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On the cellular decomposition and the Lusternik–Schnirelmann category of Spin(7)

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Abstract

We give a cellular decomposition of the compact connected Lie group $\mathrm{Spin}(7)$. We also determine the L–S categories of $\mathrm{Spin}(7)$ and $\mathrm{Spin}(8)$.

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1. Introduction

In this paper, we assume that a space has the homotopy type of a CW-complex.

Whitehead [15] constructed a cellular decomposition of $\mathrm{SO}(n)$ using the natural inclusion map $\mathbb{R}P^{n-1} \to \mathrm{SO}(n)$ (see also [7]). Yokota [16–18] constructed cellular decompositions of $\mathrm{SU}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ according to his principle that the number of the cells in the decomposition should be minimal, where the decomposition of $\mathrm{SU}(n)$ is constructed by making use of the natural inclusion map $\Sigma\mathbb{C}P^{n-1} \to \mathrm{SU}(n)$; see Remark 2.4(2). Araki [1] gave a cellular decomposition of $\mathrm{Spin}(n)$ using the decomposition

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of $\mathrm{SO}(n)$ and the double covering $\mathrm{Spin}(n) \to \mathrm{SO}(n)$, where the number of the cells of the decomposition is not minimal, that is, it does not satisfy the Yokota principle. One of our objects is to construct the cellular decomposition of Spin(7) so that it satisfy the Yokota principle.

Among the exceptional Lie groups, the second and third authors [8] constructed a cellular decomposition of G_2 which has the minimum number of the cells in the decomposition, that is, it satisfies the Yokota principle.

Recall that we have the following isomorphisms:

$$\mathrm{Spin}(3) \cong S^3, \qquad \mathrm{Spin}(4) \cong S^3 \times S^3,$$
$$\mathrm{Spin}(5) \cong \mathrm{Sp}(2), \qquad \mathrm{Spin}(6) \cong \mathrm{SU}(4).$$

Thus Spin(7) is the first non-trivial case in determining the cellular decomposition satisfying the Yokota principle, which is one of our purposes.

The other purpose is to determine the Lusternik–Schnirelmann category of Spin(7) by using the cellular decomposition.

The Lusternik–Schnirelmann category, $\operatorname{cat} X$, of a space X is the least integer n such that X is the union of $n+1$ open subsets, each of which is contractible in X. Whitehead [13] showed that $\operatorname{cat} X \leqslant n$ if and only if the diagonal map $\Delta_{n+1} : X \to \prod^{n+1} X$ is homotopic to a composition map

$$X \to \mathrm{T}^{n+1}(X) \hookrightarrow \prod^{n+1} X,$$

where $\mathrm{T}^{n+1}(X)$ is the fat wedge

$$\mathrm{T}^{n+1}(X) = \bigl\{(x_1, \dots, x_{n+1}) \in X^{n+1} \mid \text{some } x_i \text{ is the base point}\bigr\}$$

and $\mathrm{T}^{n+1}(X) \hookrightarrow \prod^{n+1} X$ is the inclusion map.

The weak Lusternik–Schnirelmann category, $w\operatorname{cat} X$, is the least integer n such that the reduced diagonal map

$$\bar{\Delta}_{n+1} : X \to \bigwedge^{n+1} X = \prod^{n+1} X / \mathrm{T}^{n+1}(X)$$

is trivial. Then it is easy to see that $w\operatorname{cat} X \leqslant \operatorname{cat} X$ using Whitehead's characterization [13] of the Lusternik–Schnirelmann category.

The strong Lusternik–Schnirelmann category, $\operatorname{Cat} X$, is the least integer n such that there exists a space X' which is homotopy equivalent to X and is covered by $n+1$ open subsets contractible in themselves. $\operatorname{Cat} X$ is closely related with $\operatorname{cat} X$, and Ganea and Takens [12] showed that

$$\operatorname{cat} X \leqslant \operatorname{Cat} X \leqslant \operatorname{cat} X + 1.$$

Ganea [3] showed that $\operatorname{Cat} X$ is equal to the invariant which is the least integer n such that there are n cofibre sequences $A_i \to X_{i-1} \to X_i$, $1 \leqslant i \leqslant n$, with $X_0 = *$ and X_n homotopy equivalent to X.

The Lusternik–Schnirelmann category of some Lie groups has been determined, such as $\operatorname{cat}(\mathrm{U}(n)) = n$ and $\operatorname{cat}(\mathrm{SU}(n)) = n-1$ by Singhof [10], $\operatorname{cat}(\mathrm{Sp}(2)) = 3$ by Schweitzer [9],

$\operatorname{cat}(\mathrm{Sp}(3)) = 5$ by the first and second named authors [4], and Fernández-Suárez et al. [2], $\operatorname{cat}(\mathrm{SO}(2)) = 1$, $\operatorname{cat}(\mathrm{SO}(3)) = 3$, $\operatorname{cat}(\mathrm{SO}(4)) = 4$, $\operatorname{cat}(\mathrm{SO}(5)) = 8$ by James and Singhof [5]. A simple argument gives that $\operatorname{cat}(\mathrm{G}_2) = 4$ (see, for example, [4]). Among these cases it is shown that $w\mathrm{cat}\, G = \operatorname{cat} G = \operatorname{Cat} G$ for $G = \mathrm{Sp}(n), \mathrm{G}_2$. As for $G = \mathrm{SO}(n)$ for $n = 2, 3, 4, 5$, one can also show the equality. We can observe that $\operatorname{Cat}(\mathrm{SU}(n)) \leqslant n - 1$ by modifying the categorical open subsets given by Singhof [10] so as to be contractible, e.g., by adding paths among contractible connected components of each categorical open subset and hence $\operatorname{cat} G = \operatorname{Cat} G$. Thus we have $w\mathrm{cat}\, G = \operatorname{cat} G = \operatorname{Cat} G$ for any G when $\operatorname{cat} G$ is determined.

Theorem 1.1. *We have* $w\mathrm{cat}(\mathrm{Spin}(7)) = \operatorname{cat}(\mathrm{Spin}(7)) = \operatorname{Cat}(\mathrm{Spin}(7)) = 5$.

Since Spin(8) is homeomorphic to $\mathrm{Spin}(7) \times S^7$, we obtain the following corollary.

Corollary 1.2. *We have* $w\mathrm{cat}(\mathrm{Spin}(8)) = \operatorname{cat}(\mathrm{Spin}(8)) = \operatorname{Cat}(\mathrm{Spin}(8)) = 6$.

The paper is organized as follows. In Section 2 we give a cellular decomposition of Spin(7) such that Spin(7) contains a subgroup SU(4), which turns out to be useful for determining the Lusternik–Schnirelmann category of Spin(7). In Section 3 we give a cone-decomposition of SU(4), which gives rise to the Lusternik–Schnirelmann category of Spin(7) in Section 4.

2. The cellular decomposition of Spin(7)

In this section, we use the notation in [8]. Let $\mathfrak{C}$ be the Cayley algebra. (We adopt the definition of the Cayley algebra from [19].) SO(8) acts on $\mathfrak{C}$ naturally since $\mathfrak{C} \cong \mathbb{R}^8$ as an $\mathbb{R}$-module. We regard SO(7) as the subgroup of SO(8) fixing e_0, the unit of $\mathfrak{C}$. As is well known, the exceptional Lie group G_2 is defined by

$$\mathrm{G}_2 = \left\{ g \in \mathrm{SO}(7) \mid g(x)g(y) = g(xy),\ x, y \in \mathfrak{C} \right\} = \mathrm{Aut}(\mathfrak{C}).$$

According to [19], for each $g \in \mathrm{SO}(7)$, there is a unique element $\tilde{g}$ up to sign such that $g(x)\tilde{g}(y) = \tilde{g}(xy)$, and $\mathrm{Spin}(7) = \{\tilde{g} \mid g \in \mathrm{SO}(7)\}$. If $g \in \mathrm{G}_2$, then $g = \tilde{g}$, so G_2 is a subgroup of Spin(7). Observe that the algebra generated by e_1 in $\mathfrak{C}$ is isomorphic to $\mathbb{C}$. SU(4) acts on $\mathfrak{C}$ naturally, since $\mathfrak{C} \cong \mathbb{C}^4$ as a $\mathbb{C}$-module whose basis is $\{e_0, e_2, e_4, e_6\}$. We regard SU(3) as the subgroup of SU(4) fixing e_0 and also as the subgroup of G_2 fixing e_1.

Let $D^i = \{(x_1, \ldots, x_i) \in \mathbb{R}^i \mid \sum x_i^2 \leqslant 1\}$. We define four maps:

$$A : D^3 \to \mathrm{SO}(8), \qquad B : D^2 \to \mathrm{SO}(8),$$
$$C : D^1 \to \mathrm{SO}(8), \qquad D : D^2 \to \mathrm{SO}(8)$$

as follows:

$$A(x_1,x_2,x_3)=\begin{pmatrix}1&&&&&&&\\&1&&&&&&\\&&1&&&&&\\&&&1&&&&\\&&&&1-2X^2&-2x_1X&-2x_2X&-2x_3X\\&&&&2x_1X&1-2X^2&2x_3X&-2x_2X\\&&&&2x_2X&-2x_3X&1-2X^2&2x_1X\\&&&&2x_3X&2x_2X&-2x_1X&1-2X^2\end{pmatrix},$$

$$B(y_1,y_2)=\begin{pmatrix}1&&&&&&&\\&1&&&&&&\\&&y_1&-y_2&-Y&0&&\\&&y_2&y_1&0&-Y&&\\&&Y&0&y_1&y_2&&\\&&0&Y&-y_2&y_1&&\\&&&&&&1&\\&&&&&&&1\end{pmatrix},$$

$$C(z_1)=\begin{pmatrix}1&&&&&&&\\&z_1&0&-Z&&&&\\&0&1&0&&&&\\&Z&0&z_1&&&&\\&&&&1&&&\\&&&&&z_1&0&-Z\\&&&&&0&1&0\\&&&&&Z&0&z_1\end{pmatrix},$$

$$D(w_1,w_2)=\begin{pmatrix}w_1&-w_2&-W&0&&&&\\w_2&w_1&0&-W&&&&\\W&0&w_1&w_2&&&&\\0&W&-w_2&w_1&&&&\\&&&&1&&&\\&&&&&1&&\\&&&&&&1&\\&&&&&&&1\end{pmatrix},$$

where we put for simplicity

$$X=\sqrt{1-x_1^2-x_2^2-x_3^2},\qquad Y=\sqrt{1-y_1^2-y_2^2},$$
$$Z=\sqrt{1-z_1^2},\qquad W=\sqrt{1-w_1^2-w_2^2}.$$

We prepare the following two lemmas.

Lemma 2.1. *The elements* $A(x_1, x_2, x_3)$, $B(y_1, y_2)$, $C(z_1)$ *and* $D(w_1, w_2)$ *belong to* $\mathrm{Spin}(7)$.

Proof. Note that the elements $A(x_1, x_2, x_3)$, $B(y_1, y_2)$ and $C(z_1)$ are exactly the same as in [8] so they belong to G_2 (see [8] for their properties). In the proof, we denote $D(w_1, w_2)$ simply by D. Obviously elements in the image of A and D commute with each other. Let D' be the matrix

$$\begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& w_1 & -w_2 & W & 0 \\ &&&& w_2 & w_1 & 0 & -W \\ &&&& -W & 0 & w_1 & -w_2 \\ &&&& 0 & W & w_2 & w_1 \end{pmatrix}.$$

Then we can show by a tedious calculation that $D'xDy = D(xy)$ for any $x, y \in \mathfrak{C}$, which gives us the result. $\square$

Let φ_3, φ_5, φ_6 and φ_7 be maps

$$\varphi_3 : D^3 \to \mathrm{Spin}(7),$$
$$\varphi_5 : D^3 \times D^2 \to \mathrm{Spin}(7),$$
$$\varphi_6 : D^3 \times D^2 \times D^1 \to \mathrm{Spin}(7),$$
$$\varphi_7 : D^3 \times D^2 \times D^2 \to \mathrm{Spin}(7)$$

respectively defined by the equalities

$$\varphi_3(\mathbf{x}) = A(\mathbf{x}),$$
$$\varphi_5(\mathbf{x}, \mathbf{y}) = B(\mathbf{y})A(\mathbf{x})B(\mathbf{y})^{-1},$$
$$\varphi_6(\mathbf{x}, \mathbf{y}, \mathbf{z}) = C(\mathbf{z})B(\mathbf{y})A(\mathbf{x})B(\mathbf{y})^{-1}C(\mathbf{z})^{-1},$$
$$\varphi_7(\mathbf{x}, \mathbf{y}, \mathbf{w}) = D(\mathbf{w})B(\mathbf{y})A(\mathbf{x})B(\mathbf{y})^{-1}D(\mathbf{w})^{-1},$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2)$, $\mathbf{z} = (z_1)$ and $\mathbf{w} = (w_1, w_2)$. As noted above φ_i for $i = 3, 5, 6$ maps into G_2 and hence into $\mathrm{Spin}(7)$. So does φ_7, since D belongs $\mathrm{Spin}(7)$. We define sixteen cells e^j for $j = 0, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 21$ respectively as follows:

$$e^0 = \{1\}, \quad e^3 = \mathrm{Im}\,\varphi_3, \quad e^5 = \mathrm{Im}\,\varphi_5, \quad e^6 = \mathrm{Im}\,\varphi_6, \quad e^7 = \mathrm{Im}\,\varphi_7,$$
$$e^8 = e^5e^3, \quad e^9 = e^6e^3, \quad e^{10} = e^7e^3, \quad e^{11} = e^6e^5, \quad e^{12} = e^7e^5,$$
$$e^{13} = e^6e^7, \quad e^{14} = e^6e^5e^3, \quad e^{15} = e^7e^5e^3, \quad e^{16} = e^6e^7e^3,$$
$$e^{18} = e^6e^7e^5, \quad e^{21} = e^6e^7e^5e^3,$$

where the product of two (or more) cells is defined by using the multiplication of Spin(7). For later use we observe that φ_i for $i = 3, 5, 7$ maps into SU(4). In fact, the matrices A, B, D belong to SU(4) by their definition.

Let S^7 be the unit sphere of $\mathfrak{C}$. Then we have a principal bundle over it:

$$\mathrm{SU}(3) \to \mathrm{SU}(4) \xrightarrow{p_0} S^7,$$

where $p_0(g) = ge_0$.

Lemma 2.2. *Let $V^7 = D^3 \times D^2 \times D^2$. Then the composite map $p_0\varphi_7 : (V^7, \partial V^7) \to (S^7, e_0)$ is a relative homeomorphism.*

Proof. We express the map $(p_0\varphi_7)|_{V^7 \setminus \partial V^7}$ as follows:

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = D(\mathbf{w})B(\mathbf{y})A(\mathbf{x})B(\mathbf{y})^{-1}D(\mathbf{w})^{-1}e_0 = \begin{pmatrix} 1 - 2X^2Y^2W^2 \\ 2x_1XY^2W^2 \\ 2(w_1X - x_1w_2)XY^2W \\ -2(w_2X + x_1w_1)XY^2W \\ 2(-y_1X + x_1y_2)XYW \\ 2(y_2X + x_1y_1)XYW \\ 2x_2XYW \\ 2x_3XYW \end{pmatrix}$$

and hence we have

$$\begin{pmatrix} 1 - a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = 2XYW \begin{pmatrix} XYW \\ x_1YW \\ (w_1X - x_1w_2)Y \\ -(w_2X + x_1w_1)Y \\ -y_1X + x_1y_2 \\ y_2X + x_1y_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

By a tedious calculation we can obtain that

$$x_1 = \frac{a_1\sqrt{(1-a_0)^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2}}{\sqrt{2(1-a_0)((1-a_0)^2 + a_1^2)}},$$

$$x_2 = \frac{a_6}{\sqrt{2(1-a_0)}},$$

$$x_3 = \frac{a_7}{\sqrt{2(1-a_0)}},$$

$$y_1 = \frac{a_1a_5 - (1-a_0)a_4}{\sqrt{((1-a_0)^2 + a_1^2)((1-a_0)^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2)}},$$

$$y_2 = \frac{a_1a_4 + (1-a_0)a_5}{\sqrt{((1-a_0)^2 + a_1^2)((1-a_0)^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2)}},$$
$$w_1 = \frac{(1-a_0)a_2 - a_1a_3}{\sqrt{((1-a_0)^2 + a_1^2)((1-a_0)^2 + a_1^2 + a_2^2 + a_3^2)}},$$
$$w_2 = \frac{-a_1a_2 - (1-a_0)a_3}{\sqrt{((1-a_0)^2 + a_1^2)((1-a_0)^2 + a_1^2 + a_2^2 + a_3^2)}}.$$

The details of checking are left to the reader. Thus the inverse map has been constructed, which completes the proof. □

In a similar way to that of Section 3 of [8], we can obtain the following theorem, which is essentially the same as Yokota's decomposition [16].

Proposition 2.3. $e^0 \cup e^3 \cup e^5 \cup e^7 \cup e^8 \cup e^{10} \cup e^{12} \cup e^{15}$ *thus obtained is a cellular decomposition of* $\mathrm{SU}(4)$.

Proof. First we show that $\mathring{e}^i \cap \mathring{e}^j = \emptyset$ if $i \neq j$. We consider the following three cases:

(1) For the case where $i, j \in \{0, 3, 5, 8\}$; both cells e^i and e^j are in $\mathrm{SU}(3)$ and $e^0 \cup e^3 \cup e^5 \cup e^8$ is a cellular decomposition of $\mathrm{SU}(3)$; see [8, Proposition 3.2]. Then we have $\mathring{e}^i \cap \mathring{e}^j = \emptyset$ if $i \neq j$.
(2) For the case where $i \in \{0, 3, 5, 8\}$ and $j \in \{7, 10, 12, 15\}$; we have $p_0(\mathring{e}^i) = \{e_0\}$ and $p_0(\mathring{e}^j) = S^7 \backslash \{e_0\}$. Then we have $\mathring{e}^i \cap \mathring{e}^j = \emptyset$.
(3) For the case where $i, j \in \{7, 10, 12, 15\}$; suppose that $A \in \mathring{e}^i \cap \mathring{e}^j$. Since $\mathring{e}^i = \mathring{e}^7\mathring{e}^{i-7}$ and $\mathring{e}^j = \mathring{e}^7\mathring{e}^{j-7}$, we can put $A = A_1A_2 = A_1'A_2'$ where $A_1, A_1' \in \mathring{e}^7$, $A_2 \in \mathring{e}^{i-7}$ and $A_2' \in \mathring{e}^{j-7}$. We have $A_1 = A_1'$, since $p_0(A_1) = p_0(A_1A_2) = p_0(A_1'A_2') = p_0(A_1')$ and $p_0|_{\mathring{e}^7}$ is monic. Then we have $A_2 = A_2'$ and the first case shows that $i - 7 = j - 7$, that is, $i = j$. Thus $\mathring{e}^i \cap \mathring{e}^j = \emptyset$ if $i \neq j$.

Next, we will check that the boundaries of the cells are included in the lower-dimensional cells. In the proof of Proposition 3.2 [8], it is proved that the boundaries $\dot{e}^3$, $\dot{e}^5$ and $\dot{e}^8$ are included in the lower-dimensional cells. Observe that the boundary $\dot{e}^7$ is the union of the following three sets:

$$\begin{aligned}\dot{e}^7 = {} & \{DBAB^{-1}D^{-1} \mid A \in A(\dot{D}^3),\ B \in B(D^2),\ D \in D(D^2)\}, \\ & \cup \{DBAB^{-1}D^{-1} \mid A \in A(D^3),\ B \in B(\dot{D}^2),\ D \in D(D^2)\}, \\ & \cup \{DBAB^{-1}D^{-1} \mid A \in A(D^3),\ B \in B(D^2),\ D \in D(\dot{D}^2)\}.\end{aligned}$$

The first set contains only the identity element, since A is the identity element. It is easy to see that the second set is contained in e^3 and that the third set is contained in e^5. We have $\dot{e}^{10} = e^7\dot{e}^3 \cup \dot{e}^7e^3 \subset e^7e^0 \cup e^5e^3 = e^7 \cup e^8$. We also have $\dot{e}^{12} = \dot{e}^7e^5 \cup e^7\dot{e}^5 \subset e^5e^5 \cup e^7e^3 = e^8 \cup e^{10}$, and $\dot{e}^{15} = \dot{e}^7e^5e^3 \cup e^7\dot{e}^5e^3 \cup e^7e^5\dot{e}^3 \subset e^5e^5e^3 \cup e^7e^3e^3 \cup e^7e^5 = e^8 \cup e^{10} \cup e^{12}$.

Finally, we will show that the inclusion map $e^0 \cup e^3 \cup e^5 \cup e^7 \cup e^8 \cup e^{10} \cup e^{12} \cup e^{15} \to \mathrm{SU}(4)$ is epic. Let $g \in \mathrm{SU}(4)$. If $p_0(g) = e_0$, then g is contained in $\mathrm{SU}(3) = e^0 \cup e^3 \cup e^5 \cup e^8$. Suppose that $p_0(g) \neq e_0$. There is an element $h \in e^7$ such that $p_0(h) = p_0(g)$. Thus we have $h^{-1}g \in \mathrm{SU}(3) = e^0 \cup e^3 \cup e^5 \cup e^8$, since $p_0(h^{-1}g) = e_0$. Therefore we have $g \in h(e^0 \cup e^3 \cup e^5 \cup e^8) \subset e^0 \cup e^3 \cup e^5 \cup e^7 \cup e^8 \cup e^{10} \cup e^{12} \cup e^{15}$. $\square$

Remark 2.4.

(1) We regard SO(6) as the subgroup of SO(7) fixing e_1. Let $\pi : \mathrm{Spin}(6) \to \mathrm{SO}(6)$ be the double covering. Then, according to the proof of Lemma 2.1, $\pi(\mathrm{SU}(4)) \subset \mathrm{SO}(6)$ so that $\pi|_{SU(4)} : \mathrm{SU}(4) \to \mathrm{SO}(6)$ is the double covering.

(2) According to [18], there is a subspace $\Sigma\mathbb{C}P^n$ of $\mathrm{SU}(n+1)$ which consists of the elements

$$M \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & e^{2i\theta} \end{pmatrix} M^{-1} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & e^{-2i\theta} \end{pmatrix}$$

for any elements M in $\mathrm{SU}(n+1)$. Obviously, the subcomplex $e^0 \cup e^3$ is $\mathrm{SU}(2) = \Sigma\mathbb{C}P^1$. It is easy to see that the subcomplex $e^0 \cup e^3 \cup e^5$ is homeomorphic to $\Sigma\mathbb{C}P^2$, since we have

$$\begin{aligned} BAB^{-1} &= BM \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{2i\theta} \end{pmatrix} M^{-1} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{-2i\theta} \end{pmatrix} B^{-1} \\ &= BM \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{2i\theta} \end{pmatrix} M^{-1}B^{-1} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{-2i\theta} \end{pmatrix} \end{aligned}$$

for some $M \in \mathrm{SU}(2)$. In a similar way, the subcomplex $e^0 \cup e^3 \cup e^5 \cup e^7$ is homeomorphic to $\Sigma\mathbb{C}P^3$. Thus the cellular decomposition of SU(4) is essentially the same as Yokota's decomposition. Moreover, according to Proposition 2.6 of Chapter IV of [11], we have $e^{2i+1}e^{2j+1} \subset e^{2j+1}e^{2i+1}$ for $i < j$; in fact we have $e^{2i+1}e^{2j+1} = e^{2j+1}e^{2i+1}$ (see [20]).

Let S^6 be the unit sphere of $\mathbb{R}^7$ whose basis is $\{e_i \mid 1 \leqslant i \leqslant 7\}$. We consider the following diagram

$$\begin{tikzcd} \mathrm{SU}(3) \arrow[r] \arrow[d] & \mathrm{G}_2 \arrow[r] \arrow[d] & S^6 \arrow[d, equal] \\ \mathrm{SU}(4) \arrow[r] \arrow[d] & \mathrm{Spin}(7) \arrow[r, "p"] \arrow[d, "\pi"'] & S^6 \arrow[d, equal] \\ \mathrm{SO}(6) \arrow[r] & \mathrm{SO}(7) \arrow[r] & S^6 \end{tikzcd}$$

where the horizontal lines are principal fibre bundles and $p(g) = \pi(g)e_1$.

Lemma 4.1 of [8] implies the following lemma immediately.

Lemma 2.5. *Put $V^6 = D^3 \times D^2 \times D^1$. Then the composite map $p\varphi_6 : (V^6, \partial V^6) \to (S^6, \{e_1\})$ is a relative homeomorphism.*

Now we can state one of our main results.

Theorem 2.6. *The cell complex $e^0 \cup e^3 \cup e^5 \cup e^6 \cup e^7 \cup e^8 \cup e^9 \cup e^{10} \cup e^{11} \cup e^{12} \cup e^{13} \cup e^{14} \cup e^{15} \cup e^{16} \cup e^{18} \cup e^{21}$ gives a cellular decomposition of* $\mathrm{Spin}(7)$.

Proof. First we show that $\mathring{e}^i \cap \mathring{e}^j = \emptyset$ if $i \neq j$. We consider the following three cases:

(1) For the case where $i, j \in \{0, 3, 5, 7, 8, 10, 12, 15\}$; both cells e^i and e^j are in SU(4) and $e^0 \cup e^3 \cup e^5 \cup e^7 \cup e^8 \cup e^{10} \cup e^{12} \cup e^{15}$ is a cellular decomposition of SU(4), whence we have $\mathring{e}^i \cap \mathring{e}^j = \emptyset$ if $i \neq j$.
(2) For the case where $i \in \{0, 3, 5, 7, 8, 10, 12, 15\}$ and $j \in \{6, 9, 11, 13, 14, 16, 18, 21\}$; we have $p(\mathring{e}^i) = \{e_1\}$ and $p(\mathring{e}^j) = S^6 \backslash \{e_1\}$, whence we have $\mathring{e}^i \cap \mathring{e}^j = \emptyset$.
(3) For the case where $i, j \in \{6, 9, 11, 13, 14, 16, 18, 21\}$, suppose that $A \in \mathring{e}^i \cap \mathring{e}^j$. Since $\mathring{e}^i = \mathring{e}^6\mathring{e}^{i-6}$ and $\mathring{e}^j = \mathring{e}^6\mathring{e}^{j-6}$, we can put $A = A_1A_2 = A_1'A_2'$, where $A_1, A_1' \in \mathring{e}^6$, $A_2 \in \mathring{e}^{i-6}$ and $A_2' \in \mathring{e}^{j-6}$. We have $A_1 = A_1'$, since $p(A_1) = p(A_1A_2) = p(A_1'A_2') = p(A_1')$ and $p|_{\mathring{e}^6}$ is monic. Then we have $A_2 = A_2'$ and the first case shows that $i - 6 = j - 6$, that is, $i = j$. Thus $\mathring{e}^i \cap \mathring{e}^j = \emptyset$ if $i \neq j$.

Next, we will check that the boundaries of the cells are included in the lower-dimensional cells. In Proposition 2.3, it is proved that the boundaries of the cells of SU(4) are included in the lower-dimensional cells. In the proof of Theorem 4.2 in [8], it was shown that $\dot{e}^6 \subset e^3 \cup e^5$, $\dot{e}^9 \subset e^6 \cup e^8$, $\dot{e}^{11} \subset e^5 \cup e^9$ and $\dot{e}^{14} \subset e^8 \cup e^9 \cup e^{11}$. By using (2) of Remark 2.4, we also obtain

$$
\begin{aligned}
\dot{e}^{13} &= e^6\dot{e}^7 \cup \dot{e}^6e^7 \subset e^{11} \cup e^{12},\\
\dot{e}^{16} &= e^6e^7\dot{e}^3 \cup e^6\dot{e}^7e^3 \cup \dot{e}^6e^7e^3 \subset e^{13} \cup e^{14} \cup e^{15},\\
\dot{e}^{18} &= e^6e^7\dot{e}^5 \cup e^6\dot{e}^7e^5 \cup \dot{e}^6e^7e^5 \subset e^{16} \cup e^{14} \cup e^{15},\\
\dot{e}^{21} &= e^6e^7e^5\dot{e}^3 \cup e^6e^7\dot{e}^5e^3 \cup e^6\dot{e}^7e^5e^3 \cup \dot{e}^6e^7e^5e^3 \subset e^{18} \cup e^{16} \cup e^{14} \cup e^{15}.
\end{aligned}
$$

Let $S = e^0 \cup e^3 \cup e^5 \cup e^7 \cup e^8 \cup e^{10} \cup e^{12} \cup e^{15}$ and $T = e^0 \cup e^3 \cup e^5 \cup e^6 \cup e^7 \cup e^8 \cup e^9 \cup e^{10} \cup e^{11} \cup e^{12} \cup e^{13} \cup e^{14} \cup e^{15} \cup e^{16} \cup e^{18} \cup e^{21}$. Finally, we will show that the inclusion map $T \to \mathrm{Spin}(7)$ is epic. Let $g \in \mathrm{Spin}(7)$. If $p(g) = e_1$, then g is contained in $\mathrm{SU}(4) = S$. Suppose that $p(g) \neq e_1$. There is an element $h \in e^6$ such that $p(h) = p(g)$. Thus we have $h^{-1}g \in \mathrm{SU}(4)$ since $p(h^{-1}g) = e_1$. Therefore we have $g \in hS \subset T$. □

Remark 2.7. Araki [1] also gave a cellular decomposition of Spin(n), but the one we have given here is a cellular decomposition with the minimum number of cells, satisfying the Yokota principle [16,18,20]. As will be seen later, it is effectively used to determine the Lusternik–Schnirelmann category.

It is easy to give a cellular decomposition of Spin(8) using a homeomorphism

$$\mathrm{Spin}(8) \to \mathrm{Spin}(7) \times S^7.$$

3. The cone-decomposition of SU(4)

Obviously there is a filtration $F_0' = * \subset F_1' = \mathrm{SU}(4)^{(7)} \subset F_2' = \mathrm{SU}(4)^{(12)} \subset F_3' = \mathrm{SU}(4)$. It is well-known that $F_1' = \Sigma\mathbb{C}P^3 = S^3 \cup e^5 \cup e^7$ and $F_2' = F_1' \cup e^8 \cup e^{10} \cup e^{12}$. Thus the integral cohomology $H^n(F_2'; \mathbb{Z})$ is given by

$$H^n\bigl(F_2'; \mathbb{Z}\bigr) \cong \begin{cases} \mathbb{Z}\langle 1\rangle & (n=0), \\ \mathbb{Z}\langle y_n\rangle & (n=3,5,7,8,10,12), \\ 0 & (\text{otherwise}). \end{cases}$$

The action of the squaring operation Sq^2 is given as follows:

$$Sq^2 y_n = \begin{cases} y_{n+2} & \text{for } n=3,10, \\ 0 & \text{for } n=5,7,8,12 \end{cases}$$

where y_n is regarded as an element of the mod 2 cohomology. To give the cone decomposition of SU(4), we use the following homotopy fibration:

$$F \xrightarrow{\Psi} F_1' \xrightarrow{\iota} F_2'. \tag{3.1}$$

Without loss of generality, we may regard this as a Hurewicz fibration over F_2'.

Firstly we consider the Serre spectral sequence $(E_r^{*,*}, d_r)$ associated with the above fibration, where the generators of $E_2^{*,0}$ for $* \leqslant 7$ are permanent cycles and survive to E_∞-terms. Hence F is 6-connected and the transgression $\tau : H^7(F; \mathbb{Z}) \to H^8(F_2'; \mathbb{Z})$ is an isomorphism to $H^8(F_2'; \mathbb{Z}) \cong \mathbb{Z}\langle y_8\rangle$. Thus $H^7(F; \mathbb{Z}) \cong \mathbb{Z}\langle x_7\rangle$ for some $x_7 \in H^7(F; \mathbb{Z})$. Similarly, the generators in $E_2^{3,7} \cong \mathbb{Z}\langle y_3 \otimes x_7\rangle$ and $E_2^{10,0} \cong H^{10}(F_2'; \mathbb{Z}) \cong \mathbb{Z}\langle y_{10}\rangle$ must lie in the image of differentials d_3 and $d_{10} = \tau : H^9(F; \mathbb{Z}) \to H^{10}(F_2'; \mathbb{Z})$ respectively, and we have that $H^8(F; \mathbb{Z}) = 0$ and $H^9(F; \mathbb{Z}) \cong \mathbb{Z}\langle x_9\rangle \oplus \mathbb{Z}\langle x_9'\rangle$, where the elements x_9 and x_9' in $H^9(F; \mathbb{Z})$ correspond to x_{10} and $y_3 \otimes x_7$ by the transgression τ and d_3 respectively. We remark that the choice of the generator x_9' is not unique. Continuing this process, we have that $H^{10}(F; \mathbb{Z}) = 0$ and $H^{11}(F; \mathbb{Z}) \cong \mathbb{Z}\langle x_{11}\rangle \oplus \mathbb{Z}\langle x_{11}'\rangle \oplus \mathbb{Z}\langle x_{11}''\rangle \oplus \mathbb{Z}\langle x_{11}'''\rangle$ whose generators correspond to x_{12}, $y_3 \otimes x_9$, $y_3 \otimes x_9'$ and $y_5 \otimes x_7$ respectively by the transgression τ and differentials d_3, d_3 and d_5.

Thus the integral cohomology $H^n(F; \mathbb{Z})$ for $0 \leqslant n \leqslant 11$ is given by

$$H^n(F; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}\langle 1\rangle & (n=0), \\ \mathbb{Z}\langle x_7\rangle & (n=7), \\ \mathbb{Z}\langle x_9\rangle \oplus \mathbb{Z}\langle x_9'\rangle & (n=9), \\ \mathbb{Z}\langle x_{11}\rangle \oplus \mathbb{Z}\langle x_{11}'\rangle \oplus \mathbb{Z}\langle x_{11}''\rangle \oplus \mathbb{Z}\langle x_{11}'''\rangle & (n=11), \\ 0 & (\text{otherwise}) \end{cases}$$

where x_7, x_9 and x_{11} are transgressive generators in $H^*(F; \mathbb{Z})$. Hence F has, up to homotopy, a cellular decomposition $e^0 \cup e^7 \cup_{\varphi_1} e^9 \cup_{\varphi_1'} e_1^9 \cup_{\varphi_2} e^{11} \cup$ (cells in dimensions

$\geqslant 11$), where the cells e^7, e^9 and e^{11} correspond to x_7, x_9 and x_{11} respectively. Then we obtain a subcomplex $A' = e^0 \cup e^7 \cup_{\varphi_1} e^9 \cup_{\varphi'_1} e^9_1 \cup_{\varphi_2} e^{11}$ of F.

Secondly, we determine the attaching maps φ_1 and φ'_1: Let us recall that $\pi_8(S^7) \cong \mathbb{Z}/2\langle\eta_7\rangle$ whose generator η_7 can be detected by Sq^2, the mod 2 Steenrod operation. Since the action of mod 2 Steenrod operation commutes with the cohomology transgression (see [6, Proposition 6.5]), we see that Sq^2x_7 is transgressive, and hence is cx_9 for some $c \in \mathbb{Z}/2$. We know that $\tau x_9 = y_{10} \neq 0$ and $\tau Sq^2 x_7 = Sq^2\tau x_7 = Sq^2 y_8 = 0$, and hence Sq^2x_7 must be trivial. Thus the attaching maps φ_1 and φ'_1 are both null homotopic and A' is homotopy equivalent to $(S^7 \vee S^9 \vee S^9_1) \cup_{\varphi_2} e^{11}$.

Thirdly we check the composition of projections with the attaching map $\varphi_2 : S^{10} \to S^7 \vee S^9 \vee S^9_1$ to S^9 and S^9_1, which can also be detected by Sq^2. Again by the commutativity of the action of mod 2 Steenrod operation with the transgression, we see that the composition map $\mathrm{pr}_{S^9} \circ \varphi_2 : S^{10} \xrightarrow{\varphi_2} S^7 \vee S^9 \vee S^9_1 \to S^9$ represents a generator of $\pi_{10}(S^9) \cong \mathbb{Z}/2\langle\eta_9\rangle$, since $Sq^2 : H^8(F'_2; \mathbb{Z}/2) \to H^{10}(F'_2; \mathbb{Z}/2)$ is non-trivial. If the composition map $\phi_1 = \mathrm{pr}_{S^9_1} \circ \varphi_2 : S^{10} \xrightarrow{\varphi_2} S^7 \vee S^9 \vee S^9_1 \to S^9_1$ is non-trivial, we replace φ_2 by the composition of φ_2 and the homotopy equivalence $\xi : S^7 \vee S^9 \vee S^9_1 \to S^7 \vee S^9 \vee S^9_1$ where $\xi|_{S^7}$ and $\xi|_{S^9_1}$ are the identity maps and $\xi|_{S^9}$ is the unique co-H-structure map $\phi : S^9 \to S^9 \vee S^9_1$; then we obtain that ϕ_1 is trivial, since $2\eta_9 = 0$. Then A' is homotopy equivalent to $((S^7 \vee S^9) \cup_{\varphi_2} e^{11}) \vee S^9_1$. Let A denote the subcomplex $(S^7 \vee S^9) \cup_{\varphi_2} e^{11}$ of A' and $\psi = \Psi|_A : A \to F'_1$.

Lemma 3.1. *F'_2 is homotopy equivalent to $F'_1 \cup_\psi CA$.*

Proof. The elements in $H^*(F; \mathbb{Z})$ corresponding to those in $H^*(A; \mathbb{Z})$ under the induced map of the inclusion coincides with the module of transgressive elements with respect to the fibration (3.1) (see [6, Chapter 6]). Thus we may identify $H^{n-1}(A; \mathbb{Z}) = \delta_F^{-1}(\iota_F^*(H^n(F'_2, *; \mathbb{Z}))) \subset H^{n-1}(F; \mathbb{Z})$:

$$\begin{array}{ccccc}
H^{n-1}(F;\mathbb{Z}) & \xrightarrow{\delta_F} & H^n(F'_1, F;\mathbb{Z}) & \xleftarrow{\iota_F^*} & H^n(F'_2, *;\mathbb{Z}) \\
\downarrow & & & & \downarrow = \\
H^{n-1}(A;\mathbb{Z}) & \xrightarrow{\delta_A} & H^n(F'_1, A;\mathbb{Z}) & \xleftarrow{\iota_A^*} & H^n(F'_2, *;\mathbb{Z}),
\end{array}$$

where ι_F and ι_A are given by ι, and δ_F and δ_A denote the connecting homomorphisms of the long exact sequences for the pairs (F'_1, F) and (F'_1, A), respectively. Thus the image of δ_A is contained in the image of ι_A^* and we also have

$$H^n\big(F'_1, A; \mathbb{Z}\big) \cong H^n\big(F'_1 \cup_\psi CA, CA; \mathbb{Z}\big) \cong H^n\big(F'_1 \cup_\psi CA, *; \mathbb{Z}\big).$$

Since the composition map $A \xrightarrow{\psi} F'_1 \xrightarrow{\iota} F'_2$ is trivial, we can define a map

$$f : F'_1 \cup_\psi CA \to F'_2,$$

by $f|_{F'_1} = \iota : F'_1 \to F'_2$ and $f|_{CA} = *$.

To complete the lemma, we must show that $f^*: H^n(F_2'; \mathbb{Z}) \cong \mathbb{Z} \to H^n(F_1' \cup_\psi CA; \mathbb{Z}) \cong \mathbb{Z}$ is an isomorphism for $n = 3, 5, 7, 8, 10, 12$. We have a commutative diagram

$$\begin{tikzcd}
 & H^n(F_2';\mathbb{Z}) \arrow[r, "\iota^*"] \arrow[d, "f^*"'] & H^n(F_1';\mathbb{Z}) \arrow[d, "="] \\
H^n(F_1' \cup CA, F_1';\mathbb{Z}) \arrow[r, "j^*"] & H^n(F_1' \cup CA;\mathbb{Z}) \arrow[r, "i^*"] & H^n(F_1';\mathbb{Z}),
\end{tikzcd}$$

where the bottom row is a part of the exact sequence for the pair $(F_1' \cup CA, F_1')$. The induced map i^* is an isomorphism for $n \leqslant 7$, since $H^n(F_1' \cup CA, F_1'; \mathbb{Z}) = 0$ for $n \leqslant 7$ and since ι^* is an isomorphism for $n \leqslant 7$. Then we obtain that f^* is an isomorphism for $n \leqslant 7$. Moreover we can show that $j^*: H^n(F_1' \cup CA, F_1'; \mathbb{Z}) \to H^n(F_1' \cup CA; \mathbb{Z})$ is an isomorphism for $n \geqslant 8$, by considering the exact sequence for the pair $(F_1' \cup CA, F_1')$, since we have $H^n(F_1') = 0$ for $n \geqslant 8$. To perform the other cases for $n = 8, 10, 12$, it is sufficient to show that f^* is surjective. In fact, we have a commutative diagram

$$\begin{tikzcd}
H^{n-1}(A;\mathbb{Z}) \arrow[r, "\delta_A"] \arrow[d, "\Sigma"', "\cong"] & H^n(F_1', A;\mathbb{Z}) \arrow[dr, "\cong"] & H^n(F_2', *;\mathbb{Z}) \arrow[l, "\iota_A^*"'] \arrow[d, "f^*"] \\
H^n(\Sigma A, *;\mathbb{Z}) \arrow[r, "\cong"] & H^n(F_1' \cup CA, F_1';\mathbb{Z}) \arrow[u] \arrow[r, "j^*"] & H^n(F_1' \cup CA, *;\mathbb{Z}),
\end{tikzcd}$$

where Σ is the suspension isomorphism. Since j^* is an isomorphism for $n \geqslant 8$, we obtain that δ_A is an isomorphism for $n \geqslant 8$. Since the image of δ_A is contained in the image ι_A^*, we see that f^* is surjective for $n \geqslant 8$, and hence f is a homotopy equivalence. □

Proposition 3.2. *We have* $w\mathrm{cat}(F_i') = \mathrm{cat}(F_i') = \mathrm{Cat}(F_i') = i$.

Proof. The cohomology of F_i' implies that $w\mathrm{cat}(F_i') \geqslant i$. The cone-decomposition

$$F_1' = \Sigma \mathbb{C}P^3, \qquad F_2' \simeq F_1' \cup CA, \qquad F_3' = F_2' \cup CS^{14}$$

implies that $\mathrm{Cat}(F_i') \leqslant i$, which completes the proof. □

4. Proof of Theorem 1.1

We define a filtration $F_0 = * \subset F_1 \subset F_2 \subset F_3 \subset F_4 \subset F_5 = \mathrm{Spin}(7)$ by

$$F_1 = \mathrm{SU}(4)^{(7)}, \qquad F_2 = \mathrm{SU}(4)^{(12)} \cup e^6,$$
$$F_3 = \mathrm{SU}(4) \cup e^6 \cup e^9 \cup e^{11} \cup e^{13}, \qquad F_4 = \mathrm{Spin}(7)^{(18)}.$$

We need the following lemma to prove Theorem 4.2.

Lemma 4.1. *We have a homeomorphism of pairs*

$$(CA_1, A_1) \times (CA_2, A_2) = \big(C(A_1 * A_2), A_1 * A_2\big).$$

(The proof can be found in pp. 482–483 of [14].)

Now Theorem 1.1 follows from the following theorem.

Theorem 4.2. *We have* $w\mathrm{cat}(F_i) = \mathrm{cat}(F_i) = \mathrm{Cat}(F_i) = i$.

Proof. The mod 2 cohomology of F_i implies that $w\mathrm{cat}(F_i) \geqslant i$. Then it is sufficient to show that $\mathrm{Cat}(F_i) \leqslant i$. Obviously we have a homeomorphism $F_1 = \Sigma\mathbb{C}P^3$. Since the cell e^6 is attached to F_1, we obtain that $F_2 \simeq F_1 \cup C(S^5 \vee A)$ using Lemma 3.1. Since we have $e^9 \cup e^{11} \cup e^{13} = e^6(e^3 \cup e^5 \cup e^7)$, the composition map

$$\begin{aligned}\big(CS^5, S^5\big) \times \big(C\mathbb{C}P^3, \mathbb{C}P^3\big) &\to \big(CS^5, S^5\big) \times \big(\Sigma\mathbb{C}P^3, *\big)\\ &\to \big(F_2 \cup e^9 \cup e^{11} \cup e^{13}, F_2\big)\end{aligned}$$

is a relative homeomorphism. Then we obtain $F_2 \cup e^9 \cup e^{11} \cup e^{13} = F_2 \cup C(S^5 * \mathbb{C}P^3)$ using Lemma 4.1. The cell e^{15} is the highest-dimensional cell of $\mathrm{SU}(4)$ and is attached to F_2. Then we obtain $F_3 \simeq F_2 \cup C(S^{14} \vee (S^5 * \mathbb{C}P^3))$. Now we consider the following composition map:

$$\big(C\big(S^5 * A\big), S^5 * A\big) = \big(CS^5, S^5\big) \times (CA, A) \to \big(CS^5, S^5\big) \times \big(F'_2, F'_1\big) \to (F_4, F_3).$$

Since we have $e^{14} \cup e^{16} \cup e^{18} = e^6(e^8 \cup e^{10} \cup e^{12})$, the right map is a relative homeomorphism. The left map induces an isomorphism of homologies of pairs so that the map $H_*(F_3 \cup C(S^5 * A), F_3; \mathbb{Z}) \to H_*(F_4, F_3; \mathbb{Z})$ is an isomorphism. Thus we obtain $F_4 \simeq F_3 \cup C(S^5 * A)$. Obviously we have a homeomorphism $F_5 = F_4 \cup CS^{20}$. □

References

[1] S. Araki, On the homology of spinor groups, Mem. Fac. Sci. Kyusyu Univ. Ser. A. 9 (1955) 1–35.

[2] L. Fernández-Suárez, A. Gómez-Tato, J. Strom, D. Tanré, The Lusternik–Schnirelmann category of $\mathrm{Sp}(3)$, Trans. Amer. Math. Soc., submitted for publication.

[3] T. Ganea, Lusternik–Schnirelmann category and strong category, Illinois J. Math. 11 (1967) 417–427.

[4] N. Iwase, M. Mimura, L–S categories of simply-connected compact simple Lie groups of low rank, Proc. of the Skye Conference, submitted for publication.

[5] I.M. James, W. Singhof, On the category of fibre bundles, Lie groups, and Frobenius maps, in: Higher Homotopy Structures in Topology and Mathematical Physics (Poughkeepsie, NY, 1996), in: Contemp. Math., Vol. 227, 1999, pp. 177–189.

[6] J. McCleary, A User's Guide to Spectral Sequences, 2nd Edition, Cambridge University Press, Cambridge, 2001.

[7] C.E. Miller, The topology of rotation groups, Ann. of Math. (2) 57 (1953) 90–114.

[8] M. Mimura, T. Nishimoto, On the cellular decomposition of the exceptional Lie group G_2, Proc. Amer. Math. Soc. 130 (2002) 2451–2459.

[9] P.A. Schweitzer, Secondary cohomology operations induced by the diagonal mapping, Topology 3 (1965) 337–355.

[10] W. Singhof, On the Lusternik–Schnirelmann category of Lie groups, Math. Z. 145 (1975) 111–116.

[11] N.E. Steenrod, Cohomology Operations, in: Ann. Math. Stud., Vol. 50, Princeton University Press, Princeton, NJ, 1962.

[12] F. Takens, The Lusternik–Schnirelman categories of a product space, Compositio Math. 22 (1970) 175–180.

[13] G.W. Whitehead, The homology suspension, Colloque de topologie algébrique, Louvain (1956) 89–95.

[14] G.W. Whitehead, Elements of Homotopy Theory, in: Graduate Texts in Math., Vol. 61, Springer-Verlag, Berlin, 1978.
[15] J.H.C. Whitehead, On the groups $\pi_r(V_{n,m})$ and sphere-bundles, Proc. London Math. Soc. 48 (1944) 243–291.
[16] I. Yokota, On the cell structures of $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$, Proc. Japan Acad. 31 (1955) 673–677.
[17] I. Yokota, On the cells of symplectic groups, Proc. Japan Acad. 32 (1956) 399–400.
[18] I. Yokota, On the cellular decompositions of unitary groups, J. Inst. Polytech. Osaka City Univ. Ser. A 7 (1956) 39–49.
[19] I. Yokota, Representation rings of group G_2, J. Fac. Sci. Shinshu Univ. 2 (1967) 125–138.
[20] I. Yokota, Groups and Topology, Shokabo, 1971 (in Japanese).