-1)(2n-7)/6} D_{n,q} \prod_{k=0}^{n-1} [k]! \Gamma (b + k; q).
\]

Next, we consider the recurrence relation for \( D_{n,q} \). By a straightforward calculation, \( D_{0,q} \), \( D_{1,q} \) and \( D_{2,q} \) can be evaluated as

\[
D_{0,q} = 1, \quad D_{1,q} = q^{-a} [a], \\
D_{2,q} = \frac{1}{(1-q)^2} \left\{ q^{-2a-1} - q^{-a+1} - q^{-a} + q^{-b} - q^{-b+1} + q \right\}.
\]

For \( n \geq 3 \), by using the Laplace expansion for \( \det[d_{i,j}] \), we have

\[
D_{n+1,q} = \det
\begin{vmatrix}
\alpha_0 & \beta_1 & 0 & 0 & \cdots & 0 & 0 \\
\gamma_0 & \alpha_1 & \beta_2 & 0 & \cdots & & \\
(1-q)\gamma_0 & \gamma_1 & \alpha_2 & \ddots & & \ddots & \\
(1-q)^2\gamma_0 & (1-q)\gamma_1 & \gamma_2 & \ddots & \ddots & \ddots & \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \\
(1-q)^{n-2}\gamma_0 & (1-q)^{n-3}\gamma_1 & & \cdots & \cdots & \alpha_{n-1} & \beta_n \\
(1-q)^{n-1}\gamma_0 & (1-q)^{n-2}\gamma_1 & \cdots & (1-q)\gamma_{n-2} & \gamma_{n-1} & \alpha_n
\end{vmatrix}
\]
From (7), we can see that this relation also holds in the case when \( n = 1 \). Thus, we have derived the recurrence relation (3). Finally, we consider a \( q \)-difference equation for the generating function. From the above relation, it follows that

\[
q^n \sum_{n=0}^{\infty} \frac{[n+1]}{[n+1]!} D_{n+1,q} z^n - z^2 \sum_{n=0}^{\infty} \frac{[n-1]}{[n-1]!} D_{n-1,q} z^{n-2} = [a] \sum_{n=0}^{\infty} \frac{D_{n,q}}{[n]!} (qz)^n + q^{-b} [b] z \sum_{n=0}^{\infty} \frac{D_{n-1,q}}{[n-1]!} z^{n-1}.
\]

Therefore, \( D(z) \) satisfies Eq. (6).

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